PROFESSOR: So today we're going to discuss Fourier analysis. You will see a lot of math, but I will try to keep an eye on the physics. Imagine that I pluck a string which is fixed at both ends. So here is x, and here is y. And this is the string at t equals 0. And imagine I let it go. Then we know, from what we have learned, that it will start to oscillate in the superposition of its normal modes which are standing waves. And so the $64 million question today is now-- in which modes will it oscillate. And what are the amplitudes of those modes.

If you write down y as a function of x and t then we know, from what we have learned to date, that you should be able to write this-- n equals 1 to infinity. Amplitude-- which I indicate with a B now rather than with an A-- but B is, of course, an amplitude-- has a dimension length times the sine of kn x times the cosine of omega n t. And because of the boundary conditions, k of n-- because the string is fixed at both ends-- k of n is n pi divided by L. And omega n is the velocity propagation times k of n. And n then is 1, 2, 4, 3, 5.

So we know this. This is nothing new. We've had this recently. So I can also write down then that t equals 0. I can write down this shape in terms of a series of signs. We'll leave the time off now. So if t equals 0, then we have y-- x at time t equals 0 would be B1 times the sine of pi x over L plus B2 times the sine of 2 pi x over L plus B3 and so on. And our task now today is-- what values of n are necessary to make that pulse and what are then the amplitudes B that we have here.

And that's what Fourier analysis is all about. Fourier was born in 1768 and he died in 1830. So this is work that was done a long time ago. Brilliant, brilliant mathematician. Now I will give you the general approach first. And then I will come back to the special case of a string which is fixed at both ends. But I really think I owe you the general format-- the general procedure first. So the idea behind Fourier then is that any periodic, single-valued function with period of 2 pi radians can be represented by a Fourier series. And I'll write down this Fourier series here.

So this is now a function of x. x is now in radians-- it's not that x, I will come back to that. So a function of x can now be written as a constant-- we call it A0 divided by 2 plus the sum, 1 to infinity, of Am cosine mx plus the sum, 1 to infinity, of Bm times the sine of mx. And this x that you see here-- it's also called x in Bekefi and Barrett-- is in radians. I will call this first term 1. I will call this one 2. And I will call this one 3. So our task now is to find the values for A and B if you know the function fx.

And I will build up with you the recipe that allows you to calculate the values for A and the values for B. What you do-- the first thing you do to find A0-- you take the integral from minus pi to plus pi dx on both sides of the aisle. This is one side of the aisle and this is the other side of the aisle. And when you do that, you will see that for all values of x-- excuse me, for all values of m, Mary-- all terms here will be 0. And the same is true for 3. For any value of m that you take-- if you do an integral over a sine over a complete period from minus pi to pi, that is a complete period-- you get 0, of course. So all values for 3 will be 0.

And so what you're left with is then that the integral from minus pi to plus pi of f of x dx is then the integral from minus pi to plus pi of A0 divided by 2 dx. And that is pi times A0. That's immediately obvious. So we now already have our first way of calculating A0. A0, if I use this result, is nothing but 1 over pi times the integral from minus pi to plus pi times a function x dx. So you tell me what the function of x is and I can calculate A0 for you. Keep in mind that A0 divided by 2 is nothing but the average value of the function over the period 2 pi. And I will come back to that later during this lecture.

How are we going to calculate now the other values for A. Well what you do now is you take the integral from minus pi to plus pi times the cosine of Nancy x dx. And you do that on both sides of the aisle. You do it on the left side, and you do it on the right side. And Nancy now is 1, 2, 3, and so on. When you do that, you will see that this term is going to be 0. That's immediately obvious. You do an integral over the cosine function times a constant, you get zero. So the first one you don't worry about. The third one, for any value of Mary and for any value of Nancy, the third one is also 0. And if you don't believe that, check that on your own.

So then you would get a cosine of Nancy x times the sine of Mary x for any value of n, any integer of n, any integer of m. If you integrate it over one whole period, you get 0. So this is also 0. If we now go to term number two, it is also always 0 except when Mary is the same as Nancy. So number two is also 0 except when m equals n. When m equals n, you get here a cosine squared. And when you integrate a cosine squared, you do not get 0. In other words, what you have to do if Mary equals Nancy, you're going to get the cosine squared of mx dx, because Mary is Nancy. And I must integrate that between minus pi and pi and that equals pi.

So now we have a recipe for A of m. Because A of m now is 1 divided by pi-- that is this pi-- that comes from the integral of the cosine squares. Integrate between minus pi and plus pi of function x times the cosine of Mary x dx. So we know now how to calculate A0. And we know how to calculate the values for A-- Am, A of m and we will of course do an example together. It's now clear what you're going to do to find 2 values for B. You're now going to integrate both sides of the aisle between minus pi and plus pi times the sine of Nancy x dx.

