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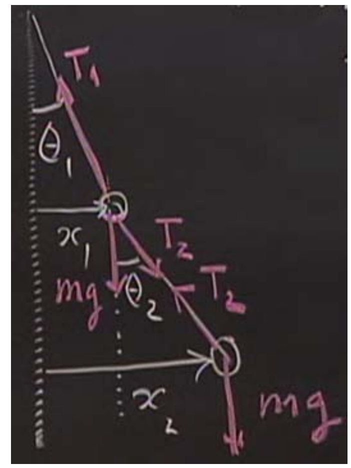
8.03SC **Fall 2012**

Notes for Lecture #6: Driven Coupled Oscillators



The system of the double pendulum, two bobs on one string, is analyzed using the recipe introduced in the last lecture. In the small angle approximation, the tension in the upper string supports both (identical) masses, and that in the lower string just one, so $T_1 = 2mg$ and $T_2 = mg$. The horizontal restoring force on the lower bob is $T_2 \sin\theta_2$, and $\sin\theta_2 = (x_2 - x_1)/\ell$, so Newton's Second Law for this bob is $m\ddot{x}_2 = -T_2 \sin\theta_2 = -\frac{mg}{\ell}(x_2 - x_1)$. Dividing by m and with $\omega_0^2 = g/\ell$, $\ddot{x}_2 + \omega_0^2 x_2 - \omega_0^2 x_1 = 0$ (**4:10**).
The first bob has two forces acting on it, so Newton's Second Law gives $m\ddot{x}_1 = -T_1 \sin\theta_1 + T_2 \sin\theta_2 = -2mg\dfrac{x_1}{\ell} + mg\dfrac{x_2 - x_1}{\ell}$, where we used $\sin\theta_1 = x_1/\ell$, so $\ddot{x}_1 + 3\omega_0^2 x_1 - \omega_0^2 x_2 = 0$ (**6:10**). Use $x_1 = C_1 \cos wt$ and $x_2 = C_2 \cos wt$ as trial solutions. Taking the derivatives and noting that we do not need to keep the common $\cos\omega t$ terms, we get $-C_1\omega^2 + 3\omega_0^2 C_1 - \omega_0^2 C_2 = 0$ and $-C_1\omega_0^2 + C_2(\omega_0^2 - \omega^2) = 0$ (**9:45**). These two equations have three unknowns, but we only need to find ω and the ratio of the C's (the overall magnitude of the C's is determined by initial conditions).

Instead of using the usual algebra, Prof. Lewin finds the solutions to this set of two equations using *Cramer's Rule*. He does this because when there are more than two objects, the algebraic method gets very long and tedious. At this point, Prof. Lewin shows a set of linear algebra equations for three equations which he has sent the students vie e-mail (**10:50**).

The next few paragraphs of this lecture note reviews the main ideas behind Cramer's Rule. The aim is to be able to solve systems of simultaneous linear equations. The method can be generalized to any number of solvable linear equations with the same number of variables as equations. Assume a system of three linear equations with three unknows, x, y, and z, and parameters a, b, and c for each equation, along with a right hand side d (each of these has a subscript identifying which equation it is in). The solution involves a single number (a *scalar*) which is combination of the parameters called the *determinant*. It involves the parameters multiplied together with appropriate signs. In the case of the system of two equations in two unknowns, $\begin{matrix} a_1x + b_1y = d_1 \\ a_2x + b_2y = d_2 \end{matrix}$, the determinant is denoted $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and is computed as $D = a_1b_2 - a_2b_1$. Phrased more generally, this formula could come about by applying the following procedure sequentially, take any entry in the upper row, multiply by $(-1)^{1+CN}$ (where CN is the column number), then multiply by what is not in the same row and column as that entry. This recipe can be expanded to larger sets of equations (the set of parameters is in fact a *matrix*) since it will apply even if the "what is not in the same row and column" is another matrix. We can in this way proceed to the calculation of 3 by 3 determinants (the above being two by two) which gives the example of Cramer's Rule shown in the lecture.

For three equations in three unknowns: $\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}$ so we have D=$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. By the above recipe, $D = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$. Note that the variables are arranged by column, x in the first column, y in the second, z in the third and that the $(-1)^{1+CN}$ makes the middle term negative. The 3×3 determinant can now be computed by expanding out the small determinants using the simple rule for 2×2 determinants. Cramer's Rule says that x, y, and z are obtained by taking the determinant from the original parameters, D, and using it as a denominator. The numerators are the determinant of a matrix found by replacing the column for a given variable by the right hand side (RHS) of the three equations, the column $\begin{matrix} d_1 \\ d_2 \\ d_3 \end{matrix}$ in this case.

