PROFESSOR: When the speed of propagation of a wave in a medium is independent of the frequency, thereby independent of the wavelength, we call that a non-dispersive medium. When the speed of propagation depends on frequency we call that a dispersive medium. And that is on our plate today.

I want to revisit the case that we have N beats, each mass m, and they are connected with little strings, little l, and the total length, say, is capital L. Perhaps you remember, you even had a problem whereby we had five beats-- It was also demonstrated by me in my lectures.

This end was fixed, and this end was fixed. So this was a very special case whereby N equals 5, and we derive the normal mode frequencies. The general result is that the normal mode frequencies, in terms of n-- n being 1, 2, 3, et cetera, equals 2 omega 0 times the sine of n pi divided by 2 (N + 1).

I don't want you to remember that, I certainly don't remember it-- let me move this. I certainly don't want to remember it, but this is something that today I will need. 2 (N + 1). Omega 0 was the square root of T divided by ml. So this little m, let me make it a capital m, each one of those beats has mass capital M.

You can see that the maximum possible frequency is 2 omega 0 because that's when the sine of this function equals 1. The boundary conditions, that it is fixed here and that it is fixed there, demand that there are certain values of k of n, which I allowed, which is n pi divided by L if the length here is l.

Now omega n, as you see here, does not increase linearly with little n because it increases with the sine of n. But the speed of propagation, v, is, of course, the ratio of omega divided by k. So that is omega n, divided by k of n. And since k of n goes up linearly with n, little n, but omega n does not go up linearly with little n, it is clear that the speed of propagation is lower for high frequencies. Because when you go to a higher values of little n, k goes linearly up but omega n goes slower. And so you see here an example of dispersion. Namely, that the speed of propagation is a lower for higher frequencies than for lower frequencies, and therefore it also depends on wavelength.

A very nice way of seeing that is to make a plot, and you will see many of these plots today, of omega versus k. Let me make the plot here. So this vertical is going to be omega, and this is k, and this is omega. So this is k1, this is k2, k3, k4, k5, let me move the k, and then here is the end of the string.

So this value here is k5, and this value here is k2. Those are those normal modes solutions. And so the maximum value that omega can have is then 2 omega 0. And if I connect, if I draw this curve now, it looks like this. It's a sine curve. So for k1, this is omega 1. For k2, this is omega 2 and so on.

So you can immediately see that the speed of propagation in the fifth mode is lower than the speed of propagation in the lower modes because this point here, which is k5 connected with omega 5, I can connect that with the origin, and then this angle is a measure for the speed of propagation because the speed of propagation is only got divided by k. So omega divided by k, the slope is a measure for the propagation speed. And you see that if I go to this point here, and I draw this line, that the slope is larger. And so the speed is larger. And so you see here, in a graphical way, which is always very nice, to use omega k diagrams, well I already argued with you here that the higher the frequency, the lower the speed of propagation. And that is what we call dispersion.

Now dispersion is very common in physics, as you will see today. Water waves, under certain conditions, are dispersive. Electromagnetic waves, which we will begin to think about today, and next lecture we will go further into that. Radio waves, and light, and infrared are electromagnetic waves. They can be dispersive in certain media, not in a vacuum, but in media. Waves on a string, as I will discuss very shortly can also be dispersive, are, in fact, dispersive.

The consequences of dispersion are very non-intuitive, and this is what I want to discuss with you first. I will start with two waves, which have a different wavelength, and I will give them, purposefully, a different speed of propagation. Think of this as being on a string because that's always very easy to see because that's a transverse motion.

So I have y1, which is a certain amplitude. And I'm going to make it a traveling wave, so we get the sine of k1 x minus omega 1 t, and so the speed of propagation is omega 1 divided by k1. And then I have a second one, which for simplicity I give the same amplitude, the sine of k2 times x minus omega 2 t.

And this is v1, and v2 is omega 2 divided by k2. And the two are not the same. If they are the same, then it is a non-dispersive medium, but they're not the same. And so now I want to add them, so I want to know what y is, which is the sum of the two. So y is y1 plus y2. So that becomes 2A times the sine of half the sum times the cosine of half the difference. And so that gives me the sine of k1 plus k2 divided by 2 times x plus this term. So that becomes minus (omega 1 plus omega 2) divided by 2 times t. So that is the sine of half of the sum-- I should put brackets here so the minus sign holds for here and for there-- and I have to multiply by the cosine of half the difference. So I get k1 minus k2 divided by 2 times x. And now I get minus (omega 1 minus omega 2) divided by 2 times t.

So look closely at what I've done here. I've added two sine curves. I get twice the amplitude, the sine of half the sum times the cosine of half the difference.

Now let us take as an example whereby k1 is approximately k2 and omega 1 is approximately omega 2. So we take the frequencies close together. You would agree with me, then, that k1 plus k2 over 2 is then, effectively, k. It's the sum of the 2 divided by 2. You would also agree with me that omega 1 plus omega 2 divided by 2 is omega, sort of the mean value between the two.

So I can rewrite this, now, that y is approximately 2A times the sine. And now I'll just write kx minus omega t-- K being, now, the mean value between the two, omega being the mean value between the two-- times the cosine of delta k divided by 2 times x minus delta omega divided by 2 times t. And now look at what we have here. This, all by itself, is a traveling wave. The sine alone is a traveling wave. And that sine alone has a velocity. This wave, as a traveling wave, has a velocity which

I'm going to call the phase velocity, with a P, that's our definition of phase velocity, which is omega divided by k. So when you see that wave moving, it will move with that speed.