And when you do that, you will see exactly the same that you have here. This will always be 0. This will now always be 0. And this will always be 0 except when Mary is Nancy. Then you get here the sine square. And the integral of the sine square will be pi. Just like the integral here of the cosine square was pi. And so you see now that we also have now the recipe to find all the values for B of m. That's going to be 1 divided by pi times the integral of minus pi to plus pi-- that is the period of the function-- times f of x times the sine of Mary x dx.

And here you see, in its most general form, the formalism of Fourier analysis. Keep in mind that whenever you have an integral from minus pi to pi, if you prefer 0 to 2 pi, that's fine because the function is periodic. So you can always replace this 0 to 2 pi, if that suits your purpose. If you look at these three recipes, you can do away with this one because if you make m equals 0 here-- originally m was 1, 2, 4, 3, 5-- but if you also include m equals 0 here, you get exactly the same that you have here. Because when m equals 0, the cosine is 1. So this is then identical. That is the only reason why we called this constant term A0 divided by 2.

We could have called that C because, after all, a constant is a constant. By which you want to call it A0 divided by 2. If you had called this C, which is the average value of the function, then you would need this C here and you would have here 1 over 2 pi. And therefore you need three recipes to do for Fourier analysis. Whereas now, if we define this constant as A0 over 2, you only need two. So that's the only reason. It's just for practical purposes. But A0 over 2 is a constant, and it is the average value of the function over the period.

Now all this may look a little bit opaque to you now. And it will become clear, I hope, during this lecture when I put this Fourier analysis to work. What you see here-- and that's the way I want you to look at it-- is that it is simply a recipe. And you and I, when we use this recipe, we execute it. And we do not always ask ourselves why is the recipe the way it is. If I apply Cramer's rule, I do not every time say to myself-- why is it really the way it is. I have seen once why it is the way it is, and I apply it. Of course, you have to know when you can use it and when you cannot use it.

So now I would like to return to the plucked string, but I first want to take a close look at it at time t equals 0. And what I'm going to do now is to pluck it in a very unusual way. And there's a reason why I do that so unusual-- because that is doable in one lecture-- to do the Fourier analysis on it. So I have a string now which is fixed between 0 and L. And I'm going to pluck it in an extremely obnoxious way. Namely, like this. So I'm forcing it like a square- very painful for the string, but that's not my problem. And let this be an amplitude A.

If I want to use Fourier analysis, and you will see that I do, and you will also see how I'm going to do that. I need a periodic function. And this is not periodic. So I'm going to make it periodic. And the way I'm going to make it periodic is the following. I'm going to pretend that the function is really this. So I'm adding this- I make it 2L, but I go much farther. I go on, and go on, and go on, and go on. So I'm going to define it between 0 and 2L.

And that's periodic. Notice that this pattern now-- up square, down square-- is periodic. So that means my period is 2L. And also notice that by doing that, my A0 divided by 2 is 0 because the average value of this function-- which is now defined between 0 and 2L and much beyond 2L and below 0-- that that function has an average value of 0.

And so my function now-- my function of x, you can write down y of x if you want to-- is plus a for x being between L and 0. And it is minus a for x between 2L and L because this here is minus a. Now I want to express this shape into Fourier series but before I can do that, I have to make some changes in the recipe because, in the recipe, we have radians. And radians are apples. Here we have x, but that's not radians. That is meters, and meters are coconuts. You have a question?

AUDIENCE: Is that 0 [INAUDIBLE]?

PROFESSOR: Between 2L and L, it is minus a. Between 0-- is that what you want?

AUDIENCE: Yeah.

PROFESSOR: Thank you very much. I appreciate corrections because it's awkward to do it later. Thank you. So radians are apples, and meters are coconuts. And so what should I do now to make the coconuts into apples? I now have to take the old x, which was in radians, which is the one that I have here. And it is the one that I have here. And I'm going to replace that in that formalism by pi x over L. And this x now is in meters. And if I have done that, you can see that if my x now becomes 2L, then this has moved over 2 pi radians.

So that's exactly what you have to do. It is unfortunate that we call this x. We could have called this formalism z or some other symbol, but in general we give x. We could have called this z. Well, you would still have written-- you would have to write down that the x there is then pi z divided by L. Now we have apples here, and we have apples there. So that means that now my Fourier series are going to look very much like what I have here, but I prefer to write them again.

So my function y, as a function of x, which of course is my function of x-- I know that it is in the y direction-- that function should now be written somewhat differently-- A0 divided by 2 plus the sum from 1 to infinity of A of m times the cosine of m pi x over

L. This is now our x. This is x in meters. Plus the sum, 1 to infinity, of B of m times the sine of mx times pi divided by L. And so now we have made a modification to the general idea, which is in radians, to be applicable to a case whereby we don't have radians, but why we have x in meters.

Before we can execute the analysis, I also have to change the recipe in terms of A of m and B of m. First of all, I don't have an integral over a period 2 pi. but I have to do an integral from 0 to-- to L, or from minus L to plus L. So where earlier I had 1 over pi-- integral minus pi to plus pi-- I have to change this boundary either from 0 to 2L, or, if you prefer, from minus L to L. I don't care. It's the same thing because the function is periodic. But I also have to change this 1 over pi which we have in front here, because that was the result of this integral.

