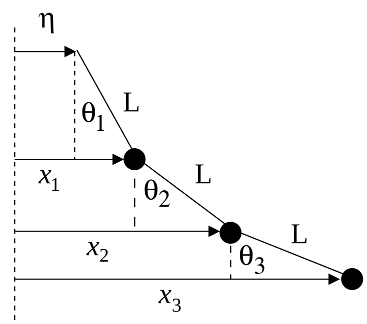
MIT 8.03 Fall 2005 — Analysis of the Driven Triple Pendulum

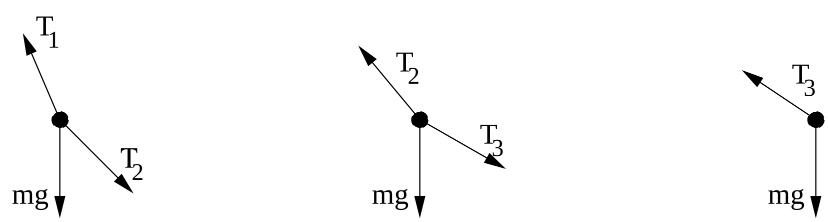
We wish to study the behavior of three pendulums, each of mass m and length L, in the configuration shown below under the influence of a variable displacement $\eta = \eta_0 \cos(\omega t)$ at the top end.

We first find the normal mode frequencies by considering the case in which the system is un-driven, i.e. $\eta(t) = 0$. We then proceed to find the amplitudes of oscillations of the pendulums as a function of the driving frequency ω.

We begin by setting a convenient coordinate system and labeling the relevant variables.



The figure below shows the forces acting on the individual pendula.



For small angles, $T_1 \approx 3mg$, $T_2 \approx 2mg$ and $T_3 \approx mg$. If all three angles are zero and the system is at rest then this follows immediately. The acceleration in the y-direction is negligible for small angles. Note that $\sin\theta_1 = (x_1 - \eta)/L$, $\sin\theta_2 = (x_2 - x_1)/L$ and $\sin\theta_3 = (x_3 - x_2)/L$. Then, the equations of motion for the pendulums (for small oscillations) are

$$\begin{aligned}
m\ddot{x}_1 &= T_2 \sin\theta_2 - T_1 \sin\theta_1 &\approx& \; 2mg(x_2 - x_1)/L - 3mg(x_1 - \eta)/L \\
m\ddot{x}_2 &= T_3 \sin\theta_3 - T_2 \sin\theta_2 &\approx& \; mg(x_3 - x_2)/L - 2mg(x_2 - x_1)/L \\
m\ddot{x}_3 &= -T_3 \sin\theta_3 &\approx& \; -mg(x_3 - x_2)/L,
\end{aligned}$$

where we have made the small angle approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. Rewriting the equations of motion in terms of $\omega_0 = \sqrt{g/L}$ and rearranging terms,

$$\begin{aligned}
\ddot{x}_1 + \omega_0^2(5x_1 - 2x_2) &= 3\omega_0^2\eta \\
\ddot{x}_2 + \omega_0^2(-2x_1 + 3x_2 - x_3) &= 0 \\
\ddot{x}_3 + \omega_0^2(x_3 - x_2) &= 0.
\end{aligned}$$

We use the (normal mode) anzatz $x_i = C_i \cos\omega t$, where $i = \{1, 2, 3\}$. Note that $\ddot{x}_i = -\omega^2 x_i$. Then, the set of coupled differential equations (x_i unknown) becomes a system of linear algebraic equations (C_i unknown):

$$\begin{aligned}
-C_1\omega^2 + 5C_1\omega_0^2 - 2\omega_0^2 C_2 &= 3\omega_0^2\eta_0 \\
-C_2\omega^2 - 2C_1\omega_0^2 + 3\omega_0^2 C_2 - \omega_0^2 C_3 &= 0 \\
-C_3\omega^2 - \omega_0^2 C_2 + \omega_0^2 C_3 &= 0.
\end{aligned}$$

We can write these equations in the compact form $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is the matrix of coefficients, $\mathbf{x}$ is the vector of amplitudes (unknowns) and $\mathbf{b}$ is the source vector. In matrix form, this means

$$\begin{bmatrix} 5\omega_0^2 - \omega^2 & -2\omega_0^2 & 0 \\ -2\omega_0^2 & 3\omega_0^2 - \omega^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & \omega_0^2 - \omega^2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 3\omega_0^2\eta_0 \\ 0 \\ 0 \end{bmatrix}.$$

