PROFESSOR: Today I'm going to couple a huge number of oscillators, and then ultimately we'll make that number infinitely large. And the way that you can couple oscillators is through springs, as we did last time, on the air track. And we can let the cars move like this. We call that a longitudinal motion. That means that the motion is along the line of the oscillators.

And then we have ways that we can oscillate things perpendicular to the direction of the oscillators, and we call that a transverse motion. So that goes like this when the oscillators are like this. And the algebra is identical. It is easier to use the transverse motion when you do the derivations. And so I will use the transverse motion.

Suppose I have a string and I put on this string beats, masses. I have a tension, T, each mass is m, and the length between these is l. And this is a fixed end. Fixed, and this end is also fixed, cannot move. And so this is number 1 this is number 2 this is number 3 and then here is the last one, which is number N. I'm going to have N of these beats on the string.

So this point here, I can call that 0. I think of this direction of x, and this is the direction of y. And so at 0 there is no beat, and here at the position N plus 1 there is no beat either. And so the question now is, what are the normal mode solutions to that system. There must be "capital N" normal modes. And if N goes to 10,000, there must be 10,000 normal modes.

At a particular moment in time you can imagine, for instance, that this one is here and maybe this one is here. And this one maybe here, and maybe this one is here. The only thing we have to keep an eye on-- that this is always 0, and the y is also always 0. That's what we call the boundary condition.

Now if you make the amplitude smallest then it's easy to demonstrate that the tension will remain constant in these pieces, modest amplitudes, and that there is no motion in the x direction either-- at least you can make it negligibly small. And so we will only concentrate on the forces is in the y direction. And so now I'm going to make a blown-up version here for particle number p.

This is particle p, and here's the location of p minus 1. And here's the location of particle p plus 1. And at a certain moment in time let's assume that the particle is there, that little mass m. And let's assume that this one is here, and let's assume that this one is here.

So this vertical displacement is yp, this is yp plus 1, and this, then, is yp minus 1. But the strings are just attached like this. And so the fact that the strings get longer, we will ignore that, because of their small amplitudes. So the tension will not change.

Draw this line. We draw this line. I will call this angle alpha p. And I will call this angle alpha p minus 1. So the tension is going to be on that point p on this little mass m in this direction, and there's a tension in this direction. And we will assume that these tensions are then the same, for reasons that I mentioned.

So I can write down, now, for that mass, for that location p, I can write down Newton's second law. And so I get my of p, second derivative d2y dt squared of position p. And I see I have one horizontal component-- one vertical component that drives it down. That is due to this tension, and then I have one that drives it away from the equilibrium. And so I'm going to get minus T times the sine of alpha p minus 1, and I get plus 3 times the sine of alpha p.

But, of course, I know what the sine of alpha p minus 1 is because the sine of alpha p minus 1 is yp minus yp minus 1 divided by l. And so I can write here y of p minus yp minus 1 divided by l. And here I have T. The sine of alpha p is yp plus 1 minus y of p divided by l.

I'm going to introduce shorthand notation. I'm going introduce that omega 0 squared is T divided by ml. As time goes on, you will get more insight in why we choose that. At least convince yourself, if you have the time, that this is the dimension one over second squared. It has the right dimension.

T, by the way, whenever I use this throughout this lecture, is never period. It is always tension. So I will stay away from the capital T when we're dealing with a period. So I can divide m out and I can rewrite this, get the m downstairs, and so we get y of p double dot.

And what I'm going to do now, I'm going to take the yp here and the yp there. And I bring to the left. I have a minus here and I have a minus here. So I get plus omega squared times y of p. And then I'm going to bring the p minus 1 in, and I'm going bring the p plus 1 in. Notice I have a plus sign here. So I get minus omega 0 squared times yp plus 1. And here I have a plus-- minus minus is a plus. So when I bring that back I get a minus but I already have the minus so I get yp minus 1. And that now equals 0. But that is Newton's second law for particle p.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Excuse me?

AUDIENCE: [INAUDIBLE]

PROFESSOR: What is your problem?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yes, thank you very much. Very good, I appreciate it. This is a 2 because you 2 p's. You have one here and you have one here. Thank you very much. Very attendant.

So now we have to do this for every single object. So we have capital and differential equations, one for each particle. And the only thing that we have to keep in mind, now, when we solve it is that the y0, which refers to that location 0 there, must always be 0, and that yN plus 1 is also 0.

So if I make a sketch to warm you up to the idea. And I made a sketch for only two particles, so this is number 1, and this is number 2, and this is a fixed end, and this is a fixed end, then you can sort of see that in the lowest normal mode you're going to see something like this, so this is omega minus, and it's going to oscillate like this. But you can also imagine that in the second mode, which is the highest one, that number 1 is up and then number 2 is down. So you get this situation and so they oscillate like this. Just to make you see, I call this little n equals 1, and I call this little n equals 2-- n referring to the mode. Mode one, I will use this n later on, and this is then mode number two. And there are only two modes because there are only two particles.