And if you do the integral in R space, you would have found L and not pi. This is half the period, and half the period of our function is L. So this 1 over pi, from minus pi to plus, now must be changed into one divided by L-- integral 0 to 2L. And so, to make sure you're not going to get confused, I'm going to rewrite the recipe for you. And I might as well do that again in red.

So we go now to our A of m is 1 divided by L times an integral that just goes from zero to 2L. And then we get our function of x, and then we're going to get the cosine of m pi x divided by L dx. And B of m is then 1 divided by L times the integral from 0 to 2L of my function x, which is my plucked string, times the sine of m pi x over L times dx. And so now, we have not changed anything in terms of the formalism of Fourier analysis, but we have adapted the recipe.

So we now have a new series written in terms of our new x. And we have the new recipe to calculate the A values, in terms of our new x, and in terms of the new period, which is now 2L. So now you may think that I'm ready to plunge into the math and to start executing it, but I'm not going to do that yet. I would like to take a look first at the values of A. We already agreed that A0 is 0, so that's not an issue anymore.

But let us look, for instance, at the function A1 times the cosine of pi x over L. That would be the first term of the cosine series. And the second term would be A2 times the cosine of 2 pi x over L. Well, let us assume that we found the value for A1. So I'm going to plot now into this function of ours, the A1 value. So it's here, here, here, here, and here. And so here is that function. And this amplitude then is A1. That's unacceptable. Why is this unacceptable? Look at my function and then look at the red curve, which is my A1 cosine function.

That red curve is supposed to help construct this function. Here it is positive, so I demand that it must be negative here, otherwise I can never make this trough. Whereas it is going to add a positive here. So this is A1-- out of the question-- can never contribute to my function. But not only can A1 not contribute, but any cosine function that you draw will always look here like this. It will always be positive here and positive there.

Whereas I will demand that it has to be positive here and negative there. Otherwise, I can never build up this function. To put it in a more intellectual way, the way that mathematicians would use it, the cosine function is an even function. That means that the function of x is the same as the function of minus x. We call that an even function. Here x is 0. A cosine function of x has exactly the same value as at minus x.

But our function is odd. Our function-- the way we defined it at 0 here is odd. And an odd function-- the function of x equals minus the function of minus x. And to put in a nutshell, you can never fit an odd function with even functions in Fourier. Neither can you fit an even function with odd functions. So that's perhaps a better way of looking at it. So we can already conclude that all cosine functions are out, for this specific case, of course.

Let's now take a look at the values for B. So keep in mind, B are the sine functions. So A0 is 0, all values for A are 0. Now let's turn to the B values. So I'm going to make-- draw the function again here. We realize, of course, it's only defined from 0 to L. The rest is only a mathematical way of doing the Fourier analysis. So we are going to define it all the way up to 2L, and then we make it periodic.

So let us put in there the function B1 sine pi-- B1 times the sine of pi x over L. That is the very first one in the series of the B's. That one is wonderful. That is exactly what we want. That would be-- this is sinusoid which would have an amplitude B1. It couldn't be better. Here it is a mountain and here it is a valley. Well we need a mountain here and we need a valley there. No surprise because a sine function is an odd function and our function is odd. So clearly, odd functions will do very well.

Let's now take a look at B2. B2 times the sine of 2 pi x over L. Let's put it in there. I'm full of expectations that B2 will do a great job. So we have here zero crossings. So here is my B2, and this amplitude is B2. Out of the question. There is no way that B2 can do me any good. Why?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yep.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Exactly. There are various ways of looking at it. One is easy. Here you see a mountain that helps building this up, but look what you see here. You want a valley. And it's not adding to the valley. That's one way of looking at it. There's another way of looking at it-- to say look, everything has to be symmetric about the line 1/2 L. If this builds up, it's got to build up there and it's not doing that. So we must conclude that B2 must become 0. And by analogy, all even values of m, B will be 0.

So this is a prediction that I make. We haven't done any fancy Fourier analysis. We have already concluded that for our specific function, for our specific string which is fixed at both ends, that all values for A will be 0 and that only odd values for B, for which the m value is odd, will we get non-zero answers. So now we're ready to actually execute the Fourier recipe. And if you insist that you want to do the A's, be my guest. You can do all the A's, and you will see that all of them are 0. So I will not waste your time on that. But I will certainly do the B's.

So we now execute our recipe. And here it is, adjusted to our x. And so we're going to get that B of m. Now the function is defined between 0 and L is plus a. So I can-- I can bring the a outside and put it here. And then I get the integral from 0 to 2L. And then I get the sine of m pi x divided by L dx. That is not between 0 and 2L. You should have screamed-- that is only between 0 and L. It's only between 0 and L that it is plus a, but when we integrate from L to 2L, then it becomes minus a. So the next part of the integral is minus a. So I bring the a outside. So I get minus a divided by L times the integral from L to 2L times the sine of m pi x over L dx.

