Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

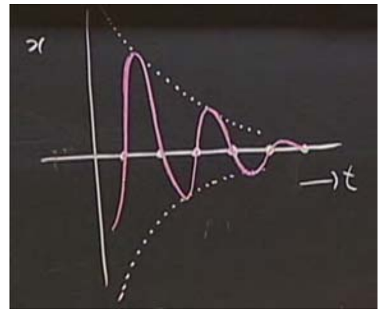
Notes for Lecture #2: Damped Free Oscillations

The lecture starts by adding oscillations using the trigonometric identity:
$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$. With $x_1 = A\cos\omega_1 t$ and $x_2 = A\cos\omega_2 t$, you find:
$x_1 + x_2 = 2A\cos\left(\dfrac{\omega_1+\omega_2}{2}t\right)\cos\left(\dfrac{\omega_1-\omega_2}{2}t\right)$ The second term is a "slow" one, varying at the difference of the frequencies, while the first is the "fast" one at their average, which is close to the original frequencies if they are close in value. The slow term modulates the amplitude of the fast one and produces *beats*. The frequency of the beats is Δf, the difference between the two frequencies. It is not $\frac{1}{2}\Delta f$ because the sound is loud when the "slow" wave is large whether it is positive or negative. If the two oscillations do not have the same amplitude, then the beats do not give complete cancellation, as demonstrated (**6:00**).

Damping of oscillations can be caused by a friction or drag force. The main terms of the friction force due to air drag are $\vec{F}_{fr} = -C_1\vec{v} - C_2 v^2\hat{v}$, respectively the viscous and pressure terms (**13:30**). Here we will consider only the viscous term appropriate to low speeds, and unless you are interested, you do not need to explore the general case further. As a first example, a horizontal spring with viscous drag is considered(**16:00**). Changing notation of C_1 to b, we find the equation of motion $m\ddot{x} = -kx - b\dot{x}$. Defining $k/m = \omega_0^2$ and $b/m = \gamma$, this can be written as: $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$. We want to find the frequency of the actual oscillation ω, which intuition tells us should be lower (longer period) than for completely free oscillation.

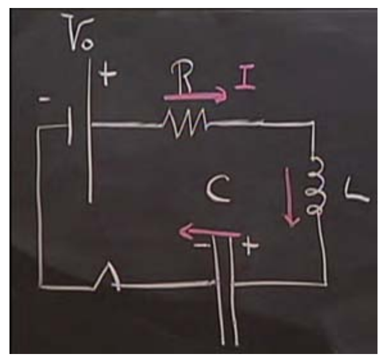
This problem can be solved using complex numbers (**21:00**). In the complex plane, $\ddot{z}+\gamma\dot{z}+\omega_0^2 z = 0$ and a solution is sought in the form $z = Ae^{j(pt+\alpha)}$. Plugging into the differential equation gives $(-p^2 + j\gamma p + \omega_0^2)z = 0$. Independently, the real and imaginary parts must be 0. The parameter p must itself be complex for this to happen in a reasonable way. Put $p = n + js$ (**25:00**), with n and s real. We need p^2 and this can be calculated as any binomial term, recalling that $j^2 = -1$: $p^2 = n^2 - s^2 + 2jns$. Substituting gives $(-n^2 + s^2 - 2njs + j\gamma n - \gamma s + \omega_0^2)z = 0$. Since z is not zero, the part in brackets must be zero, and both in its real and imaginary parts. The imaginary part is $-2njs + j\gamma n = 0$ which means that (**27:30**): $\gamma/2 = s$. We can plug in this value for s, in which case the real part is $-n^2 + \dfrac{\gamma^2}{4} - \dfrac{\gamma^2}{2} + \omega_0^2 = 0$, giving $n^2 = \omega_0^2 - \dfrac{\gamma^2}{4}$. At this point (**28:40**) p is fully determined, and $z = Ae^{j(pt+\alpha)} = Ae^{j(nt+jst+\alpha)} = Ae^{-st}e^{j(nt+\alpha)}$ so $z = Ae^{-\gamma t/2}e^{j(nt+\alpha)}$ (**29:50**). The first term is an exponential decrease which multiplies the second, which is an oscillatory term. Since n is a frequency, we replace it with ω, with $\omega^2 = n^2 = \omega_0^2 - \gamma^2/4$, less than ω_0^2 as expected.

An error in the real part (should remove the complex exponential) is made initially. The correct formula for the real part is $x = Ae^{-\gamma t/2}\cos(\omega t + \alpha)$ (**33:30**). Another correction is that $T = 2\pi/\omega$ (**35:50**).The period T does not change with time, only the amplitude changes. We now introduce the quality factor $Q = \omega_0/\gamma$(**36:45**), and rewrite $\omega^2 = \omega_0^2 - \dfrac{1}{4Q^2}$. One use of Q is to study how the amplitude decays as a function of number of oscillations, N(**39:00**), giving $A(N) = Ae^{-N\pi/Q}$, meaning that Q/π is the number of oscillations one has to wait for the amplitude to decrease by a factor of e (recall $e \approx 2.7$).



A demo uses a Styrofoam ball (**42:00**) decaying from 27 cm amplitude to 10 cm amplitude (factor e). In this case Q is about 35, with a decrease in amplitude by a factor e after about 10 oscillations.

Consider an "RLC" circuit (**48:00**), one which has a resistor, R, capacitor, C, and inductor, L, along with a switch and battery (Note that "8.02" is the MIT Electricity and Magnetism course). The resistor "resists" the flow of electricity, a capacitor "stores" electric charge, and an inductor "stores" magnetic field, which arises from current flowing in the inductor. Recalling the energy approach to oscillation, we could expect that the C acts like potential energy, while the L, reflecting flow, acts like kinetic energy. It is by exchange between these two forms that oscillation can take place. In turn, R acts like damping.