We now wish to find the normal mode frequencies in the source-free case, i.e. $\mathbf{Ax} = 0$. Thus,

$$\det \mathbf{A} = 6\omega_0^6 - 18\omega_0^4\omega^2 + 9\omega_0^2\omega^4 - \omega^6 = 0.$$

I, Igor Sylvester, solved this equation using MAPLE. The exact solutions are too complicated to present here. Instead, I give the following approximations for the normal mode frequencies:

$$\begin{aligned} \omega &= \{\omega_1, \omega_2, \omega_3\} \\ &\approx \{0.6448\omega_0, 1.5147\omega_0, 2.508\omega_0\}. \end{aligned}$$

You may ask what happened to the other three roots. Since the determinant is a polynomial of degree six, it should have six roots. In fact, the complete solution to $\det \mathbf{A} = 0$ is given by $\omega^2 = \{\omega_1^2, \omega_2^2, \omega_3^2\}$. Since negative frequencies do not add new physics, we use only the positive values of ω. Since we know the roots of $\det \mathbf{A}$, we can write it in the convenient form:

$$\det \mathbf{A} = (\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)(\omega_3^2 - \omega^2).$$

The relative amplitudes of oscillation at the normal modes are:

$$\frac{C_2}{C_1} = \frac{2\omega_0^2\left(\omega_0^2 - \omega^2\right)}{2\omega_0^4 - 4\omega_0^2\omega^2 + \omega^4} \qquad \frac{C_3}{C_1} = \frac{2\omega_0^4}{2\omega_0^4 - 4\omega_0^2\omega^2 + \omega^4}.$$

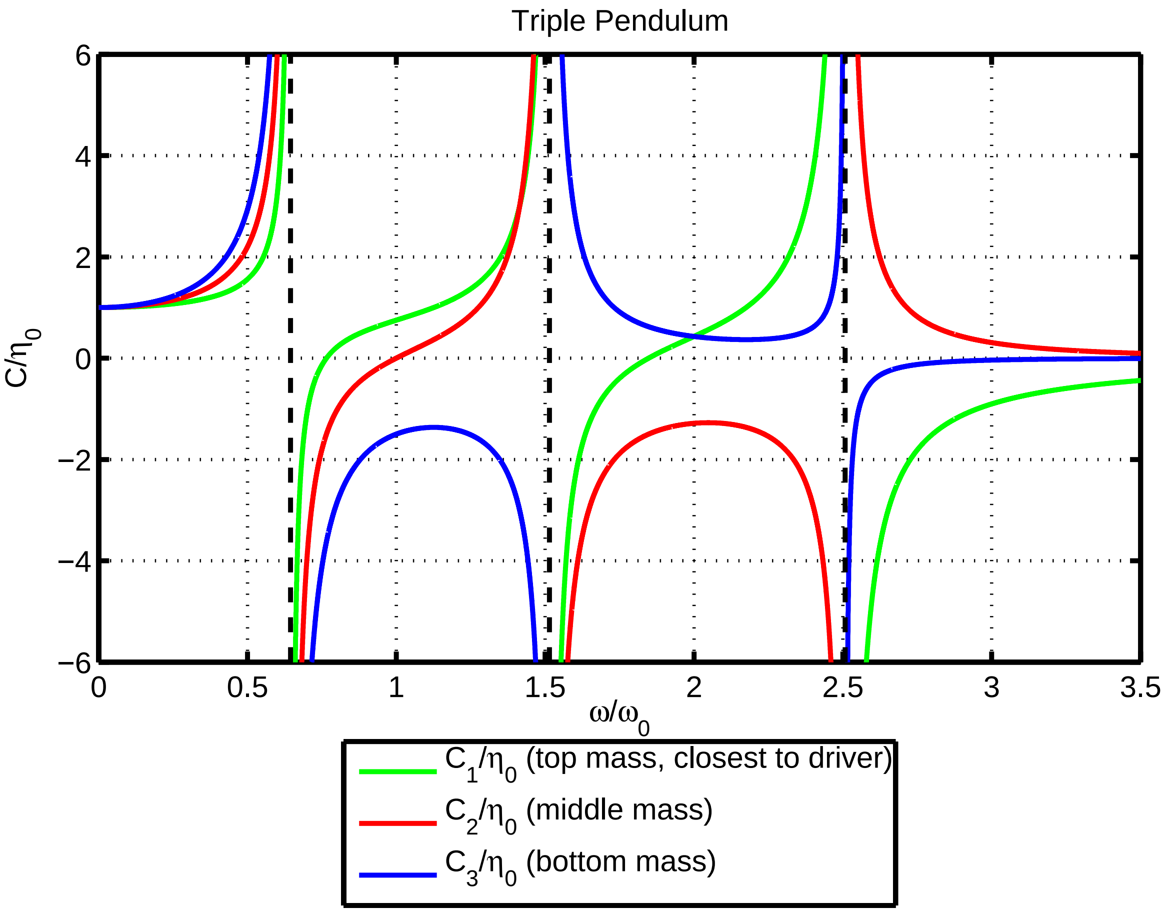
triple.tar triple.tar The approximate values are