So the function comes in two parts. More, there are Fourier analysis where the function comes in more than two parts, believe me. This is an easy case. That's why I chose it. This integral is trivial, of course. If I do an integral, I get minus the cosine. So I get an a divided by L out. And I get m pi downstairs, and I get L upstairs. And then I get the cosine of m pi x over L. And I have to evaluate that between 0 and L. And here I get a minus sign out. So I get plus a divided by L. I get m pi downstairs, I get L upstairs. And then I get the cosine of Mary pi x divided by L. And I have to evaluate that between L and 2L. That's all I have to do. That is the integral, a very simple one.

Now when n is odd, this one alone-- don't look at this-- this one alone is minus 2. You have enough knowledge to confirm that. You can see if you substitute x equals L, you for instance, you take m equals 1, which is odd, you get the cosine of pi. That's minus 1. And then the cosine of 0 is 1 but you have to subtract that. You get another minus 1. So you get minus 2. And so this one, if you do that, you get plus 2. Check it. And you will see that it is plus 2 because the borders-- the boundaries-- L to 2L are different. It's not the same as 0 to L.

But if m is even, you get a 0 here and you get a 0 there. And so what comes out is exactly what we predicted-- that all even values of Mary will give 0 values for B, and that only the odd values will give me values that are not 0. Before I write this out in a complete Fourier form, you should appreciate the fact that this entire integral here gives me exactly the same answer as this entire integral here. Because minus times minus 2 is plus 2 and this plus times plus 2 is also plus 2.

And that is always the case when you have a string which is fixed at both ends-- that the 0 to L integral always gives you the same answer as this imaginary L to 2L, which you introduced to make the function periodic. It is for that, and only for that reason, that Tony French gives you a much easier way to calculate Fourier components in strings. And he says all you have to do is say that B of m is 2 divided by L. He multiplies the recipe by two.

So instead of having-- where is my recipe? Instead of having 1 divided by L, he says no, you should really be 2 divided by L. But then he says all you have to do is now integrate between 0 and L of the function of x times the sine times Mary pi x divided by L times dx. And this is equation 6-32 in French. If that's all we knew about Fourier analysis, we would have a very narrow picture because it is an extremely special case. But it's true that whenever you have a string, which is fixed at both ends, this will do. I felt an obligation to you to show you the Fourier formalism, which is a beautiful formalism, in more general terms.

So we are now ready to write down the complete Fourier series. Notice that the sum of these two becomes 4a divided by m pi. So B of m is going to be 4a divided by m pi, but m is only odd. And so with that in mind, I can write down now here our function, which was one of our big steps-- that y-- I put a zero here because this at time equals 0 when I have this string in this crazy shape-- can now be written as 4a divided by m pi. I could write it down as simply a pi. And then I got here the sine pi x divided by L plus 1/3-- that is that m equals 3, which is 3 times lower because you have an m3 here-- times the sine of 3 pi x divided by L plus 1/5. And then you get the sine of 5 pi x divided by L, and so on and so on.

So now I would like to put in my function-- I would like to put in the first two terms. And I will do that here on the blackboard. So I'm going to concentrate now only on my function 0 to L, in reality the string is only here. And now I would like to put in there-- and this is a-- and now I would like to know what is B1. Well B1 is going to be 4a divided by pi, which is approximately 1.27a. Well, let's put it in there. This is a, so I have to put in 1/3 more so it comes up to this point roughly, so there it is. I love it. It's not quite a square yet, but we're getting there. We're on our way to building up the square.

The next one is B3. B3 is 4a divided by 3 pi. So it is 1/3 of 1.27, which is about 0.42a. And B5 is 1/5 times B1. Let's put in the B3. So B3 has an amplitude which is about 0.4, which is about this much-- it's about the height, 0.4, a little lower. And so here we have the maximum and then we have a 0 here, we have a 0 here, and so we have here. And then we go through this point. So the curve is something like this. So this now is B3 and this is B1. Now look at B3. Look at the beauty of Fourier analysis. This is higher than it should be.

B3 says, I'll take care of that. I will subtract something there. I'll take that off there. Isn't that beautiful? So you're going to flatten this out already. Already with two terms, B1 and B3, you're already beginning to see the building up of this crazy square. And look here. B1 says, sorry, I cannot fill this up for you. This is the best I can do. B3 says, I'm going to help you. I'm going to help stuff there. And is going to do the same there. And so you see that if you now keep adding odd values of m that you will gradually approach that square, which is an amazing concept.

And what I will demonstrate to you is making a square. And we will make it like this, just like we did here. We can do that on an oscilloscope. We can have a signal that makes this trace. Admittedly, this is time, but the way you will see it, it is space-- so you can think of it as being space. It's no different from the function that we had. And then we will show you the B1 value. In this case, since it is in time, it actually has a frequency. It's 440 hertz. We can make you listen to it.