But the cosine term has a very different velocity. It has a velocity delta omega divided by delta k. And that is the same as omega divided by k if the waves travel with the same speed. So that v, which we call the group velocity is delta omega divided by delta k. And if the two are the same, then it's a non-dispersive medium.

But if the two are different, then you're going to see that this wave moves with a different speed than that wave. The consequences are even more bizarre. The wavelength of the sine curve is, of course, given by 2 pi divided by k. That's the definition of wavelength.

And if you think of the wavelength, the repetition pattern of the cosine function, that is 2 pi divided by this k. So that is 4 pi divided by delta k. And if delta k is very small, then this wavelength, the repetition of the cosine is way longer than the wavelength of the sine curve. If I made a curve of the sine curve you would see this. And that moves with the velocity, phase velocity, say, in the positive direction.

If I make a plot of the cosine, that would be like this. And that would move with the group velocity. And if the two are the same, then this overall pattern, which is the product of the two, which I will put in shortly, moves with the same speed. If I multiply this with this, and I will try to do that here, you will get this.

So what you're looking, here, at is exactly this function. The sines are in here, and the cosine is there. Some kind of a beat phenomenon, of course. And it is in this envelope that moves with the group velocity, as I already have here. It is the individual sines that move with the phase velocity. And if the two are the same, it's a non-dispersive medium. If, however, the two are not the same-- and that's what I wanted, I wanted the two to be different, then you will see that the group velocity is different from the phase velocity.

The group velocity can be larger than the phase velocity, and so then this red envelope moves faster than the individual sines. It can also be smaller than the phase velocity-- I will demonstrate that to you. When you look at omega k diagrams, in general, you get the full story. I'll make you an omega t diagram-- omega k diagram of a non-dispersive medium, which is very boring.

So this is omega versus k, and this is a non-dispersive medium. A straight line. It's clear that v phase, which is omega divided by k, is this, indicated by this angle, by the slope. And is the same. Independent of omega, it's everywhere the same. That's a non-dispersive medium. That is the definition of a non-dispersive medium. That always, regardless of frequency, have the same phase velocity.

And so you will see, since the group velocity is the omega dk, which is the tangent of this line, you will see that the group velocity is the same, of course. Because if you take the tangent of this line, no matter where you take it, that is the same slope.

So this is also the group velocity, and so this is a non-dispersive medium. When we dealt with strings, so far, we always made the assumption that the speed of propagation was independent of frequency, and so we always treated strings as if they were non-dispersive. dispersive And we derive that from the wave equation.

I can show you another omega k relationship now. For instance, the omega k relationship could go like this, that would be a dispersive medium now. You should be able to tell me, now, whether the speed is higher or lower at high frequencies. Well, the speed of propagation is clearly lower at high frequency because it's dictated by this angle, and this angle is smaller here than it is there.

But the group velocity is also dependent, now, now on frequency. For instance, right at this frequency here, so right at this frequency, the group velocity is given by the slope, the omega dk. And that group velocity, which is the omega dk, has a different angle, the slope, than this slope. And so you see that, for all these points here, the group velocity is different from the phase velocity. And the exact values of the phase velocity, and the exact values of the group velocity follows from the shape of that curve. So you have to know that curve, and that curve is called, in general, the dispersion relation. So any relation between omega and k gives the whole show away.

Suppose you had omega k relationship that goes like this, and you will see today that that's possible. So now, you have enough knowledge to tell me that at high frequency, the phase velocity is higher than at low frequency because the slope is higher. You also can see that the group velocity at high frequency, which is the tangent, is higher than the phase velocity. Pick any point here, and you will see that the tangent has a higher slope than the line straight through here. So the group velocity at any point is larger than the phase velocity.

It is even possible in physics, but it is a rare case, that you have this situation. And I will demonstrate it to you, but it's a computer simulation. So it's always easy to do that on a computer. It is even possible that there are parts of the omega k diagram which go like this. They may go up here, but they go down again. If that's the case, that is remarkable. What you have, this slope would indicate the phase velocity, but this slope would now indicate the group velocity. And that group velocity, now, is in the opposite direction of the phase velocity. So the sine wave goes like this, and the group goes back. Even that as possible. That's a rare case but it does happen.

So what you have here is that the wavelengths, remember wavelength increases in this direction because k and lambda are inversely proportional, so the wavelength increases, now, with increasing frequency. If my frequency increases, the wavelength increases, which is something absurd. We always think that if the frequency increases, that the wavelength would get shorter. Well there are media where that is not the case. That is what dispersion is all about.

I can create a pattern like this with two waves for you, which I will do, and I do that in a graphical sense, that I have two transparencies and one transparency has black bars on it, many, as you will see shortly, and they have a separation, d. And I have many of them. Maybe 100, I didn't count them. And I have another transparency which has bars which are 5% thicker. And also, the opening between them is 5% larger.

So here we have 1.05d. And I line these up, so this is the very first one on my sheet, this is number one, and this is number one. So they line up. This one, the opening, the separation is 5% larger. So that means when I reach here on my sheet number 20, then I reach here number 19. But they are, again, on top of each other. They are in phase, so to speak.