Analysis proceeds by looking at the electrical current flowing in the circuit (**49:00**). The current in a circuit is due to the flow of electric charge and is denoted I. Charge itself is denoted q. At any point in the circuit, the rate of flow of charge is the current, so $I = \dfrac{dq}{dt}$. The analysis follows the standard E&M procedure for such circuits using Faraday's Law, $\oint \vec{E}\cdot d\vec{l} = -\dfrac{d\phi}{dt}$, where the circle through the integral sign means that the integral is not evaluated between two limits, but rather following a path in space. Here, that path is our circuit. The circuit has a magnetic field through its area, and the total of magnetic field times area (this is in fact an integral too) is called the *magnetic flux*. Faraday's law says that if the flux inside a path (in this case our circuit) changes, then its rate of change will cause an electric field to build up around the perimeter.

The quantity $\oint \vec{E}\cdot d\vec{l}$ across a circuit element equals the voltage across it. In a capacitor, if there is charge on each plate of q (positive on one plate, negative on the other), then there is a voltage across the capacitor of $V_C = q/C$, where C is the capacitance (**50:50**). Across a resistor, the voltage drops by the product IR (this is known as Ohm's Law). Summing the voltages to determine $\oint \vec{E} \cdot d\vec{l}$ is analogous to "Kirchhoff's Law", which is mentioned disparagingly. The flux is contained inside the inductor, and is given by $\phi = LI$, where L is the value of the inductance. So, the right side of the equation is $-\frac{d\phi}{dt} = -\frac{d}{dt}(LI) = -L\frac{dI}{dt}$. Doing the sum of the voltages, we get $+IR + 0 + V_C - V_0 = -\frac{d\phi}{dt} = -\frac{d}{dt}(LI) = -L\frac{dI}{dt}$ (**51:45**). With $I = \frac{dq}{dt}$, this can all be brought to an equation in q: $L\frac{dI}{dt} + R\frac{dq}{dt} + \frac{q}{C} = L\frac{d^2q}{dt} + R\frac{dq}{dt} + \frac{q}{C} = V_0$. Dividing by L and going back to the 'dot' notation for derivatives we get $\ddot{q} + \frac{R}{L}\dot{q} + \frac{q}{LC} = \frac{V_0}{L}$. If we put $R/L = \gamma$ and $1/LC = \omega_0^2$, this starts to look very familiar (**52:45**): $\ddot{q} + \gamma\dot{q} + \omega_0^2 q = V_0/L$ which is almost identical to that for a spring with damping, as discussed earlier in this lecture. The resistance plays the role of the damping (heat is dissipated in the resistor) and the natural frequency is determined by the capacitance and inductance, not surprising since the oscillation is due to transfer of energy between them.

The difference is that, unlike in the case of the spring, the right hand side is not zero (**54:00**). For the spring, the oscillation in position eventually damps down to zero. However, for the capacitor, the oscillation in the charge does not damp down to zero. Looking at the differential equation for charge after a long time (when the derivatives will be damped down to a very small value) gives $q_{final} = V_0C$. So, the solution in this case can be found by adding this end result to the previous solution. So, $q = q_1e^{-\gamma t/2}\cos(\omega t + \alpha) + q_{max}$ (**55:30**). This is a decaying oscillation which ends up at $q = q_{max}$. The parameters q_1 and α must be determined from initial conditions. Assuming the circuit starts with no charge on the capacitor and no current flowing(**56:30**), you get $q_1 = -q_{max}/\cos\alpha$ and $\tan\alpha = -\gamma/2\omega$. For high-$Q$ systems (which many RLC circuits are), $\omega \approx \omega_0$ and so (**59:00**) $q_1 \approx -q_{max}$ and the resulting time dependence is very similar to that shown for a damped mass on a spring (see top figure on Page 2) except that the charge decays to q_{max} while the position of the mass decays to zero (**1:00:00**). A demo follows, driving an RLC circuit with a square wave (**1:06:00**) so many instances of the oscillation can be seen.

What if $\gamma^2/4 > \omega_0^2$? Then $\omega^2 = \omega_0^2 - \gamma^2/4$ (**1:08:30**) or equivalently, $n^2 = \omega_0^2 - \gamma^2/4$, is negative and so $n = j\left(\frac{\gamma^2}{4} - \omega_0^2\right)^{1/2}$ Putting this back into $z = Ae^{j(pt+\alpha)}$ where $p = n + js$ gives (for x being the real part of z: $x = A_1e^{-\left[\frac{\gamma}{2} + \left(\frac{\gamma^2}{4} - \omega_0^2\right)^{\frac{1}{2}}\right]t} + A_2e^{-\left[\frac{\gamma}{2} - \left(\frac{\gamma^2}{4} - \omega_0^2\right)^{\frac{1}{2}}\right]t}$ (**1:11:30**) which is a pure exponential decay, corresponding to *overdamping* (the oscillatory case discussed above is called *underdamped*). Again, the constants (A_1 and A_2) are determined by initial conditions.

A final case (**1:14:30**) is $\omega_0 = \gamma/2$, called *critical damping*. The solution is $x = (A + Bt)e^{-\gamma t/2}$, again a decay. A damped torsional pendulum demo (**1:16:00**) ends the lecture.

8.03SC Physics III: Vibrations and Waves
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These viewing notes were written by Prof. Martin Connors in collaboration with Dr. George S.F. Stephans.