And then we will show you the B3 value which has three times higher frequency. And then we will show you to the B5, and the B7, and the B9. And then we will add them all up and you will see it begins to look like a square. And so let us set that up. Supposed to see that on the central there. What you see here is simply the B1. So you don't see very much, do you. You just see a sine curve. Well you're looking at this sine curve. Now I'm going to show-- can we hear it, Demarcus, the 440 hertz.

Now I'm going to show you the B3. Notice it is three times smaller in amplitude. That's the way we set it-- has a three times higher frequency. And now I'm going to show you B5, which is five times smaller than the-- hold it, no, no, no, no, no. OK, it's five times smaller than B1. And the frequency, of course, is five times higher, you can hear it. There it is.

[SOUND WAVE]

PROFESSOR: And now I'm going to show you 7.

[SOUND WAVE]

PROFESSOR: Seven times smaller than B1. And here is B9, which is nine times smaller than B1.

[SOUND WAVE]

PROFESSOR: And now I'm going to show them-- all five to you.

[SOUND WAVE]

PROFESSOR: And it's really beginning to look like a square-- a square pulse. It is clear that B1 is very, very important. So if I take B1 out, which I will do now, it doesn't even look like a square. That shows you how important that first term is. If I take B3 out that is less disastrous, but it is still pretty disastrous. But you already begin to see something that looks like a square, slowly. And it should be clear to you now that in order to get the real sharp edges here, you need very high harmonics. You have to go to m values of 49, 101, 201. They're all odd. You have to go to very high values. And the higher you go, the closer you will get to the square.

So this is a nice moment to have a break. And I know you're dying to do your fifth mini quiz. I will hand it out now and then we will all start at the same time with the mini quiz, so that each one of you has the same amount of time. I owed you the histogram of the exam. You see that here. I think it's a wonderful histogram, as far as I'm concerned. It is too early to talk passing and failing, and A, B, C's, and D's. But if I knew nothing else-- no other grades-- then I would have to put you in the danger zone if you scored less than 45. That's all I can say. I can add nothing more than what I already wrote you by email.

So let us now return to our string. Our string is still at t equals 0. So our string is still in this position dying to be released. That was the goal remember. We would pluck it and we would let it go. At time t equals 0, this is the Fourier analysis. These are the Fourier components of that string. But now I say, OK, go. And what happens now is that this one is going to oscillate with omega 1 t. So this one's going to oscillate omega 2 t-- and this one, ah, omega 3 t. And this one is going to co-oscillate with cosine omega 5 t.

Each one of these are standing waves. And so each one of these are going to do this with their own amplitude and with their own frequency. And omega m is going to be v times k of m. And the speed of propagation is the square root of T divided by a mu. And that's what's going to happen. So now you may wonder what you're going to see if you let the spring go. And chances are that if I asked you what do you think will happen-- that you may think that this will happen with the string as a whole. But that's not true.

And the best way, actually, to answer the question, what you're going to see, is to go back to a simple case of a triangular pulse on a string. If I have a triangular-- I'll do that here-- if I have a triangular pulse on a string. I exaggerate, of course, the amplitude highly. Fixed here at 0 and fixed here at L. And I let it go. Yes, I can decompose that in terms of the Fourier components. And each of these Fourier components are going like standing waves, up and down. It doesn't give me much insight, but it is intriguing that that's what happens.

I can also think of it-- that nature is seeing a disturbance. And he says, well, we're going to propagate this disturbance. Nature doesn't know the difference between left and right. And so, clearly, what will happen, if this has an amplitude A, there will be a pulse with an amplitude 1/2, which goes in this direction. And one with the same 1/2, which goes into that direction. And that is exactly what you're going to see when you let it go. So a little later in time, you will see this going in this direction, going in this direction.

At the moment that the top of this mountain reaches L, and the top of this mountain reaches 0, you're going to see nothing, absolutely nothing. Because the reflected one and the incident one exactly cancel each other. And then a little later in time, they will be on their way back. And then a little later in time, you will see this one but completely flipped over when these two triangles meet each other. And so that's when you will see this thing upside down.

So there are two very different ways of looking at what happens when you release a plucked string. One way is to say they are oscillations of standing waves, a la Fourier. And the other way is to say, well, we're going to always cut it in two pulses, each of half the amplitude. And let them go back and forth and let them reflect at the ends. Now keep in mind that standing waves are the result of the superposition of travelling waves. So the two different ways of looking are, of course, connected. They have the same underlying physics.