So my number 20 coincides with number 19 here because this one is 5% larger spacing. The wavelength is 5% larger. That means halfway in between the black lines will, from this sheet, will exactly eclipse the open spots of this one. And so these black ones here will all occult, to use the astronomical phrase, occult the openings, and so this whole central portion will be black. Black will be on top of light, and light will be on top of black, and black will top of light.

Here, however, it's back to what it was here. So you're beginning to see this kind of pattern. Think of this, for now, as being dark, and think of this now as being light. This kind of pattern is what you're going to see if I put the two transparencies on top of each other, as I will do very shortly.

But now comes the wonderful thing. Not only will you see such a pattern. But imagine, now, that I move one of the sheets, say the upper sheet, over a distance half d. So I move it only over this distance. Remember here, the black ones were occulting the open ones. But if I shift that by half d, it will be reversed. The black ones will no longer occult the open ones. And so this whole black area will instantaneously turn to light again.

And that means that this overall pattern moves 20 times faster than the motion of one of these sheets. If I move this sheet over the full distance, d, this entire pattern that you see here, which I think of as being the group velocity moves 20 times faster. And that's what I'm going to show you. And the reason why I get 20 times, of course, because I have chosen the wavelengths to be 5% apart and I am going to give them a different phase velocity.

The first thing that I want to do is to give them the same phase face velocity. So here you see those two sheets on top of each other. I've each marked them, one with a red mark and one with a blue mark. I think the blue one is the one that has

5% larger spacings. And you see exactly what I predicted. You see those dark bends. So these are the areas here where-- these are the areas, the dark areas, where the black band of one occults the opening of the other. And then, in between, you see light here, light here, and you see light there.

If I can manage to move them both with the same speed, so I take them firmly in my hands, and I'm going to move them both with the same speed, then notice that the bars, the individual bars, move with exactly the same speed as the whole pattern. So that means the group velocity and the phase velocity are the same. I'll try that once more, there we go. I have it firmly in my hands now.

So you see the group velocity and the phase philosophy are the same. But now I'm going to move one relative to the other. Watch closely. Now I'm going to move one relative to the other. And you can see by the-- you can tell by comparing the red spot with the blue spot, I have to align them, compare the red plot with the blue spot and you can see the relative motion.

I'm moving them now. And look how fast the group velocity is compared to the motion between the two. You can actually hardly see that the red one is moving relative to the blue one. You can hardly see that. But look how fast the group velocity is. You see that? 20 times faster. You see how the red one separates from the blue one? It's a very low speed, that is the phase velocity. It's the difference between the phase velocity, that's really what I should have said. It's difference between the phase velocity. And so, then, the group velocity goes 20 times faster. So now I want to return to our continuous strings.

When we dealt with continuous strings we derived the wave equation. And I still remember how we derived it. We took into account that the tension is responsible for the restoring force. So we made the wire like this, and then we had tension, and then we worked out to the the math. And then we found that in that case d2 y dx squared was 1 over v squared times d2 y dt squared.

And we found, by substituting the solution into the wave equation, that the speed of propagation was the square root of T divided by mu. T being the tension, mu was the mass per unit length. In this speed is no k, there is no omega, so it's a non-dispersive medium. It says that you tell me what omega is, and this is going to be the speed. You tell me what omega, that's going to be the speed. It's independent of frequency. It's a non-dispersive medium.

And the reason for that is because we only took into account that the tension in the string was responsible for the restoring force. So that gave us, then, that omega was vk, that came out of this wave equation, and so we found that omega squared is v squared times k squared. That is the result of the calculation when the tension is exclusively responsible for the restoring force. And you get a non-dispersive medium.

However, there is something that we did not take into account. And that is the stiffness of the wire. Imagine, for now, that the tension is 0. So you take a piano string, or you take the string from a violin and assume there is no tension at all. And you take it in your hands, and you bend it. There's no tension. You just bend it. It wants to straighten out. And that's the result of stiffness.

And it is due to the stiffness, now, that you get an extra restoring force which we have ignored. We didn't take that into account. And the restoring force, due to stiffness turns out to be proportional to k to the power of 4. It's inversely proportional to the wavelength to the power of 4. It's an approximation. In other words, our wave equation that we had is no longer valid because this wave equation only took into account the tension.

If now you go through the whole procedure again, which is slightly more complicated, you will find now that you get a different relation between omega and k. And now you see that the medium becomes dispersive. Now you get that omega squared equals v squared times k squared plus alpha times k to the power of four. And this is the result of that stiffness.

Beckafee and Barrett call this alpha, A squared. That's fine, of course, that's just a matter of definition. A is a positive number, and what this tells you is that the higher the frequency, the higher the speed of propagation. I can make for you an omega k diagram. So let's have here omega and let's have here k.

So what we had before, when we only took the tension into account, we had this. Omega is linearly with k. And that slope is the phase velocity. It's also the group velocity. All wavelengths have the same speed. But now, because of this term, it's going to curve up, because alpha is positive. So now you are going to get this.

And now you see in front of your own eyes that the higher the frequency, when you were here, the phase velocity here is higher than at low frequency here. This slope is lower than that slope. And so now you have a dispersive medium. And also, the group velocity, right here, is even higher than the phase velocity. Because this slope, here, which is the tangent, is higher than the connection to 0. And so a piano string is dispersive. And that has major consequences.

