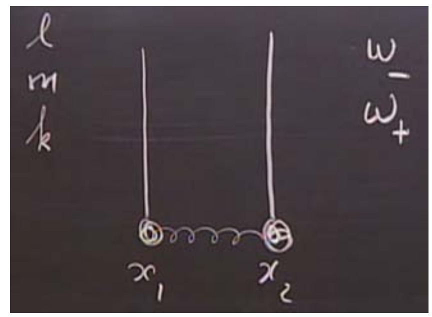
Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

Notes for Lecture #5: Coupled Oscillators

Previous lectures have considered single systems, with and without outside sources of damping and/or forcing. Most realistic physical systems have more than one component and the interaction between these objects can cause vibrations. This is known as *coupling*. Damping is left out for simplicity. In the first example, with pendulums connected by a spring, this would be a very good approximation, since the damping is small. In the demo (**2:00**) oscillation seems to move between the two pendulums. *Normal modes* will have each object at the same frequency, and in the no-damping case the coupled objects are either in phase or out of phase (180° apart). For two objects, two normal modes are expected and their frequencies are denoted ω_- (in phase) and ω_+ (out of phase). General motion will combine normal modes, so we have (**5:00**):



$$x_1 = x_{0-}\cos(\omega_- t + \phi_-) + x_{0+}\cos(\omega_+ t + \phi_+) \quad \text{and} \quad x_2 = x_{0-}\cos(\omega_- t + \phi_-) - x_{0+}\cos(\omega_+ t + \phi_+)$$

The constants x_{0-} and x_{0+} are the amplitudes of the normal modes and can be zero. Prof. Lewin first excites the $-$ mode (**8:15**), with identical motion of the pendulums, then the + mode, with opposed motion (**9:50**). Since the spring does nothing in the $-$ mode, the frequency is just that of a single pendulum $\omega_-^2 = \omega_0^2 = g/l$.

For the opposed mode, a more detailed analysis is needed. The spring is stretched by $2x$, with x the displacement of an individual pendulum, so $F_s = -2kx$. We also have string tension $T = mg$ (for small angles), with a horizontal component of $T_x = -mg\sin\theta = -mgx/\ell$. The two natural frequencies are $\omega_0^2 = g/l$ for the pendulum and $\omega_s^2 = k/m$ for the spring plus one mass. Newton's Second Law gives $m\ddot{x} = -2kx - mgx/\ell$, or $\ddot{x} + 2\omega_s^2 x + \omega_0^2 x = 0$ (**13:30**), which is so similar to previous differential equations that we can easily find $\omega_+ = \sqrt{2\omega_s^2 + \omega_0^2}$ (**15:00**).

As sample initial conditions at $t = 0$, let $x_1 = C$ and $v_1 = 0$, and $x_2 = 0$ and $v_2 = 0$ (**17:00**). This means both masses are released from rest, but with mass 1 offset and mass 2 hanging vertically. Plugging these $t = 0$ conditions into the equations for x_1 and x_2 (and their derivatives) and solving the resulting four equations give ϕ_1 and ϕ_2 both 0. Therefore, $C = x_{0-} + x_{0+}$ and $0 = x_{0-} - x_{0+}$, so x_{0-} and x_{0+} are both $\frac{C}{2}$. The full solutions (**20:30**) are $x_1 = \frac{1}{2}C\cos\omega_- t + \frac{1}{2}C\cos\omega_+ t$ and $x_2 = \frac{1}{2}C\cos\omega_- t - \frac{1}{2}C\cos\omega_+ t$. Since the amplitudes of the two cosine terms are identical, we can use a trig identity to find $x_1 = C\cos\left(\frac{\omega_- + \omega_+}{2}t\right)\cos\left(\frac{\omega_- - \omega_+}{2}t\right)$ and $x_2 = C\sin\left(\frac{\omega_- + \omega_+}{2}t\right)\sin\left(\frac{\omega_- - \omega_+}{2}t\right)$. This resembles beats with a fast term multiplying a slow term. Since the slow terms (with $\omega_- - \omega_+$) are out of phase, oscillations will "move" from one mass to the other. This is demonstrated (**25:20**) for varying amounts of coupling (including the "trivial" condition of no coupling) by repositioning the spring on the pendulum supports.

A general "recipe" is given for solving for the normal modes (**32:00**):

1. Give each object a displacement from equilibrium.
2. Apply Newton's Second Law to each object (which for n objects gives $n+1$ unknowns, the amplitudes of motion of each object plus the frequency of the normal mode).
3. Use solutions $x_1 = C_1\cos\omega t$, $x_2 = C_2\cos\omega t$, ... For normal modes, the objects will all be in or out of phase so no ϕ terms are needed. Out-of-phase just changes the sign of the C's.
4. Now substitute in the $F = ma$ equations and solve for ω and for ratios of C's (overall magnitude is given by initial conditions and you can only find n parameters from n equations).