And now I want to demonstrate this to you using a program that was developed by Professor Wyslouch when he was lecturing 8.02 and also Professor Nergis. I think she's in the audience. Nergis, are you hiding, or are you hiding. Nergis has also worked on this program. It's a wonderful program. I'm going to put a triangle, first on a string, the width of the triangle is about 1/3 of the length. And then I will release it and I will let it go. And we will see the Fourier components. And at the end, when the whole show is over-- after 1/2 the time for the fundamental period, we will wait 1/2 period of the fundamental-- we will inspect very closely the Fourier components. Because of the symmetry of this problem, all even values of m will have 0 values of B, and you will be able to see that. Now were we going to change the light setting for this? Or were we not. No? I thought we were.

[LAUGHTER]

PROFESSOR: Why don't you turn that off also. Oh that's too difficult, perhaps, so throw the bar. No-- the bar, the bar, the bar. Thank you.

[LAUGHTER]

PROFESSOR: All right, so what you're going to see first is the triangle. The numbers of Fourier components that we have

GUEST SPEAKER: Professor.

PROFESSOR: Is 25. Did I do something wrong?

GUEST SPEAKER: [INAUDIBLE].

PROFESSOR: Excuse me?

GUEST SPEAKER: We have no image?

PROFESSOR: We have no what? You broke the line?

[LAUGHTER]

PROFESSOR: Oh, that's nice. Thank you. The number of terms we have is 25. We have m equals 1, m equals 3, m equals 5, all the way up to 49. That's when we cut it off. All even values for m have no B value. And if you're ready, I'm ready.

So let's first run this one. There you see the triangle, and the blue lines are the Fourier components. They are each doing their thing-- their own thing, standing waves up and down. But the net result is something that you're quite familiar with. You see two pulses with half the amplitude of the original one. They move through each their own side, they reflect, and they come back. And then we'll stop when we have half the period of the fundamental.

Now look, look at the individual Fourier components. This is B1. This one is B3. You see it helps building up that triangle. This is B5. It helps building up the triangle. This is B7. And this is B9. And here they all conspire. Amazing. They are 25 sinusoids that are conspiring there. And they show you nothing. And the same here. And it's hard to believe that all these sinusoids can add up to 0 here. And how beautifully they connect here. And the only reason why this tip is not very sharp is that we only have 25 terms.

If I would run it with 400 terms, then you would see this sharper. But it would take a lot more time. Let's look at the Fourier components which you will see here. So this is B1, which we have arbitrarily called 1. You see all the even values. This is B2, B3-

- ah, sorry-- B2, B4, B6, B8. All the even values are 0. The B3 is negative, in this case. That was not the case for our square. Remember in the case of the square, B1, B3, B5, B7 were all positive. They all had the same sign. That's not the case with the triangle. They alternate.

The next one that I want to run for you is a square which has a width half the length of the string. And at the center-- at the middle of the string. And again, two pulses are each going their own direction, with half the amplitude. They're being reflected and strange things happen at the ends. And now there comes a time-- 1/4 over period of the fundamental-- that you see nothing. And now they're coming back from the reflection. Very different from what you expect, isn't it?

Now look at B1. That's a real biggie. Look at B1. Boy, here. Is that a surprise? No, because clearly B1 is not quite the square but it's almost a square, right? You just have to subtract something here. Well, B3 says that's OK. I will subtract there. And B3 will subtract here. And B1 is a little-- a little shallow here. It's a little bit too, not generous enough. And then B3 says, OK I'll help you building it up. And so if we look at the Fourier components, you see now that again, all the even values are 0. B1, by definition 1, and now you see this one is negative. And B5 is also negative. Not so obvious. And B7 is positive, and B9 is positive.

And now I'll show you one whereby I'm going to offset the square. The square is only 1/4 of the length and I'm going to offset it. And when I offset it from the center, you need both odd and even values. And you're going to see that. So now you cannot just get away with only even values of Mary. So here you see the wave, splitting up in two, just like you expected. It has a width 1/4 of the length of the string. This one has already reflected. This one is now on its way to being reflected. This one is coming back, there it is. And this one is already on its return.

And, of course, they're going to meet again. And now we are 1/2 period of the fundamental later. And now look. This now is the first harmonic. But now there's also a second harmonic, m equals 2 has a B value. Here it is. This is the B2 value. Large amplitude, you notice that? And look how important B4 is. I think this is the B4 value. Yes indeed, here is the B3. This is the B4. It has a huge amplitude. And look how important it is because right here where the pulse is, you need this bulge to push this further out because the B1 alone cannot reach that point.

Let's take a look at the Fourier spectrum here. Here you see B1, which we by definition call 1. You see B2 is there. It's large-- it's a little larger than 0.7. And then you have a B3, which is negative. And a B4, which is negative. And a B5, which is negative. Here B4 is off scale. And now you get B6 positive, B7 positive. B8 is 0 and B16 is 0. Well, that's probably the result of the way we set up the square which has a width of 1/4 of a length. And we offset it from the middle by 1/4 of a length.

Remarkable example of Fourier analysis now, which is way more complicated than if you nicely center it. All values of B now, all values of m, are important. You will now understand when I said-- if we pluck the string of a harp or the string of a violin, and you pluck it this way, fixed here and fixed here, the tone that you hear is different than when you pluck it this way. You now understand that, because the Fourier components of this one and this one are very different than the Fourier components from this one. And so when you let the string go, the frequencies that will be produced are the frequencies of these normal modes of these B values.