Now the values for alpha, as you can imagine, depend, of course, entirely on your string. If you have a very thick string, then it is very stiff. It's very hard to bend. So alpha will be high. Also if Young's modulus of the wire is very high, then it is extremely difficult to deform it. So alpha will also be high. So there is no such thing as one value for alpha for all your piano strings. They will be different for all the strings.

But I do want to do a calculation to give you an order-of-magnitude idea of what effect this would have on a piano. So I choose a value for alpha which is not entirely absurd. Although way higher values of alpha are possible. And then I want to calculate with you, quantitatively, what difference that would make for a particular string in the piano.

And so the case that I have chosen, I take alpha is times 10 to the minus 2. I take the tension which is very common for piano strings, which is 250 newtons. I take the length of the piano string, which is one meter, just for simplicity. And I give it a mass-per-unit length of 10 grams per meter. So that would be 10 to the minus 2 in our SI units. 10 to the minus 2 kilograms per meter. And I want to explore with you the tenth harmonic of this string.

Well T divided by mu, by the way this can be written, I will write it down again, in-- this is also T divided by mu times k squared plus alpha k to the fourth. This v, remember, was T divided by mu. So T divided by mu, that's easy enough, that is 2.5 times 10 to the fourth. I have to know what k is. Well lambda 10-- lambda 1 is 2 meters, right, fixed at both ends. If it's 1 meter, the length, then lambda one is 2 meters so lambda 10 is 10 times smaller. So that is 2 divided by 10. So k 10, which is of course, by definition, 2 pi divided by lambda 10, is then 10 pi.

And so now I have all the ingredients to compare this omega squared, this term, with that term to see by how much the frequency changes over the case where we have no dispersion. So this first term now, and you can check that, of course, for yourself. So I will write down omega squared again, which is that term. Is T over mu. And then I have to multiply it by k squared, and then I find 2.5 times 10 to the seventh. And then I can calculate alpha k to the fourth, and I have all the ingredients on the blackboard, and you find that this term is 10 to the fourth.

If I calculate, roughly, what omega is, I'm not interested in the exact value of omega, you will see shortly why. You'll find that omega is about 5,000 radians per second in this case. Which translates into a frequency of about 800 hertz. The value itself is not so important, but what is important, now, is that as a result of this extra number the frequency will go up by 0.02%.

And this, of course, you can check immediately for yourself with your calculator. And 0.02% increase means, in this case, one sixth of a hertz. So at the 10th harmonic, if you expected exactly 800 hertz, this is of course not exact, but I take a round-off number, then it will be one sixth of a hertz higher because of the dispersion relation. But if you make alpha ten to the minus 1, which is by no means impossible, then it would go up by 1.6 hertz.

And so the bottom line, now, is that for a real piano string omega n is no longer n times omega one. And that is the reason, that is why pianos go sharp, as musicians say. That means at the higher harmonics the frequency is a little higher than linearly proportional with n. And that's what's called-- makes the piano go sharp.

Professor Wyslouch developed, several years ago, when he was lecturing 8.03, a toy model which he uses on the computer. And the toy model has the following dispersion relation. It has no direct connection with the string, no direct connection with any physical thing, but it is simply used for the purpose of demonstration. And that is that omega squared equals v squared times k squared times 1 plus alpha k squared. And if you're not careful, you may think that it is the same as this. But it's not. It's very different.

The difference is that this term here has a v square in it. And this term here does not have this v squared. So totally different dispersion relationship. In this program that you're going to see, Bolek Wyslouch shows you six waves, all six have the same amplitude, but they differ-- the shortest wavelengths and the longest wavelengths differ by 12%. So two neighboring wavelengths, at 2.5% apart. And the velocities of the individual waves, the phase velocity, is then dictated by this dispersion relation that is omega divided by k. So these velocities are not the same if you put in alpha non-zero, of course. If you make alpha 0 then they are the same.

First, we have to understand what you're going to see when the waves are not moving. If I take six waves with different wavelengths-- there I showed you only two waves with different wavelengths, and you saw all these beautiful beats. If you do that with six waves, and you line them up, at say, the center of your screen, that the amplitudes all line up with each other, you'll see six times the amplitude at the center of the screen.

But when you go away from the center you do not see anything like this. What you're going to see, the computer will show you, it's not so intuitive, but you'll see something like this, and then it bulges a little, and it bulges again, and that's simply a matter of geometry. And that has nothing to do with dispersion. We haven't even touched dispersion. We simply take six waves, and we draw six waves all with different wavelengths, 2.5% apart between adjacent wavelengths, we put them on top of each other like I did there, and you see a pattern. And that pattern you will see.

And to make sure that you remember that pattern for the next 10 minutes, I'm first going to show you what happens when alpha is 0. That means all the waves will move with exactly the same speed. And the envelope, crazy as the envelope will be, is moving with exactly that same speed because the envelope is the group velocity. And then when you have really remembered, when you really remember that envelope, then I'm going to change alpha.