So let us now proceed with the equation that we have. And let us write down, for this system, the two differential equations. And so look at this one. We're going to substitute for p first number 1, which is this one. That's one differential equation. And then we're going to put in for p number 2, we get a second differential equation. So if you're ready, then we're going to get y1 double dot plus 2 omega 0 squared times y1 minus omega 0 squared. And then we get y2, that's this particle, and then we get plus yp minus 1, which is plus y0, which happens to be 0, by the way, because y0 here is 0.

And now we go to the-- and this is 0-- now we go to the second particle. And so we get y2 double dot plus 2 omega 0 squared y2 minus omega 0 squared. And now we get, first, p plus one, which is number 3, which is y3, which happens to be 0 because y3 is this point, and that's 0 in this specific case with only two objects. And then we get here plus yp minus 1 and that is-- 5 plus 1, we have 2-- that is y1. And that is 0.

So these 2 coupled oscillators will have to be solved. And in the normal mode situation, we are clearly going to put this in as our trial function, cosine omega T. They must oscillate with the same frequency, omega, otherwise we wouldn't be dealing with normal modes.

And so these are our trial functions that we're going to put in these equations. And we will put them in number 1, we're going to put them in number 2 and then we will put them in particle number p. And then we can all-the-way go to N, and if N is 10,000, we have to write down 10,000 differential equations on the black board. And that will take the rest of the hour.

So I'm going to number 1 here, so I get a y1 here, which is A1, the second derivative always gives me a minus omega squared, so you get minus omega squared times A1. I ditched the cosine omega T because each term will have a cosine omega T. Then I get plus 2 omega 0 squared times A1. And then I get minus omega 0 squared times A2, right? Because now I get a 2 plus A0, which happens to be 0, but I just put it there-- you will see shortly why I want to keep it there.

I got a particle number 2, I got minus omega squared times A2, becomes a little boring, 2 omega 0 squared times A2 minus omega 0 squared. And now I have a y3 so I get an A3, which happens to be 0, but that's not so relevant right now. And then I get plus A1.

And the reason why I started off with 1 and 2 is that now you see how we can put it in the p's particle.. So the p's particle, now, is going to be rather easy-- maybe I should do that in color-- minus omega squared Ap plus 2 omega 0 squared Ap minus omega 0 squared. And now we're going to get Ap plus 1 plus Ap minus 1. And that equals 0. Also this equals 0 and also this equals 0.

And so now you see the differential equation for particle number p. And so you can go on now to particle capital N, and now you have to solve N differential equations. That's a zoo. That's a terrible thing.

Now we will take a shortcut which is not very rigid, but it really will save a lot of math. And that is we will use our intuition. Something that we know, sort of, from experience. If you had a lot of beats on here, fixed and fixed here, and you ask yourself what's going to happen in the lowest possible mode, then you just know that you get something like this. It goes like this, and like this, and like this.

And you know that in the second normal mode, the one that follows, that has a higher frequency, you expect that this side goes up, this goes down, and that it will oscillate like this. So we use that experience, which is not very rigid, in order to decide on our trial function.

This would be mode 1, and this would be node-- mode. Mode 1 and this would be mode 2. And so now I'm going to put in, as a trial solution, A for particle p, which is in mode n, as in Nancy, this is the mode.

I want a sinusoid in there that is always 0 here and 0 there. And they can have an amplitude, of course, which I can freely choose. So this C of n is the amplitude of this sinusoid. So this is C1, and this value is going to be C2. Each one can have its own amplitude.

And then I get the sine of pn pi over N plus 1. So let's look together at this equation, so that we have a full understanding of what we are trying to put in there. Notice that p equals 0, that the sine is always 0. That's obvious, because we wanted that, because this point, the 0 point, is not moving.

Notice also, that if you put in p equals N plus 1, that AN plus 1 is also always 0. So put in pN plus 1, the sine of a multiple times pi is always 0. Because n is now our mode, it's going to be 1, 2, 3, et cetera. These are the modes that we're looking for.

For instance, if we take n equals 1. Let's take n equals 1. So we have particles, and they are all in mode 1. So then we get that Ap, in mode 1, would have C1, which is the amplitude of that sinusoid. And then you would have the sine of n pi divided by N plus 1. And you can indeed convince yourself that that exactly this sinusoid with an amplitude C1. And you can also convince yourself that p0, A0, and particle n plus 1 is again 0, of course. And if you take n equals 2--

AUDIENCE: [INAUDIBLE]

PROFESSOR: Excuse me?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, boy-- yeah. The n is a p, right? Thank you very much. Is that what you were saying? Is that what you were saying?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah. Thank you very much. Today is not my day. Yeah, that is a p. So for particle number-- when p is 0, you see, this goes to 0, and when p is n plus 1 this again goes to 0. So you see here that Ap2 is now C2 times the sine of 2p pi divided by n plus 1. And that, exactly, is this curve. Notice that if you put in p equals 0, you get a 0. If you put in p is capital N plus 1, you get a 0, but you will also find now a 0 precisely in the middle. When p is capital N plus 1 divided by 2.