This recipe is used to solve again our system of two pendulums connected by a spring (**37:00**). The spring force is $|F_s| = k(x_2 - x_1)$ and the differential equations are $m\ddot{x}_1 = -mgx_1/l + k(x_2 - x_1)$ and $m\ddot{x}_2 = -mgx_2/l - k(x_2 - x_1)$, with very careful attention paid to signs. Now we rearrange and use the notations introduced before which is $\omega_0^2 = g/l$ and $\omega_s^2 = k/m$. This gives (**44:00**):

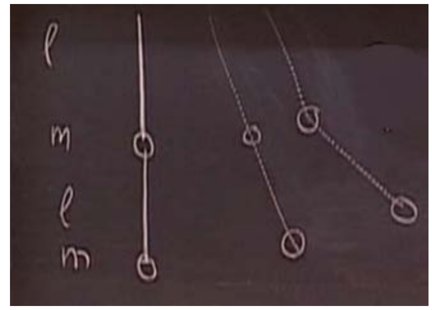
$$\ddot{x}_1 + (\omega_0^2 + \omega_s^2)x_1 - \omega_s^2 x_2 = 0 \qquad \text{and} \qquad \ddot{x}_2 + (\omega_0^2 + \omega_s^2)x_2 - \omega_s^2 x_1 = 0$$

Note that the last term in each equation is the coupling term. The solutions $x_1 = C_1\cos\omega t$, $x_2 = C_2\cos\omega t$ are plugged in. With only second derivatives, the cos terms end up common and can be cancelled giving $-\omega^2 C_1 + (\omega_0^2 + \omega_s^2)C_1 - \omega_s^2 C_2 = 0$ and $-\omega^2 C_2 + (\omega_0^2 + \omega_s^2)C_2 - \omega_s^2 C_1 = 0$. These two equations are solved for the ratio C_1/C_2 (**50:00**):

$$\frac{C_1}{C_2} = \frac{\omega_s^2}{-\omega^2 + \omega_0^2 + \omega_2^2} \qquad \text{and} \qquad \frac{C_1}{C_2} = \frac{-\omega^2 + \omega_0^2 + \omega_s^2}{\omega_s^2}$$

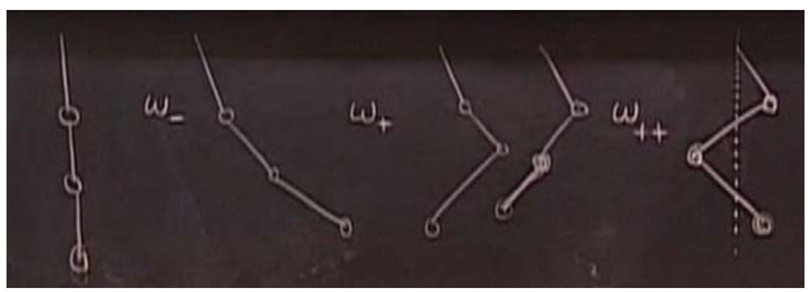
These two equations must have two solutions for ω (what we earlier called ω_- and ω_+). Multiplying through, $\omega_s^4 = (-\omega^2 + \omega_0^2 + \omega_s^2)^2$. Taking the square root, $-\omega^2 + \omega_0^2 + \omega_s^2 = \pm\omega_s^2$, where the $\pm$ is important. Now, solve for ω, using an inverted $\pm$ to keep track of where it came from, to find $\omega^2 = \omega_0^2 + \omega_s^2 \mp \omega_s^2$. All the algebra simplifies (**53:20**), and we get the two solutions $\omega_- = \omega_0$ and $\omega_+ = \sqrt{\omega_0^2 + 2\omega_s^2}$. Substituting back, we find $\left(\frac{C_1}{C_2}\right)_- = +1$ and $\left(\frac{C_1}{C_2}\right)_+ = -1$. This more general procedure gives the same answers as found before.

Now a seemingly simple (but nonsymmetric) case of the double pendulum is discussed (**56:30**). Denoting the top bob as "1", $\dfrac{C_2}{C_1} = 1 + \sqrt{2} \approx 2.4$ for the lowest mode, and $\dfrac{C_1}{C_2} \approx -2.4$ for the highest mode (note the inverted fraction in the two solutions).



The normal modes are also resonance frequencies (natural frequencies) of the whole system, and so it is relatively easy to get the system oscillating at those frequencies. Demos follow (**1:00:40**); first the low ω case with $\frac{C_1}{C_2} = 1 + \sqrt{2} \approx 2.4$, then the $\frac{C_1}{C_2} \approx -2.4$ higher mode case. Exact frequencies are not found but it is clear which mode has higher frequency. The relative amplitudes and the fact that the two bobs are in (out) of phase for the lower (higher) frequency are also obvious.

Now a more complex system is introduced, 4 springs and 3 cars, with the same spring constant k and mass m (**1:03:30**). There must be an ω_-, an ω_+, and an ω_{++}. The first of these has all the objects in phase, and the middle one must be at $\sqrt{2}$ larger displacement than the others (stated but not derived). In the ω_+ mode, the rightmost has the opposite displacement from the leftmost, and the middle one is at rest. In the ω_{++} mode, the outer two are in phase and the central one is out of phase with displacement $-\sqrt{2}$ of the others. If these initial conditions are set up on the air track (**1:09:00**), the normal modes occur. The demo is impressive in matching what is expected.



Getting braver, some general characteristics of the triple pendulum are examined (**1:13:30**). The lowest mode must have all bobs in phase, the highest mode has the middle one out of phase with the other two. The middle mode is harder to figure out since there are two possibilities. Using the resonance property, the answer can be found experimentally with the answer being that the top two are in phase and the bottom one is out of phase with a large amplitude (**1:17:00**).

8.03SC Physics III: Vibrations and Waves
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These viewing notes were written by Prof. Martin Connors in collaboration with Dr. George S.F. Stephans.