If I make alpha positive, it will turn up. That's the general idea, if you plot, it goes like this. If you make alpha negative, which is not the case for a string, just in this toy model, I can put alpha minus, right? If you put also minus, then you're going to get this, and if you go to very high values of k it will even turn over. Because if alpha becomes high enough negative, ultimately it will turn over.

And if I can do that, if I can choose a value for k, very high, with a negative alpha, this is the situation that was required, remember, for the phase velocity to be in the opposite direction as a group velocity. So I can also show you that, which is really cute. So you see the envelope move in a different direction than the individual waves.

So Marco, did we decide on five? So the first thing that I want to run with you is then alpha equals 0. You're going to see the six waves, here it comes. Now look very closely, the red one is the superposition of six waves. The blue ones are the waves. The blue ones all travel with the same speed, take my word for it. I put alpha as 0.

And so this red curve, which is the superposition, which has nothing to do with dispersion, it has to do with geometry, that red curve moves with exactly the same velocity as the blue ones. The group velocity is the same as the phase velocity. And this crazy shape of the red one, which is due to geometry, to the overlay of the six waves, that shape is not changing. So hold onto it, make sure you remember that five minutes from now. That shape is not changing. And the reason why it is not changing is because the waves move with same speed, all of them. So that shape cannot change.

All right, so now I'm going to make alpha 1,000, and I make it positive. And I also want you to see it a little longer in time because I want you to see that now that the phase velocities are different for the individual waves. I want you to see that the overall envelope is now going to change. Which is no surprise, of course, that is now the result of dispersion. So this overall envelope will now change because, now, you've changed the geometry, how they line up with each other.

So what you're going to see here now is an example whereby the six waves all have different velocity. The group velocity is larger than the phase velocity because I made alpha positive. And the shape of the envelope is slowly going to change. You cannot see, of course, with your naked eye-- neither can I-- that the six waves don't move with the same speed. It's impossible to see that. Neither can you see that the group velocity is different from the phase velocity. You just don't have the resolution in your eyes.

But notice now that the overall red shape already has changed somewhat. It's no longer as crispy, it's no longer as sharp. That's why I'm giving it a little more time than the first one. So that you can actually begin to see that the shape, the red, shape is changing in time. And that is the result of dispersion.

Here look, look, this red one is coming out. It's way more drawn out now, and that is the result of dispersion. Alpha, by the way, is only 100 that I put-- 1,000 in this toy model.

Hey, look at that. You see that? Very, very different. And now I want you-- oh let's look at the omega k plane. This program also shows you the omega k plane. Now this is not omega 0 here, so be very careful when you look at this. You see only a very short part of the omega k plane. Something here, you see a very high value for k, you see a short section there. So you see omega, vertically plotted, and you see k horizontally plotted.

And the fact that it looks like a straight line there doesn't mean that it is non-dispersive. Because it looks, it's only a very small section. So this looks like a straight line, but the whole thing, of course, is on a curve, and that's why it is non-dispersive.

And so now I'm going to show you a value for alpha which is negative. And I'm going to give it the same amount of time. And so we start at T equals 0, we start with the same original shape that you're used to now, which is that crispy shape.

And now if you look very closely at the red one, look very closely. I will keep pointing at the top of the red one. When you see is moving in the opposite direction, do you see that? You see that? It's moving back. So the group velocity is now in the opposite direction as the phase velocity. And the shape is changing. Look how fast the shape is changing, much faster, even, than in the first case. So we are now, in our omega k diagram, we're now here. So we have a linear portion there, well it looks like a linear here, but this is of course part of this curve. Look how dispersive this is.

So the change of the envelope is the result of dispersion. And the fact that the group velocity is opposite the direction of the phase velocity. And we can look at the omega k plane when this is done. Boy, this whole envelope is falling apart, right? It's totally falling apart. And that's only due to the fact that the six waves no longer have the same phase velocity.

So we're going to look at the omega k plane when this is coming to a halt. I expect it's very close now, and oh boy yea, there we go-- so here is your omega k plane. Again it's a little section here you could see slightly curved but even if it looks like a straight line it's because of this curve. So that's where Bolek chose a very high value for k that he picked. And it was the negative value for alpha then, that gives you that result.

So as we just saw now, the fact that there is dispersion changes the shape if you have the superposition of many different waves. Now suppose that you pluck a string. Or, as we did during the last lecture, you have a square wave on top of a string. And you do a Fourier analysis of this square wave and you could say to the Fourier analysis, OK, show me what you are going to do if we look in time. And each one of those Fourier components is going to oscillate with its own frequency. And because that was non-dispersive, we always use that omega n was n times omega

1.

And therefore, if we waited one full period of the first harmonic, the shape, whatever it was, was back to what it was before. It never changed. Because after one full oscillation of Nancy one, Nancy two has made two oscillations, Nancy three had made three oscillations. So you're exactly back where you were.

That is no longer the case when you have dispersion. Because now one wave takes a little longer, or a little shorter than the other wave. So they no longer meet up with each other. Another way of putting it is that the traveling waves, of course standing waves are the result of traveling waves, the traveling waves have different velocities because of the dispersion. And that, now, will make any sharp features in a pulse disappear. Because the sharp features, if you have a rectangular pulse, the sharp features require high frequencies to make that steep. And there will be a large difference, therefore, between the phase velocity of the high frequencies and the low frequencies. And so you are going to see your sharp pulse fall apart, and that's what I'm going to demonstrate to you.