And so that is the consequence of the introduction of this function. And of course the ratios of the individual p's for your particular mode-- take for instance mode number 1, when the system is oscillating like this, the ratios of the amplitudes of the individual p's is then given by this sine. p1 has its own amplitude, p2 has its own amplitude, p3 has its own amplitude. So p1 will be here, p2 will be there, p3 will be there, p4 will be there, and so on. And so then the amplitude first goes up and then the amplitude goes down again.

But now comes the question, what is omega n? What is the frequency omega at these normal modes, n equals 1, n equals 2, n equals 3, n equals 4, and n can go all the way down, then, to capital N.

And for that, we have to return to this equation. And I'm going to write it, now, slightly differently. I'm going to take the A's to one side. So I'm going to write down here A p plus 1 plus 1 p minus 1 and divide that by Ap-- just rearranging. And then you will find that it is minus omega squared plus 2 omega 0 squared divided by omega squared. Take a look at this, and convince yourself that this equation is identical to this one.

If you look in French, French will take you from here one step further, which is pure trigonometry. And I decided not to go that route, but you can use the values for Ap that we have defined, namely this one. And so you can now massage the trigonometry, and you can find that this ratio is very simple. It's twice the cosine. So this whole thing is twice the cosine of n pi over n plus one. That is correct. n pi over n plus 1. So there's no physics there, it's purely a matter of trigonometry.

We can now put to an n here, Nancy, because we know now that we're going to get solutions as a function of the mode number n. And then, with a little bit more trigonometry, and you really want to check up on French there, which is page 141, he then comes up with the normal mode frequencies which was our goal.

So I will give you the result, but it really is implicit, already, in here. You will get 2 omega 0, the result is by no means intuitive, times the sine of n pi divided by 2 times N plus 1. And of course, I'm going to look through that result with you. As of now, it looks very opaque.

So this then is the solution to omega n. This is the solution. I will take one color to show you how these link. This is the solution for A. If you know the mode, and you know which particle it is, and you have specified the amplitude of the sinusoid, then this tells you each particle, what the amplitude is. If you know the mode n, then you know that this is going to be the frequency.

And so you can write down, now, that y, which is the displacement as a function of time-- A is amplitude, y is displacement for article p in mode n. And now we can put in the amplitude that we know, that is the pn amplitude Apn. And now it's going to oscillate with cosine omega n times t.

And of course, you can always add a phase angle depending upon at t equals 0, what the particle is doing. And so this one gives you the amplitude, this one gives you the frequency, and this, then, is the time dependence of the displacement of particle number p in mode n.

What I want to do now is to take a specific example, which I also will try to demonstrate, which will give you tremendous insight. We can actually do it on the left here. Because this is all very opaque, but when you see an example worked out, and you see, actually, how it oscillates, then it comes to life.

I'm going to have five beats on a string. 1, 2, 4, 3, 5, fixed here, fixed here. 1, 2, 3, 4, 5, the tension is T, the mass of each one is m, and the separation between them is l. And so N is 5, so keep in mind that N plus 1 is 6. The reason why I write that down is because you're going to need the N plus 1. And then omega 0 is the square root of T divided by ml.

So I'm interested in knowing what the frequency is in the lowest possible mode, which is going to resemble something like this. And so that frequency, omega 1, is then 2 omega 0 times the sine n equals 1, capital N plus 1 is 6. 180 degrees divided by 12 is 15 degrees so that's the sine of 15 degrees. I write it now in degrees because I have a better feeling for degrees than I have for radians. And that is all 0.51.

And now got to omega 2, and I get the same thing, except I get 30 degrees. So I get 0.5-- sorry, I get exactly omega 1-- omega 0, right? Because the sine of 30 degrees is 1/2 and that eats up this one. And I get omega 3. And now I get the sine of 45 degrees. And that is approximately 1.41-- not exactly-- but as the square root of 2, so that is about 1.41 times omega 0.

Then I go to omega 4. So I get 60 degrees, and that can only be approximated, again, by about 1.73 omega 0. And then I have omega 5, which is the last 1, so I get the sine of 75 degrees. And that then becomes 3.73-- no, 1.93, 1.93 times omega 0.

What I want to concentrate on, because that's part of the demonstration, is not so much on the meaning of omega 0-- that's just some arbitrary thing that I have called omega 0, but I want to concentrate with you on the ratios of the higher frequencies to the lowest one.

And so I call the lowest one omega 1. Simply call this omega 1. If that one is omega 1, then the next one is 1.93 omega 1, again, not exactly, but approximately. It is this 1 divided by 0.51. It's the ratio now of the frequencies.

And if I take this one and divide it by that one, then I get 2.73 omega 1, and I take the next one, I get 3.35 and the last one, then, is 3.73. So the bottom line is that the ratio of all these frequencies are not at all very nice numbers, as you may have expected, but the ratios are quite bizarre. 1.93 times higher than the lowest one, 2.73 times higher 3.35 times higher, and 3.73.

And so the general solution to that system is then the linear superposition off all these normal modes. That's the general solution. You give them very modest amplitudes. And you can choose the amplitude each one of them. That's effectively like saying you're choosing C1, C2, C3, C4, C5.