But the various B values are very different here from there. And so the sound that you hear is distinctly different when you pluck a string from the side than when you pluck it in the middle. You may remember that during that lecture, I told you that the hammer of a piano hits the string about 1/7 of the length of the string from one side. And that is done to suppress the seventh harmonic. For some reason that beats me, people don't like the seventh harmonic, so that's the way they kill it.

So it does depend on where you pluck and where you hit. And now you can put that in context because now you understand that you analyze the thing here, in terms of Fourier components. And each one will then oscillate at their own frequency, with their own amplitude. So what we have discussed now, at length, is a Fourier analysis in space, in x in meters. And we decomposed a function, in the sum of many harmonics, which together make up then the shape of the string.

Now if we look at sound-- sound, of course, is something that is a function in time. Suppose I look at the sound signal made by a tuning fork of 440 hertz. And I have here that signal in time. Suppose I have one second of data, 440 hertz. So this is now time, but if you want to think of this as being in space, be my guest. Now there are special programs, special algorithms, that take this one second of data, perform a Fourier analysis. They have a name-- we call them fast Fourier transforms.

And it's going to tell me what the amplitude of B1 is, and A1, of B2 and A2, of B3 and A3, and so on. If I now think in terms of hertz, rather than in terms of omega, there is nothing at 1 hertz. So those A's and B's are 0. There is nothing at 2 hertz, so those A's and B's are 0. But by the time I reach 440 hertz, the system says YIPPEE, there's a lot of power there, at 440 hertz. And what we now do, we make a plot of the square of A plus the square of B. And the reason why we square them is that energy is always proportional to the amplitude squared.

And so you take the A squares and the B squares and you add them, we call this a power density spectrum, a Fourier spectrum. And here you have your omega or you have your f value in hertz. The difference is only 2 pi. And if you now do a Fourier analysis of that signal, you will see at 440 hertz , a huge value. And you will see almost nothing anywhere else. So you have now done a Fourier analysis of a time signal. And you have decomposed it into its Fourier components, which in this case, is only 440 hertz.

If you did the same with the middle A of the piano, which is 440 hertz, you would see a big value here, but you would see also something at 880. And you may even see something at the third harmonic. And the same is true for a-- for a piano. And it is also for a violin. If you take a violin and you ask the violinist to produce the 440, you always get a little bit more of the 880. And you get some of three times the fundamental. And this is something that we can demonstrate.

We have here a, admittedly, somewhat poor man's version of what's called the Fourier analyzer. We take in sound for about 1/5 of a second. And then we ask the computer to Fourier analyze that for us, and show us the Fourier spectrum in terms of where the frequencies were. And we will see that here, for which I think we also invented a special light condition, didn't we? The scale that you see, from left to right, is two kilohertz. I'll first make you see the 440 hertz.

[SOUND WAVE]

PROFESSOR: You see that huge spike? That's at 440 hertz. So the end of the scale is 2 kilohertz.

[SOUND WAVE]

PROFESSOR: 256 hertz.

[SOUND WAVE]

PROFESSOR: That's lower.

[SOUND WAVE]

PROFESSOR: I hear a flute. And remember during the lecture on musical instruments, we believed-- and I demonstrated that-- that you could really make this one resonate only in its fundamental. I will now show you with this Fourier analyzer, that there was also a little bit of the second harmonic, maybe even the third harmonic. Look at it now.

[FLUTESOUND]

PROFESSOR: You see that? That was the fundamental second and third harmonic even. Look again.

[FLUTESOUND]

PROFESSOR: Ah. So this is a beautiful way of analyzing the frequencies in sounds. I will whistle for you.

[WHISTLE]

PROFESSOR: Amazing. Would you have guessed it? Very nicely separated. What is the frequency?

[WHISTLE]

PROFESSOR: It's about here, that's about 1 kilohertz, about 900 hertz.

[WHISTLES]

PROFESSOR: And the second harmonic, not so obvious. My whistle. Very high frequency.

[WHISTLE]

PROFESSOR: It is so high that it's off scale.

[LAUGHTER]

PROFESSOR: And I did that purposely because I know you hate the whistle, so I didn't want you to see what it really looks like.

[LAUGHTER]

PROFESSOR: We can listen to radio. And then we do-- can do a Fourier analysis on what we hear.

RADIO SPEAKER 1: Hi Kenny. Hi there. Have you got a question for Mr. Nader?

PROFESSOR: Oh, I don't have a question for you, I'm sorry.

[LAUGHTER]

[RADIO STATIC]

RADIO SPEAKER 2: --think what you're doing is--

[RADIO STATIC]

[CLASSICAL MUSIC]

PROFESSOR: It's not the right type of music.