This toy model will be used to do the Fourier analysis of a square wave. I will first show you the square wave the way I showed it last Tuesday, when there was no dispersion. You will see the evolution in time, and I waited half the period of the fundamental. When the square wave came out upside down. I will show you that to remind you of what that was like, and then I will add some--

All right. I think I have to find that program, moving triangle. OK, this is the one. All right this is it, moving square at the center. So are we ready for this?

So this is what I showed you last Tuesday. The square is half the length of the string. We Fourier analyze it, all these Fourier components are going to oscillate in their own way, which is like saying we have two pulses moving in opposite direction, and they're reflecting, and they come back. I have 100 terms here this time. Last time I only had 50 terms. I have 100, and only the odd ones contribute, remember?

So we effectively have 50 terms, Mary 1, Mary 3, Mary 5, all the way to Mary 99.

And so look, this is one half-period of the fundamental. And your shape is back. Well upside down, but that's because I only ran it half a period. The shape is back from where it was.

And the reason is that it is a non-dispersive medium. And so the reason is that the Fourier components oscillate in this fashion. But now I'm going to use Bolek's toy model and I am going to-- these are the Fourier components, which are not so important for you right now.

And so now I'm going to add dispersion. And the dispersion that I'm adding is alpha is 0.01, one-hundredth, and we do the same pulse. We get exactly the same Fourier analysis. The blue curves are identical, except they oscillate no longer according to omega n equals n omega 1. But they now oscillate according to our new toy relation.

And now look what's going to happen. So the higher frequencies now, the traveling speed of the higher frequencies is different from the traveling speed of the lower frequencies. And if now you wait one half-period of the fundamental, you're going to see, when the thing is upside down, it really doesn't look like a square anymore. You can sort of still recognize it a little. The Fourier components, the blue components have not changed. They're exactly the same as they were, and this is now what you have. So that is the consequence of dispersion. So I think this is a great moment for break, we will reconvene in 5 minutes. So I summarize for you here, which is really something you will have to remember, and there is really not much more to it than that, that the phase velocity is omega divided by k. And that the group velocity is d omega dk. And so there's no such thing as one phase velocity for a non-dispersive medium. There's no such thing as one group velocity, it depends on the curve omega versus k. So you have to know what we call the dispersion relation in order to say what the phase velocity is for what frequency.

And dispersion is extremely common in physics. If we take a deep water waves, deep water waves, and I would imagine that what is meant by that is that the water is deeper than the length of your wavelength. That is probably the operational definition. It's very dispersive. The dispersion relation, which I will not derive-- I didn't derive it either for the string-- omega squared equals gk, g is just the gravitational acceleration, plus s divided by rho times k to the power of 3.

And s is the surface tension, which is for water about 0.072 newtons per meter. I don't know whether you've dealt with surface tension in 8.01, but if not, you just can accept this s. And the density of water, rho, is about 1,000 kilograms per cubic meter. That, of course, is a little different for seawater, but that's a nice number, sort of, to use.

Now this s term is only important for very short wavelengths. If you take the case that lambda is much, much larger than 1 centimeter, then I can show you that the s term is unimportant. So let's take lambda equals 1 meter. If lambda is 1 meter and k is 2 pi divided by lambda, so you can calculate what gk is, it's about 62 in SI units. Because g is 9.8.

And you can also calculate, now, s divided by rho times k to the third. Remember k is 2 pi divided by lambda. And this now is 0.02. So it's insignificant. It's very small.

And so for this case, you might as well, for first-order approximation, forget that term. And so you can write down, now, that omega divided by k, which is the phase velocity, in this case comes out to be 1.25 meters per second.

And what you also find, and I will show you that very shortly, is that the group velocity is half the phase velocity, and so is half this value. And if you want to know what frequency the water is hopping up and down, well, the frequency of course is the phase velocity divided by lambda. For that is about 1.25 hertz. So it's a little faster than one oscillation per second.

If we have the case that lambda is much larger than 1 centimeter, I will continue here, so lambda is much, much larger than 1 centimeter, I can use the dispersion relation and I can calculate now what the phase velocity and the group velocity is, in general terms.

So I know, now, that omega, to a high degree of approximation, is the square root of gk. Because I forget the s term, and so omega divided by k, which is the phase velocity, is therefore the square root of g divided by k. Do we agree? And notice that this is therefore proportional to the square root of the wavelength because 1/k proportional with lambda.

So what it means is that the larger the wavelengths, the higher the phase velocity. If lambda is larger, k is smaller, the higher the phase velocity. What now is the group velocity? Well, the group velocity is d omega dk.

So you take this one, and you take the derivative of the square root of g times k. So you first get your 1/2, and then you get your square root of g, but now we have to divide by the square root of k. So you get times the square root of g over k. But the square root of g over k was the phase velocity. So it's half the phase velocity. And that's what I already anticipated here.

So for water waves, which are much larger than 1 centimeter, you see that the higher the velocity, the larger the wavelength. And you also see that the group velocity is half the phase velocity.

So let's now take the situation that lambda is, say, much smaller than 1 centimeter. It doesn't even have to be all that much smaller, but just smaller than 1 centimeter. Then it's the second term that completely dominates, so now you get that omega squared, to a very good approximation, is s divided by rho times k to the third.