And you also can give them initial velocities if you want to. So at T equals 0, they do not all have to stand still. You have that choice, too, of course. So you can change the relative phase between the five different modes.

What I will do, is, I will, to make life simple, I will generate all five normal modes for you. And I will start them off all at 0 speed, when I show you the simulation. And the first one that I'm going to show you then is number 1. And then I'm going to show you number 2.

I want you to appreciate that if I showed you the superposition of 1 and 2, so I let it oscillate in this mode and in this mode, I start off at a certain position of these particles, and they start to oscillate in this mode and that mode, that the shape that I have will never, ever become the same as it was at time 0. And why is that? So you just let it is oscillate, and you can wait 100 billion ability years, and you will never see the same shape. So I start with a certain shape and it will never, ever, ever come back to that shape. Why is that?

AUDIENCE: Is the ratio an irrational number?

PROFESSOR: It's right, that the sine of 15 degrees is the killer-- that is not the ratio of two integers. And therefore you will never get it back to the same position. Maybe approximately, but you never get it back.

This demonstration is going to be a cocktail between very low tech and very high tech. And I will start with myself, which is very low tech. And that is this.

I was sitting in my office, and I said to myself, "gee, what will I see?" So I took a pencil, and I just sketched, very roughly, a sinusoid, right here. And I know, according to these solutions, if you accept them, that these beats, these particles, must lie on that sinusoid. And C1 is then the choice of that amplitude of that sinusoid.

In the second mode, you pick another value for the amplitude, say C2, and then the beats have to lie on that sinusoid, and so on. What you will see, however, is that these beats are connected with straight wires. So you will not see those nice arcs. What you will see is of course this. The red lines are the actual strings. And so for instance, if you go to the second mode, we call that the second harmonic, if like, then notice that this point here and this point here never reach the amplitude C2. The sinusoid does, and the C2 in that equation does, but those points will never reach that because their location is such that they never make it to that point here.

This won't stand still then, that was intuitive. Because we have that here. And if you go to higher frequencies, particularly the very highest one, neighboring beats always are out of phase with each other. You see that, up, down, up, down, up. And again this little particle will never reach the amplitude C5. This one does, this one does not. This one does not, this one does not, this one. But this one does.

And so as I'm going to show you this simulation we will keep this going because it will be great to anticipate what we may be seeing. So the first thing that I'm just going to show you is one complete oscillation in the normal mode number 1, which I have set to be 15 seconds. With the help of [INAUDIBLE], who has guided me greatly in this demonstration.

So let me, first of all, give us the right light conditions, and now I will start the last 15 seconds. I have given the amplitude a 2, which is very large, and, of course, that's unrealistic, these high amplitudes, but I want you to see the relative position of these particles.

There we go. Now you will see, it will make one complete oscillation. And that of course is no surprise. If I clicked only once, it will stop now and that will be. If I click twice, it will start again. No, thank goodness it only did once.

Now I'm going to show you the second one. And I give it the same amplitude. So C2, I give it 2. And I want you to count how many oscillations it makes before it comes to a stop. It will again be exactly 15 seconds. And so we will have to agree that the number of oscillations that it makes is now 1.9. It misses the 2, it will not get back to the two complete oscillations.

And so, you're going to look at this mode. So these two particles will never reach the value 2. The 2 is marked here. And it will go down, up-- excuse me-- down, up, down, and it stops just short of two oscillations.

So we'll do that now. So I make the amplitude of the first one 0, and now we get the amplitude of number 2. And I make that 2, and there we go. Now count. There's one oscillation, and it will stop just short of two. You can't tell that, of course, because you don't have that resolution. But it stopped just short of two.

And now we go into this one, again, give it an amplitude of 2 and now we're going to count, and you will definitely be able to see that it just misses is 2.75. Because 2.75 is something that you can eyeball.

So we're going to number 3 now. So number 3 already has two interesting points, which don't move at all. This point won't move, and that point won't move. They will reach the plus 2 and the plus 2.

And 0, and we get a 2. And there we go. That's one, that's two, that's two and a half, ah! And that's 2.73. You see, it's just short of 2.75. So that is this number.

Now we're going to number 4. For number 4, this one will stand still, and the others do not have the maximum amplitude of 2. So again, count-- the 2, there we go. 1, 2, 3, and it stops there. It stops at 3.35.

So now, the last one, before we're going to cocktail them, is this one. So again, this point will be plus 2. So this one will never reach that. This one will reach the plus 2. OK. One, --see, it doesn't make it to 2, it doesn't get that high-- 2, 3, 3 and 1/2, and it stops. Just under the 3.5 it stops. Sorry, just over 3 and 1/2 it stops. Sure, this would have been 3.75. It's just under, there.

Now I will cocktail for you 2 and 4. If I cocktail 2 and 4, then this point will stand still, and this, of course, is going to be starting up. And this will start up, because I give them both the same amplitude, I will not go to plus 2 now, but I'll make it plus 1. Otherwise you get too large values and it becomes a little bit unrealistic in what you see there.