With D=$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, we have $x = \dfrac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{D}$, $y = \dfrac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{D}$, and $z = \dfrac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{D}$

We now return to the discussion in the actual lecture. Despite knowing that we will in the end just get the ratio of the C's, we start by trying to solve for individual C terms. Recall that the equations we are trying to solve (slightly rearranged) are:

$$(3\omega_0^2 - \omega^2)C_1 - \omega_0^2 C_2 = 0 \quad \text{and} \quad -\omega_0^2 C_1 + (\omega_0^2 - \omega^2)C_2 = 0$$

Then $D = \begin{vmatrix} 3\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega^2 \end{vmatrix}$ (**11:40**), and we can use Cramer's Rule to get in turn:

$$C_1 = \frac{\begin{vmatrix} 0 & -\omega_0^2 \\ 0 & \omega_0^2 - \omega^2 \end{vmatrix}}{D} \qquad C_2 = \frac{\begin{vmatrix} 3\omega_0^2 - \omega^2 & 0 \\ -\omega_0^2 & 0 \end{vmatrix}}{D}$$

which are both 0 if D is not 0. These would correspond to trivial solutions, correct but uninteresting cases of no motion (**13:30**). The only way to get non-trivial solutions is to require $D = 0$.

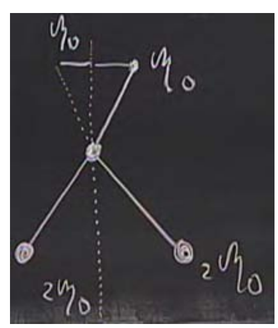
So, we have $0 = D = \begin{vmatrix} 3\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega^2 \end{vmatrix} = (3\omega_0^2 - \omega^2)(\omega_0^2 - \omega^2) - \omega_0^4$ which is actually just a quadratic in ω^2. The formula $x_\pm = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ gives the solutions of $ax^2+bx+c=0$, and can be used to find the two possible values of ω^2. They are $\omega_-^2 = (2-\sqrt{2})\omega_0^2$ and $\omega_+^2 = (2+\sqrt{2})\omega_0^2$ (**15:30**). Roughly, $\omega_- \approx 0.76\omega_0$ and $\omega_+ \approx 1.85\omega_0$. By some messy algebra from the original equations, the C ratios may also be determined. For the lower mode, $\dfrac{C_2}{C_1} = 1+\sqrt{2} \approx 2.4$, while for the higher mode $\dfrac{C_2}{C_1} = -\dfrac{1}{1+\sqrt{2}} \approx -0.41$ (**17:30**). The full solution (with the magnitudes of the C's) can be found using initial conditions. In addition, any linear superposition of these two solutions is a solution, with the relative contribution again determined by initial conditions. Normal modes were shown in the last lecture and these predicted relative amplitudes were observed.

We now proceed to a driven system, with driving at the suspension point with $\eta = \eta_0 \cos\omega t$, as seen before for a single pendulum (**19:00**). The changes initially appear small, changing only the formula for $\sin\theta_1$, which was $\sin\theta_1 = x_1/\ell$ and becomes $\sin\theta_1 = (x_1-\eta)/\ell$. The differential equation $m\ddot{x}_1 = -T_1\sin\theta_1 + T_2\sin\theta_2$ becomes $m\ddot{x}_1 = -2mg\dfrac{x_1-\eta}{\ell} + mg\dfrac{x_2-x_1}{\ell}t$. With $\omega_0^2 = g/\ell$, rearranging terms, dividing by m, and using $\eta = \eta_0\cos\omega t$, we get $\ddot{x}_1 + 3\omega_0^2 x_1 - \omega_0^2 x_2 = 2\omega_0^2\eta_0\cos\omega t$ (**21:15**). Note that ω is now imposed and we look for steady state solutions where only the amplitudes need to be determined. The $\cos\omega t$ term still divides out, so $\ddot{x}_1$ gives $(3\omega_0^2-\omega^2)C_1 - \omega_0^2 C_2 = 2\omega_0^2\eta_0$ while the second equation is unchanged $-\omega_0^2 C_1 + (\omega_0^2-\omega^2)C_2 = 0$ (**23:00**).

Now in Cramer's rule, $C_1 = \dfrac{\begin{vmatrix} 2\omega_0^2\eta_0 & -\omega_0^2 \\ 0 & \omega_0^2-\omega^2 \end{vmatrix}}{D}$ and $C_2 = \dfrac{\begin{vmatrix} 3\omega_0^2-\omega^2 & 2\omega_0^2\eta_0 \\ -\omega_0^2 & 0 \end{vmatrix}}{D}$.

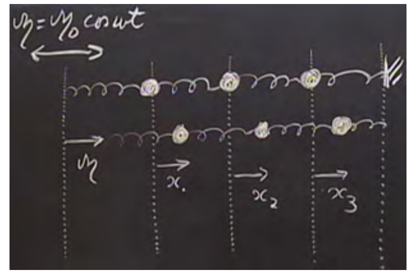
Inserting the determinants, $C_1 = \dfrac{2\omega_0^2\eta_0(\omega_0^2-\omega^2)}{(\omega^2-\omega_-^2)(\omega^2-\omega_+^2)}$ and $C_2 = \dfrac{2\omega_0^4\eta_0}{(\omega^2-\omega_-^2)(\omega^2-\omega_+^2)}$ (**26:10**).