So now the phase velocity, which is omega divided by k, is now the square root of s divided by rho times k, which now is proportional to the square root of 1 over lambda. So now you have the reverse. So now you see that the larger the wavelength, the smaller the phase velocity.

None of this, of course, is intuitive. You have no feeling for that, I have no feeling for that. What I'm asking you in problem set number six is to prove that, in this case, the group velocity is 1 and 1/2 times the phase velocity. Not all obvious, but that's what will come out. Here the group velocity was 1/2 the phase velocity. And the phase velocity was larger for larger wavelengths. Here it is reversed and the group velocity is 1 and 1/2 times the phase velocity.

Shallow water waves, which is also part of your problem set, are non-dispersive-- not so intuitive. So if you have very shallow water waves, which means, then, that the amount of water that you have, the height of the water is substantially lower than the wavelengths. Then you get non-dispersive relation, which is not at all obvious either, of course.

Sound, in this room is non-dispersive, thank goodness. Imagine for a minute that the high frequency will travel with substantially higher speeds than my low frequencies. Then only the people in front, here, knew what I was saying and the one in back had no clue because the higher frequencies might arrive much earlier than the lower frequencies. And you could not make sense of what I'm saying.

Imagine you go to a concert. If sound were very dispersive, then you would have to pay a very high price for the tickets in the front row, and the tickets in the balcony, they would give them away for free because you would have no clue what you were listening at anyhow, because the different frequencies of the violins and of the bass would reach you at different times.

So we were fortunate that sound, in this lecture hall, is non-dispersive. The next lecture, we will enter the domain of electromagnetic waves. In other words, you're going to love it. You're going to see Maxwell's equations. And I know you love Maxwell's equations. You remember 8.02, the good old days.

And we will derive, using Maxwell's equations, that there is such a thing as electromagnetic waves, and that they move in vacuum with a speed which we call c in physics, which is 300,000 kilometers per second. That can actually be derived.

So in free space, that means in a vacuum, electromagnetic waves are non-dispersive. Omega equals c times k, that's a non-dispersive relationship, so the frequency of electromagnetic waves is c, the speed of light, divided by lambda. And so we will encounter, with 8.03, radio waves, and we will actually produce them. And let's say we take a frequency of about 1 megahertz, 10 to the sixth hertz, that will give you a wavelength of about 300 meters-- you can use this equation.

If we go to radar, which we have right here in front of us, the frequency is about 10 gigahertz, 10 to the 10 hertz. And so we have a wavelength of about 3 centimeters. And if you go to infrared radiation, you have a wavelength tens of microns. And if you go to visible light, you get frequencies of up to 5 times 10 to the 14 hertz. And you get wavelengths of about 1/2 a micron. And when you go to ultraviolet, and x-rays, and gamma rays the frequencies go higher and higher, and the wavelengths get shorter and shorter. Non-dispersive in a vacuum. They all travel with exactly the same speed.

Light in glass, or in water, for that matter, is not non-dispersive. It is very dispersive. As we will derive in 8.03, the speed of electromagnetic radiation is c divided by the square root of the dielectric constant kappa e, and the relative permeability kappa m, which you may or may recognize from your 8.02 days. And in a vacuum, this is 1, and this is 1.

But that's not the case for water, and that's not the case for glass, and in fact this kappa e is strongly frequency-dependent. So if I take water, as an example, for which kappa m is very close to 1 because water is not ferromagnetic material. And all materials that are not ferromagnetic have a kappa m, or relative permeability very close to 1.0000.

So forget kappa m now, but I'll show you the enormous frequency dependent of kappa e, of the dielectric constant. It's huge for water. If you look in the frequency range, f is about 0 to 10 to the 10 hertz, so that is our 10 gigahertz. That is radar, kappa e is about 78. So that means the speed of radar in water is 1/9, roughly, of c. So it's only 10%, roughly, it's 10 times slower.

If we go to visible light, so this is the radar, and this is of course even lower frequency that goes through the radio. If you look at visible light, you may say that light, by definition, has to be visible. Well, we physicists make a distinction between light in general, which is all electromagnetic radiation, and a certain part of the spectrum that we can see. And that we call the visible part.

So that's visible light which has a frequency of, very roughly, 5 times 10 to the 14 hertz. It's huge. And kappa e there is roughly 1.77. Look at the enormous difference. At that very high frequency, the electric dipole of water cannot follow the changing electric field, and therefore it just sits there and it has almost no impact. And that's why the number is so low.

So that means that, in water, visible light has a speed which is about 30% lower than c. That is dispersion. And later in the course we will see how a prism, made of water or made of glass, makes it possible to decompose white light into colors. That is only possible because of dispersion. That is only possible because the speed of light, for blue light, is different from the speed of light for red light.

We will also explore, in 8.03, the transport electromagnetic waves through metal pipes. We call them wave guides. Here I have a metal pipe. The opening here is about 1 and 1/2 centimeters across. Light has no difficulties getting through. I can look at you, and I can see you, and I can see you. You may be able to see me, I don't know. I have no problems. Light, has no difficulty getting through here.

But radar, with a wavelength of 3 centimeters, would have great difficulties getting through here. I will derive, in the future, what happens with radar when you send it through metal plates. You will see a dramatic example of dispersion.