So I'm going to give number 2 a 1, as amplitude, and I'm going to give number 5 a 0, and number 4 also a 1. And now, already, you're going to begin to see that the motion that you're going to see becomes, already, sort of, a little bit chaotic, a little erratic. So it's a superposition now, of two normal modes. This one, which I start off at a 1, here, and this one, which I start off as a 1 here. And at T equals 0 I release them both with 0 speed. Make sure that I have the 1 in there, yes I do. There we go.

So this motion is already not so predictable. But it's still sort of symmetric, for obvious reasons, because this one stands still. And now what I want to do is start them, all five. Now, if we're starting all five, the start will be very asymmetric because, look, particle number 1, positive. Because I set them all off positive. Positive, positive, positive, positive, positive. So it will start at very high. Now look at the number five, positive, negative, positive, negative, positive. So number five will start very low, and number 1 will start very high.

And then, when it starts to oscillate, it will take more than the age of the universe to come back to that same shape. But it is extremely erratic. You and I, no one, can really relate anymore to what's going on. And it is even impossible to imagine that the motion, in a way, is very simple-- namely, the superposition of five very well-behaving normal mode solutions. It is a linear superposition or five very simple normal mode solutions, but the net result is total, utter, chaos. At least that's the way it appears to us. But it can be it can be dissected into five very simple modes.

So these were transverse motions. And the same idea holds for longitudinal motion. So you can have five beats with six springs, and then the oscillation is in this direction. We call that a longitudinal oscillation. In this case, the displacement is perpendicular to the oscillators. We call that transverse. But they algebra, as you can imagine, is identical. Except that the displacements are then all in this direction, for the longitudinal one, but in this direction.

We will shortly enter the domain of waves to make you see the idea, the big difference between transverse waves and longitudinal waves. Sound, which is a pressure wave, in my direction to you, the air is a pressure wave. It's doing this, so the air oscillating in the same direction that it moves. That is a longitudinal wave. So this is a nice moment to break. We'll break five minutes, and we'll start exactly five minutes from now.

All right, thank you very much for the performance. That was prearranged, by the way. So we're now ready to make the step to continuous medium whereby n goes to infinity, well you could argue that it goes to as many atoms as we can line up on a string-- goes to infinity.

And it should not come as a surprise of course that now you're going to get that the entire string, which is now continuous mass so you no longer have individual beats, that the entire industry is now going to oscillate as a sinusoid in its lowest mode. So this is n equals 1. And then it's going to oscillate like this for n equals 2, n equals 2, and n equals 3. Going up like this, and this goes like this. So that should not come as a surprise. I will not pursue that today. We'll get back to that later.

What I want to mention, though, what is interesting, and that is that the ratios of these normal mode frequencies will now be 1, 2, 3, 4, 5 and so on. So now you get that the second mode is twice the frequency of the first, which is what we didn't have there. That's the big difference between number of n, which is finite, and an infinite number of these oscillators.

What I want to pursue today-- I will get back to this in the future. What I want to do today is to generate a disturbance in a medium which has an infinite number of coupled oscillators, which is a string. To generate in there. So I take a string and I wiggle the end. And then I want to evaluate with you what's going to happen.

And so for this I need some assistance. Will someone-- Nicole, would you mind?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Just hold this firmly in your hands. Now most of you may think that this is a spring with a P, as in Peter. But no, it is a string with a T, as in Tom. You will see that. I'm going to use this as a string. I'm going to put tension on it, T, which is what we needed, also for the end-coupled oscillators.

And the amount of mass that we have, we express that, normally, in terms of the mass-per-unit length. Remember, in the other case we had little m divided by l. Well we call that now mu. So that's how much mass per unit length we have.

And what I want to do now is to shake my hand, and then you tell me what you see. You ready? There we go. Are you ready Nicole? What you see? Just tell me what you see right after I do this. What did you see right after I did this?

AUDIENCE: [INAUDIBLE]

PROFESSOR: The disturbance moved. That's number one that we have to understand. Why does it move?

Now look what happens at Nicole's side. I generate a pulse which is like this. I will call that a mountain for now. And only look at the moment that the mountain reaches her, and something comes back at me. And then stop looking because thing begin to wander back and forth. And tell me what comes back at me.

So I'm going to send a mountain to Nicole. What came back at me?

AUDIENCE: [INTERPOSING VOICES].

PROFESSOR: A valley? Now I'm going to send a valley to Nicole. What do you think is coming back? Very good. It's hard-- it's actually, you know, I don't know why it is, but it's very hard to generate a valley. Let me do a mountain again. This is a mountain that comes back as a valley. And I'll try a valley. OK, I'll try to do a valley. So I go down and up. Yeah, that was a good one.

AUDIENCE: [LAUGHS].

PROFESSOR: And you that--

AUDIENCE: [LAUGHS]

PROFESSOR: Yeah. Well, because of you, it worked. Thank you very much, you did a great job.

So now, we have to understand two things. And that is why does it propagates. And why does a mountain come back as a valley, and why does a valley come back as a mountain.