[RADIO STATIC]

[JAZZ MUSIC]

PROFESSOR: That's it.

[JAZZ MUSIC]

[RADIO STATIC]

RADIO SPEAKER 3: People always hear things that are said. And maybe this will give them a context to understand that. And then also at the tail--

PROFESSOR: Too complicated for me.

[LAUGHTER]

PROFESSOR: Any one of you want to sing? Come on. Be brave. Anyone want to sing-- yeah, come on. Give it a try. It's nice to be here, believe me. You will see all these students. OK. It's your decision.

PROFESSOR: All right. So you see how you can do a Fourier analysis of sound and then by decompose the signal, in terms of its Fourier components. Neutron stars were discovered in 1967 by Jocelyn Bell. Jocelyn was a graduate student in Cambridge, England at the time, and her supervisor, Anthony Hewish, had built a new type of radio telescope. And she was in charge of analyzing the data. And she was sure that she had received a periodic signal which came almost every 1.3 seconds.

And they realized that that could have been the discovery of the century because they thought that they were receiving signals from intelligent life elsewhere in the universe. And so they called it the little green man. And then a few months later, Jocelyn discovered another one. And so they called the first one little green man 1 and then little green man 2, LGM 1 and LGM 2. And then when they discovered a third one, they abandoned the idea of LGM's. We now call these pulsars and we know that the period is the spin period of the neutron stars.

In 1974, Anthony Hewish received the Nobel Prize for this discovery. It is a shame. And it is scandalous that Jocelyn did not share in the Nobel Prize because she actually made the discovery. I've known her very well. I've discussed it with her many times. She was a graduate student. Maybe that was the reason why. The Nobel Committee didn't think it was appropriate to give a Nobel Prize to a graduate student. Ridiculous, but perhaps true. She was a woman. And there is, perhaps, a very sad case of sex discrimination again. We will never know, but she did not share in the Nobel Prize, which she should have.

Neutron stars have a mass about 1 1/2 times that of the sun. They are 100,000 times smaller. They are only 20 kilometers across. So to have a density which is 10 to the 15 times higher than the sun. 10 to the 15 times higher than water, which is higher than nuclear density. And I had some email contact with Jocelyn just a few days ago. She is now in Oxford. And I said I really would like to show the class your picture. And so she showed me-- she sent me a picture which I'm going to show you.

[LAUGHTER]

PROFESSOR: Are we still connected? Where is Jocelyn? Is Jocelyn hiding? No, there she is. There she is. She sent me a picture at the time that she made the discovery. You see the radio telescope here. And you see her standing there, very modestly. She's a very modest woman. And she made one of the most important discoveries of the past century. And did not share in the Nobel Prize.

We now know of hundreds of pulsars in the sky. And so the question now is, how do you find these pulsars? You can observe the sky and radio waves, with radio telescopes. You can do it also with x-ray observatories. And what you do now is you take the data-- just like we took the data of our sound signal-- and you perform a Fourier transform. You ask the data, what are the Fourier components that are hidden-- that I cannot see, but that are hidden.

And in 1998, Rudy Wijnands in Amsterdam, and independently, Ed Morgan here at MIT, were analyzing x-ray data from a known x-ray source in our galaxy. And they were performing a Fourier analysis, which is standard nowadays in astronomy. And they discovered that the neutron star was rotating with a spin period of 2 1/2 milliseconds. They noticed, in the Fourier power spectrum, a huge spike at 401 hertz. It means that at the equator of the neutron star, the speed going around is about 1/10 of the speed of light.

And I asked Rudy, who worked with me for several years here at MIT, to send me some of the data that he obtained from which he could then finally derive that we were dealing with a neutron star rotating around in 2 1/2 milliseconds. And so he said, "Well, Walter, why don't you show the class only 1/5 of a second of my data." 1/5 of a second that you see here-- 2/10 per second. So this is the time scale. And when one x-ray arrives, you see a vertical bar there. And, at this time, they were so close together that you see there two x-rays.

If you count the total number of x-rays in that 1/5 of the second, you'll observe 33 x-rays. During that time, the neutron star was rotating 80 times around already because it rotates every 2 1/2 milliseconds. So when you look at this data, you have no idea that here is an underlying neutron star with a period of 2 1/2 milliseconds with a frequency of 401 hertz. But now you take 3000 seconds of data, and if you have taken 8.03, you know how to perform a fast Fourier transform. Those programs, by the way, are readily available on the market.

And here you see a power density spectrum-- vertically it's the sum of A squared plus B squared. And horizontally, it's the frequency in hertz. And look what you see. At 401 hertz, you see a huge spike. And that is the underlying 2 1/2 millisecond neutron star. This is the frontier of astrophysics. Nowadays, you cannot even think of astronomy or astrophysics without Fourier analysis. It has an enormously important impact on our research. My graduate students and my post-docs perform Fourier transforms every day. And so what we have discussed today is not just intellectually interesting. It is at the forefront of research. See you Thursday.