And this is a result of the boundary conditions of the electromagnetic waves with the pipe. But I want to tell you the results, because it is such a remarkable example of dispersion. And I want to demonstrate it to you, because it's very relevant in this lecture, because it all comes down to dispersion.

We have there two aluminum metal plates that you see there. Here is one, and here's the other. Let's call this the direction of z, and let's say they are a separation, a, apart. So that's the separation between them.

We have a 10 gigahertz transmitter, which is a radar transmitter, here. And here we have a receiver, which can receive the 10 gigahertz. And so we're going to send it from here to the other end. The wavelength, lambda, is 3 centimeters. So this is 10 to the 10 hertz. The lambda is 3 centimeters. When a is less than 1.5 centimeters, radar cannot go through anymore. If it's larger than 1.5 centimeters, it can. And I will demonstrate that.

But what's behind that is, and that's really what I want to get across, is a dispersion relation. Which I will derive, but not now, but I will derive it a few weeks from now, when we reach that point. But I want to show you what that dispersion relationship will be if we have done our homework.

Omega n, whereby n equals 1, 2, 3, 4, 5 equals c times the square root of n pi divided by a squared plus k of z squared. Let's first look at this term, even though you don't see directly where it comes from. You can connect to it already. It has to do with boundary conditions. Remember, when we had a string that was fixed at both ends we got terms like this. Remember, when we had sound in a box we had n pi divided by l in the x direction and pi divided by l in the y direction. So you recognize here that has to do with the boundary condition of the electromagnetic radiation without exactly understanding why it is the way it is.

And so the phase velocity v phase, in the z direction, is omega divided by kz, and the group velocity, in the z direction, is d omega d kz. I will concentrate now only on the case that n equals 1. So I make this a 1 because that gives me the lowest frequency possible in this dispersion relation.

And I'm now going to graph that thing, omega versus k, omega versus k. This is non-dispersion. I can write here omega equals kz times c. That is the case when k is huge, very short wavelength. Way shorter than the dimension of the pipe. Then that system acts like a non-dispersive medium. You're somewhere here. But if you go to low frequencies, this is the curve.

And what you see here, there is a value, which we call the cutoff frequency omega, it's normally called omega c, below which no radiation can go through the pipe. This is not the solution anymore to my wave equation. You're stuck to this line for the mode n equals 1.

So what is required that the frequency for this radiation, well first let give you what omega minimum is, omega minimum is when you make, in the extreme case, kz 0, and n equals 1. So you get c pi divided by a. That is the minimum value for omega that can propagate through these two plates. And so a required frequency is that f is larger than c divided by 2a, that means a has to be larger than 1.5 centimeters. And that is so non-intuitive.

And this is what I want to demonstrate to you. So we have there those two plates, and the plates are now something like 2 centimeters apart. You can check this after the lecture. We modulate the 10 gigahertz with a nasty triangular sound of the 550 hertz. And the reason why that is nasty, because you now understand that anything that is modulated triangular requires high harmonics because think in terms of Fourier space. And higher harmonics, not so nice in your ears.

So if this were a nice sinusoid, you would hear a beautiful 550 hertz tone, but because we modulated with a triangle, it's a nasty tone. But the only reason why we want you to hear something is that when this receiver receives the radar signal, we want to hear this modulated signal. Unfortunately you and I cannot hear 10 gigahertz. Our ear are not designed to follow a frequency of 10 billion hertz. It cuts off at 20 kilohertz, if you remember.

So we're going to show you this transmitter first. The transmitter is the signal upstairs so it is 10 gigahertz, but it is a triangular modulation which is 550 hertz. Here is the receiver. When I turn on the receiver you will hear that signal coming out here. And at the bottom, you see the signal received by the receiver. And the fact that it is a little larger than the upstairs is simply a matter of how we adjust our amplifiers.

Nicole, can you be my witness and look here, so that you can tell class what I'm doing. I'm going to take these two plates now-- yea please come up here and just watch here-- and tell them that what I'm doing is completely honest. You see the opening here now of these two plates. And these plates are about 1.8 centimeters apart, almost 2.

And now I'm going to make it a little less than 1 and 1/2 centimeters. Watch there and listen. It's completely gone. And would you tell class that the opening is still sizable, right? The opening is just a little less than 1 and 1/2 centimeters. And it instantaneously goes away. There it is. Why don't you squeeze it, Nicole. And it's gone. Thank you, Nicole. You were expert.

Now there is one more thing that is important. And that has to do with your sleepless nights. Because, you know, I'm very concerned about you getting enough sleep.

Look at that dispersion relation. Look at this curve. This line indicates the speed of light c. That line indicates that when omega divided by k is c, that's the speed of light. So what now is the phase velocity, for instance at this frequency, so it has this value for k, well it's this.

This slope is larger than this one. It's higher than the speed of light. Not only is it higher than the speed of light, the phase velocity, but by the time I reach this point here the phase velocity goes to infinity. Because look at this slope, it's 90 degrees.

So we now have the case that the phase velocity is larger than c, now the phase velocity can be much, much, much, much, much larger than c, approaching infinity. And what, now, is the group velocity when we reach this point of the cutoff frequency. That is the tangent. And so this is the slope of the group velocity. So the group velocity goes to 0 in that extreme case. I sure as hell hope that you can sleep tonight. See you next Tuesday.