Continuous medium, infinite number of coupled oscillators. I start here with a piece of that rope. Let's call this position x. And I call this position x plus delta x. I call this y. I call this angle theta plus delta theta. And I call this angle theta.

We have a tension, T on the line, and mu is the mass per unit length. So you tell me, what the mass for 1 meter is, and I know what mu is. It's the length-- the mass per unit length. Well, if our displacements are not absurdly high, then we can make the same assumption that we made with the beaded string. That the tension is the same on both sides. It's an approximation, but for modest amplitudes, it's a very reasonable approximation. So we have a T, here and we have T there. And they are then, to a reasonable approximation, the same.

Just like with the beats, for modest amplitudes we don't have to worry about motion in the x direction. The only thing that matters is the motion in the y direction. So I will concentrate exclusively on the motion in this direction, which drives it back to equilibrium. And so f of y, on this segment, is then minus T sine theta, because this component is down, minus T sine theta plus T sine theta plus delta theta. Because this component, in the y direction, is driving it away from equilibrium.

But for small angles, and we have to have small angles otherwise all our assumptions are wrong-- the T's are not the same. But for small angles, the sine of theta is the same as theta, in radians. And so this becomes a theta, this becomes a theta plus delta of theta. And so this thing becomes T delta theta. That's an approximation for small angles. Now I will apply Newton's Second Law. The amount of mass that is in here is dm. And I will calculate, shortly, what dm is. It's a little bit of mass. We're going to make dx go to zero-- infinitesimally small amount of mass. And so that mass, times y double dot, must now be this force that we just calculated. So it must be t delta theta.

But what is dm? Well we know that the length of the string is delta x, so dm must be delta x times mu. Because mu is the amount of mass per unit length. And if my length is delta x then dm is delta x. So I can write this now as delta x times mu times y double dot equals T times delta theta. We're getting there.

Now since we're in the limiting case, we're going to make delta x zero. The tension of theta, so that'll be coming, then, in this direction. The tension of theta is dy dx, right? That is dy dx. And the reason why I use partial derivatives is that I think of it as the time not changing. At any moment in time this is dy dx. That's the only justification for the partial derivatives.

I take the derivative on this side and on this side in x. So the left side, I take d tangent theta dx, and I do it on the right side. Now the derivative of the tangent of theta, of the function, is one over the cosine squared of theta. That can't take you more than 20 seconds to confirm that. You can do that in many different ways. So this is the derivative of the function itself. And then of course I have to multiply it by d theta dx. Because I take the whole function derivative in dx.

And so here I get, then, d2y dx squared. But for small angle approximation, cosine squared of theta is 1. And so I'm going to substitute, now, this result into my differential equation.

I read this as delta theta, which is here, and I read this, in my mind, as delta x, which is here. Now mathematicians would probably never do that, but physicists have no problems with that. So I'm going to write, now, here mu times delta x and here I write d2y dt squared. I use partial derivatives, because I'm not changing x. That's the justification for the partials.

And now I get T, and now this delta theta I'm going to write for it this, times delta x. So now you get delta x times d2y dx squared. And now I'm doing something that mathematicians would never do. I'm going to divide out delta x. Don't tell your 18.0... whatever people that I did that.

So now what you have is that mu divided by T times d2y dt squared, constant value of x, is now d2y dx squared. And believe it or not, this is a big moment in our life.

You have here a differential equation of y, which is a function of x and T, whereby here you take the double derivative in time, and here you take the double derivatives in space, in location.

What is a possible solution to this differential equation? You can just see it. By looking at it, you immediately see what the solution must be. Any function, any single-valued function-- you can come up with any one, I don't care which one-- any single-valued function of x plus or minus a constant times T will satisfy this differential equation. Just look at it. You can see immediately that it works.

Take the second derivative in time. You get a C square out, and you get the second derivative of the function. Take the second derivative in x, you only get the second derivative of the function and that's all. So all it requires is that C is the square root of T divided by mu. Then I bet you a month's salary that any single-valued function will satisfy this differential equation.

What is the dimension of that C? What is the dimensional of that C.

PROFESSOR: Meters per second.

PROFESSOR: Meters per second. It's a velocity. Because if I have apples here I must also have apples there. And so this can only be an apple if C has the dimension of a velocity.

So therefore you might as well write this as plus or minus vT. And you might as well as well write v for here, the velocity. And we might as well change, now, this differential equation in a way more uniform way, which is what I'm going to do now, which is one over v squared times d2y dt squared equals d2y dx squared.

And this equation is what is generally called the wave equation. It will be with us until the end of the course, until death do us part. It is really a big moment because you're going to see this equation many times for many different systems, but now you have seen it being derived for this very specific case.

Let's now evaluate the meaning of that v. Well, if I have a, here, x, and here y, and I pick just a function-- it could be a sine, it could be a cosine, I pick one is way more imaginative. I pick this one.

That's my function. It has to be single-valued, though, we have to be careful. It must be single-valued. You cannot go back. That's my function. And so that's my function, f times T equals zero.