The denominator is D, but written in a different form. Above, we noted that the $D=0$ equation was a quadratic in ω^2 and denoted the solutions ω_-^2 and ω_+^2. So, D must factorize into the product shown as the denominator. These are obviously quite complicated functions of ω, although it is clear that they blow up at frequencies ω_- and ω_+.



A discussion of the graph of the C's, normalized by dividing by η_0, follows (**27:30**). In the low frequency limit, the whole double pendulum is just translated slowly back and forth and essentially does not swing. In this $\omega = 0$ case, $C_1 = C_2 = \eta_0$, and since they are normalized to η_0, they both start at 1 on the graph. Approaching the first resonance, $C_2 > C_1$ approaching $C_2 = 2.4C_1$, as expected, and they are in phase (**32:00**). At the point where the frequency is exactly that of a single pendulum, $\omega = \omega_0$, $C_1 = 0$ and $C_2 = 2\eta_0$. Amazing as it seems, the first bob is stationary and the lower one moves

(**35:00**). A demo follows, including even this strange situation (**40:30**).



Now the three oscillator problem is considered (**43:00**). The system consists of three oscillating cars coupled to each other and to the ends by four springs. One end is fixed and the other end is driven, again with a motion $\eta = \eta_0 \cos\omega t$. The displacements from equilibrium are denoted x_i and we consider an instant in time when $x_1 > \eta$, $x_2 > x_1$, $x_3 > x_2$ (**45:30**). This is done to help visualize the direction of the forces but will not change the resulting differential equations. We can then quickly write the differential equation for the first object: $m\ddot{x}_1 = -k(x_1 - \eta) + k(x_2 - x_1)$. Similar force analysis gives $m\ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2)$ and $m\ddot{x}_3 = -k(x_3 - x_2) - kx_3$ (**49:40**). Imposing ω, we put $x_1 = C_1 \cos\omega t$, $x_2 = C_2 \cos\omega t$, and $x_3 = C_3 \cos\omega t$. Doing algebraic grinding, we should get, with $\omega_s^2 = k/m$,

$$D = \begin{vmatrix} -\omega^2 + 2\omega_s^2 & -\omega_s^2 & 0 \\ -\omega_s^2 & -\omega^2 + 2\omega_s^2 & -\omega_s^2 \\ 0 & -\omega_s^2 & -\omega^2 + 2\omega_s^2 \end{vmatrix}$$

Using Cramer's rule and the differential equations, we get (**57:20**):

$$C_1 = \frac{1}{D}\begin{vmatrix} -\omega_s^2\eta_0 & -\omega_s^2 & 0 \\ 0 & -\omega^2 + 2\omega_s^2 & -\omega_s^2 \\ 0 & -\omega_s^2 & -\omega^2 + 2\omega_s^2 \end{vmatrix} \qquad C_2 = \frac{1}{D}\begin{vmatrix} -\omega^2 + 2\omega_s^2 & -\omega_s^2\eta_0 & 0 \\ -\omega_s^2 & 0 & -\omega_s^2 \\ 0 & 0 & -\omega^2 + 2\omega_s^2 \end{vmatrix}$$

$$C_3 = \frac{1}{D}\begin{vmatrix} -\omega^2 + 2\omega_s^2 & -\omega_s^2 & -\omega_s^2\eta_0 \\ -\omega_s^2 & -\omega^2 + 2\omega_s^2 & 0 \\ 0 & -\omega_s^2 & 0 \end{vmatrix}$$

We could now solve for the resonance or normal mode frequencies, by putting $D = 0$. These frequencies turn out to be $\omega_-^2 = (2 - \sqrt{2})\omega_s^2$, $\omega_+^2 = 2\omega_s^2$, and $\omega_+^2 = (2 + \sqrt{2})\omega_s^2$ (**59:30**).

A multiple-curve graph showing the values for the Cs as a function of ω is presented. The $\omega = 0$ case is not surprising, all three springs are stretched. At ω_s (called ω_0 on the graph), the first car stands still while the two adjacent cars move with the amplitude of the driver, and out of phase with it (**1:03:00**). A demo clearly shows this strange behavior after the transient motion dies out (**1:08:00**). Next, near one of the resonances, the middle car should move only a little while the outer cars move a lot and out of phase with each other (**1:11:00**). Although D is close to zero (as is true for all normal/resonant modes), the determinant in the numerator of C_2 is also close to 0, giving a small finite motion.

Finally, the triple pendulum is examined with plots and demos (**1:12:00**). As before, the $\omega = 0$ case has in-phase motion with the amplitude of the driver. Near the first resonance, the bobs swing in phase with larger amplitude toward the bottom. At a certain frequency ($\approx 0.77\omega_0$, where $\omega_0^2 = g/\ell$), the top one will stand still and the others move. This is too hard to demonstrate. But at ω_0, the middle one will stand still, the top one move at about $0.7\eta_0$, and the bottom one about $-1.57\eta_0$ (**1:17:15**). This can be demoed since ω_0 can be "learned" by swinging a single pendulum (Prof. Lewin calls this "burning into his chips").