$$\begin{aligned} \left.\frac{C_2}{C_1}\right|_{\omega=\omega_1} &= 2.29 & \left.\frac{C_2}{C_1}\right|_{\omega=\omega_2} &= 1.35 & \left.\frac{C_2}{C_1}\right|_{\omega=\omega_3} &= -0.65 \\ \left.\frac{C_3}{C_1}\right|_{\omega=\omega_1} &= 3.92 & \left.\frac{C_3}{C_1}\right|_{\omega=\omega_2} &= -1.05 & \left.\frac{C_3}{C_1}\right|_{\omega=\omega_3} &= 0.12. \end{aligned}$$

We now re-consider the forced case, i.e. $\mathbf{Ax} = \mathbf{b}$ and solve for the amplitudes C_i using Cramer's rule:

$$\begin{aligned} C_1 &= \frac{\begin{vmatrix} 3\omega_0^2\eta_0 & -2\omega_0^2 & 0 \\ 0 & 3\omega_0^2 - \omega^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & \omega_0^2 - \omega^2 \end{vmatrix}}{\det \mathbf{A}} = \eta_0 \frac{3\omega_0^2\left(2\omega_0^4 - 4\omega_0^2\omega^2 + \omega^4\right)}{\left(\omega_1^2 - \omega^2\right)\left(\omega_2^2 - \omega^2\right)\left(\omega_3^2 - \omega^2\right)} \\ C_2 &= \frac{\begin{vmatrix} 5\omega_0^2 - \omega^2 & 3\omega_0^2\eta_0 & 0 \\ -2\omega_0^2 & 0 & -\omega_0^2 \\ 0 & 0 & \omega_0^2 - \omega^2, \end{vmatrix}}{\det \mathbf{A}} = \eta_0 \frac{-6\omega_0^4(\omega - \omega_0)(\omega + \omega_0)}{\left(\omega_1^2 - \omega^2\right)\left(\omega_2^2 - \omega^2\right)\left(\omega_3^2 - \omega^2\right)} \\ C_3 &= \frac{\begin{vmatrix} 5\omega_0^2 - \omega^2 & -2\omega_0^2 & 3\omega_0^2\eta_0 \\ -2\omega_0^2 & 3\omega_0^2 - \omega^2 & 0 \\ 0 & -\omega_0^2 & 0 \end{vmatrix}}{\det \mathbf{A}} = \eta_0 \frac{6\omega_0^6}{\left(\omega_1^2 - \omega^2\right)\left(\omega_2^2 - \omega^2\right)\left(\omega_3^2 - \omega^2\right)}. \end{aligned}$$

A graph of the amplitudes is shown on the next page.



It is interesting to note that the first mass is stationary, i.e. $C_1(\omega) = 0$, if

$$\begin{aligned} \omega &= \left\{\sqrt{2+\sqrt{2}}, \sqrt{2-\sqrt{2}}\right\}\omega_0 \qquad \text{(only real solutions)} \\ &\approx \{1.849, 0.764\}\,\omega_0. \end{aligned}$$

Even more interesting is that the second mass is stationary if $\omega = \omega_0$. If you take the triple pendulum and drive it at a frequency equal to that of a simple pendulum of mass m and length L then the second mass will not move! You SHOULD ask yourself the question:
How on Earth can the two masses (2 and 3) move if the upper mass (1) does not move at all? The answer is simple: **it is not possible!** It is only possible in our dream-world of zero damping. In the presence of damping, no matter how little, the peculiar state is unstable. You will be able to go through that “special” state by varying omega, but you cannot “stop” there.

I strongly recommend that you read Professor Lewin’s own words at the end of the solutions to Problem 3.5.