Let us take, for v, always a positive number, for simplicity. I'm going to call it even speed. Speed is always positive, right? And I want to know now, if I look a little later in time, when there is a minus sign here, what that function looks like.

So at T equals zero I gave it to you. What would it look like a little bit later in time? If there was a minus sign there. Any suggestions?

AUDIENCE: [INAUDIBLE]

PROFESSOR: The function has shifted in what direction?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Use your hands. Who thinks it's in this direction, who thinks it's in this direction. Very good. It's in this direction. So you will see, a little later in time, you will see it here. And what is it doing? It is moving with speed v in that direction.

Now we're going to evaluate the plus sign. What will happen if we now look at the function a little later in time. A little later in time. it has moved in this direction. And it's moving with speed v in this direction.

So now, you can look through the meaning of this equation. You now understand why when I wiggled here, why the string had no choice. It must propagate that function that I generated, and it must propagate that with the speed square root of T divided by mu. We derived the speed of propagation for that string. Mu is the mass per unit length, T is the tension. If I ask you, is it obvious that the higher tensions gives you a higher speed-- maybe. Is it completely obvious to me? Sort of, not quite, but yeah, I accept that.

Is it obvious, if I make mu large, that I make it a very thick, very heavy per meter, that the propagation speed is slower? Yeah, maybe. Now that I know the answer I would say, yeah, it's quite obvious. But it's not so trivial.

So in any case, we have derived two things. We have derived that there is such a thing as a speed, but we even have derived the speed itself. The square root of T over mu. So if we had done the experiment again, with a higher tension, then the pulse would've moved faster.

But now there is something else that we have to explain. Why on earth is a mountain coming back as a valley. And why is a valley coming back as a mountain. And that now is the result of boundary conditions. Some people who have lectured 8.03 make a very simple statement. They say 8.03 is only about two things, this equation and boundary conditions. And all the rest follows. It's quite accurate.

So we have here the string. That Nicole and I were holding. And here is the end. That's where Nicole was. I hope I spelled that correctly. And we know that that end must stay fixed, cannot move. I'll put the line a little lower. I'll put it here. This is the end.

And my pulse came in, this is the pulse. And let us evaluate the moment in time that this part of the pulse reaches Nicole. You ready for that?

So this part is here. And the part that, yeah, maybe it's in heaven, is here. I have to make this a little steeper to make it look alike. Make it a little steeper. And this, yeah. Who knows what happened with that.

But Nicole knew very well that this point cannot move. Therefore, she very sneakily, without telling you and me, generated a pulse that came back to me, which make sure that all moments in time this point stood still. So at this very moment in time, she must have generated the pulse which had this displacement. So that this part exactly the same as this, and so that her hand stands still.

But she must have done that at every moment in time. She must have done that when this part arrived, when this part arrived, when this part arrived, when that part arrived. So that means, he must have generated a pulse on her side that is a valley that now looks like this. So this part is here. And at this moment in time, all she has to do is generate this pulse.

And so the net result is that if you took a photograph of this string at this moment in time, you would see something very bizarre. It is the sum of this with this. And you try to draw what that looks like. For one thing, this point will be here. That's for sure. And then whatever you're going to see here, well you try to add the two up.

And this thing is moving in my direction. With speed v, because she is generating a valley. And so the consequence of the boundary condition is, since this point is fixed, a mountain must come back as a valley, and a valley must come back as a mountain.

And given a little bit of time, when this point here has passed Nicole completely, then there is, of course, a very nice healthy pulse on the way back to me, which is mirrored now this way. The mountain is a valley, but it also has mirrored this way. See, that's why I made the pulse purposefully asymmetric. And so that is what is happening.

So now I want to do this experiment again with Nicole, of course, because she knows how to do it. And you're going to look at this with completely different eyes. Your eyes were closed when we did it the first time. You were blind, let's face it. But now you've seen the light.

This is a big moment in your life because you now know, first of all, why it propagates. And now, when it arrives there you know that the mountain becomes a valley. So I'm just going to do exactly the same thing, only to allow you to look at it now through different eyes. And that's what education is all about, regardless of whether it's physics, or whether it is art. Without education you cannot appreciate it. Now you can. Watch it. You ready for this, Nicole?

You see? It moves, it has no choice. And the mountain comes back as a valley. I will do that once more, very clear. Boy, you deserve an A for this course, that's clear. Oh whoa, you don't want to-- you don't-- I'll make it a B.

AUDIENCE: [LAUGHTER].

PROFESSOR: I can change the boundary conditions I don't have to keep these points fixed. And I can do that in following way. Here is my string. I have here a metal rod. We put oil and grease on it so that it's completely frictionless in this direction. And we mount here a massless ring, massless. But the tensions, of course, is there. And mass per unit length is mu. None of that changes. But here is a massless ring, and here is a rod with zero friction.

Those are very different boundary conditions. This point can now move up and down. And it will. However, the shape of that string right here is now very special. At all moments in time, what will the shape of this string be when we photograph it. No matter when you photograph it. You can photograph it before the pulse is there, after the pulses is there, at any moment in time. What will this point look like?

It comes in at 90 degrees. dy dx, if this is y, and this is x, at that location dy dx must be zero. If it were not zero, so this would be zero. This would be zero. This is zero. This is zero. This is zero. That is all zero.

If it weren't zero, if it was this, then there would be a force on this ring, because the tension would be in this direction, but the ring has no mass. And so the acceleration of the ring would be infinitely high, which we don't allow. So therefore, in the extreme case that you can go to this situation, you will now see something very different. You will see that the string, at all moments in time will have to be like this.

If now I send in a mountain, what do you think will come back?

AUDIENCE: A mountain?

PROFESSOR: A mountain comes back. A mountain goes in, mountain comes back. Who thinks if a mountain goes in, a mountain comes back. Very good. And that's the consequence of the fact that it is open now. Because the only reason why a mountain goes in and a valley came back, there was only one reason, the end could not move.

But now the end can move, and I will demonstrate it to you. And now the mountain will come back as a mountain. We refer to this, in physics, as a closed end. And we refer to this as an open end. And when you have an open end, and this is the pulse that comes in, say it has amplitude A, then what come back at some point in time is again a mountain going in this direction with speed v.

This comes in with speed v, we call that the incident pulse, and this we call that the reflected pulse. This has amplitude A, and this has very interesting consequences. Namely, at the moment that this point here reaches that massless ring, the massless ring must go up by an amount 2A. Because it generates-- that ring generates this pulse. And so the ring generates this pulse, but this one is also there, and remember, you have to add the two together like this one was added to this, that gives me zero. Now we have to add this A to that A, and so what you'll see is that if here is your ring, it will go up 2A. So it will make a huge excursion, goes twice as high as the incoming one, and then it will go back to 0, and then the mountain rolls back. And needless to say, that we of course would like to demonstrate that.

Now, to make a rod which is nearly frictionless, it's difficult but we can use a lot of oil and a lot of grease, and a lot of soap. So that was not our major hangup. But when we looked at amazon.com, and we wanted to buy a massless ring--

[LAUGHTER]

Marcos and I really tried, but it didn't work. We couldn't buy a massless ring. And so therefore, it is not so easy to demonstrate this in the way that I have there.

So we will demonstrate it to you in another way. And that is with this instrument. I will first explain it. This is not a string. These are rods, all the same length, and they are connected here with some metal. And so you can move these and then a propagation, the pulse that you generate will propagate. So they're coupled, they are-- I don't know how many there are, do you know how many there are? Ok let's say I count 40, then there's 40 coupled oscillators.

And now I have the option, with this machine, that I can hold this one fixed, which is then a closed end, but I can also let this one open, and then it's an open end. And so if I hold this one closed, and I send in a mountain here, then a valley will come back. But if I keep it open, then I send in a mountain, then a mountain will come back. And you should be able to see that the and gets a huge amplitude at the moment that it reaches the maximum. And so that is what is on our plate.

Now, and we will make it extremely romantic for you, believe me. We're going to do this in a very romantic way. I told you. So here I have a clip, a clip here, so I will first lock this in place so that this end could not move. That's what I will do first. And from this side I will then generate a mountain. The speed with which it propagates is actually quite decent, not as fast as it was with the string.

And so I want you to see that, first of all, it propagates and that it comes back as a valley. So the end, here, is now fixed. It's a fixed end. You ready? Mountain, and now it's a valley. Did you see it? OK, now it's always a pain, because the system is a very high q system, so it doesn't want to damp out.

I can try to send in--

AUDIENCE: [LAUGHTER]

PROFESSOR: Yeah, I know exactly what you're thinking. We are aware of this. If you try to calm it down, you make it worse sometimes. I will now generate the valley, which is a little harder, I don't know why it is, why it's a little harder, I have talked to my psychiatrist about it. It's easier to go up and down then to go down and up. I don't know why that is. So I'll go down and up, make a valley, and then when it comes back it's a mountain.

There is goes, and it comes back as a mountain. Did you see it? Did you?

AUDIENCE: Yes.

PROFESSOR: If you didn't, just say so, and we can do it once more, but I don't think we have to. Now comes the big thing. Now I'm going to make this end freely moving. So now it's an open end, and I will generate a mountain now, and I want you to not only appreciate that it comes back as a mountain, but, above all, that the end will have twice the amplitude at one moment in time, when the top of the mountain reaches that end. And then, of course, it will go back to zero and the regular mountain will roll back to me.

So if you're ready for this-- there goes the mountain, [INAUDIBLE].

AUDIENCE: [LAUGHTER]

PROFESSOR: And it comes back. What is so funny about that? Did you see that huge amplitude? I'll do it once more because I don't want you to forget that. Now let's give it just 10 seconds to die-- oh boy, look, there's a whole, like an ocean.

I will it once more. I'm not going to try a valley because that's where the problem comes in with me. I will simply go up, and then let's look again at the end and see whether we can see that double amplitude of the mountain. Mountain, whoa, biggie! Whoa, man! Whoa. OK, have a good weekend.