Massachusetts Institute of Technology

Physics 8.03

Exam 2 Solutions

Tuesday, November 23, 2004

Solution for Problem 1

(a) Boundary condition is $E_y = 0$ at $x = 0, L$

$$\vec{E}(x,z,t) = E_0 \sin(k_x x)\cos(\omega t + k_z z)\hat{y} \tag{1}$$

for the n^{th} mode, $k_x = \frac{n\pi}{L}$, where $n = 1, 2, 3\ldots$

(b) Faraday's law $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\begin{aligned}\vec{\nabla} \times \vec{E} &= \frac{\partial E_y}{\partial x}\hat{z} - \frac{\partial E_y}{\partial z}\hat{x} \\ &= k_x E_0 \cos(k_x x)\cos(\omega t + k_z z)\hat{z} + k_z E_0 \sin(k_x x)\sin(\omega t + k_z z)\hat{x}\end{aligned} \tag{2}$$

$$\Rightarrow \vec{B}(x,z,t) = \frac{k_z}{\omega}E_0 \sin(k_x x)\cos(\omega t + k_z z)\hat{x} - \frac{k_x}{\omega}E_0 \cos(k_x x)\sin(\omega t + k_z z)\hat{z} \tag{3}$$

again for the n^{th} mode, $k_x = \frac{n\pi}{L}$, where $n = 1, 2, 3\ldots$

(c) $\omega^2 = c^2(k_x^2 + k_z^2) \quad \Rightarrow \quad \omega = c\sqrt{(\frac{n\pi}{L})^2 + k_z^2}$

$$k_z = \left[\frac{\omega^2}{c^2} - \left(\frac{n\pi}{L}\right)^2\right]^{1/2} = \frac{1}{c}\left[\omega^2 - \left(\frac{n\pi c}{L}\right)^2\right]^{1/2} \tag{4}$$

Phase velocity

$$v_{p_z} = \frac{\omega}{k_z} = \frac{kc}{k_z} = \frac{\omega c}{\left[\omega^2 - \left(\frac{n\pi c}{L}\right)^2\right]^{1/2}} \tag{5}$$

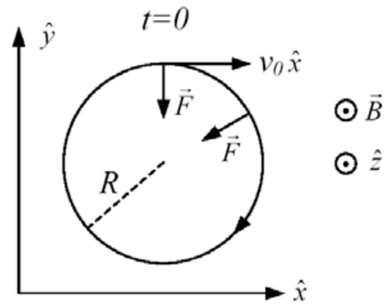
Group velocity

$$v_{gr_z} = \frac{d\omega}{dk_z} = \frac{k_z c}{k} = k_z\frac{c^2}{\omega} = \frac{c}{\omega}\left[\omega^2 - \left(\frac{n\pi c}{L}\right)^2\right]^{1/2} \tag{6}$$

(d) $n = 1, \quad k_z = 0, \quad \omega = \frac{\pi c}{L}$

(e) Now $\omega = k_z c \quad \Rightarrow \quad v_{p_z} = c$ and $v_{gr_z} = c$.
There is No dispersion!
There is No cut-off frequency!

Solution to Problem 2



(a) The force applied to this charged particle is

$$\vec{F} = q(\vec{v} \times \vec{B}) = m\vec{a} \tag{7}$$

which equals the centripetal force for the circular motion, that is,

$$m|\vec{a}| = m\frac{v_0^2}{R} \tag{8}$$

At $t = 0$, $\vec{v} \times \vec{B} = v_0 B_0(-\hat{y})$, therefore

$$qv_0B_0 = m\frac{v_0^2}{R} \quad \Rightarrow \quad R = \frac{mv_0}{qB_0} \tag{9}$$

(b) $v_0 = \omega R \quad \Rightarrow \quad \omega = \frac{qB_0}{m}$

(c) The acceleration of the particle

$$\begin{aligned} \vec{a}(t) &= \frac{v_0^2}{R}\Big[-\sin(\omega t)\hat{x} - \cos(\omega t)\hat{y}\Big] \\ &= -\frac{v_0 q B_0}{m}\Big[\sin(\omega t)\hat{x} + \cos(\omega t)\hat{y}\Big] \end{aligned} \tag{10}$$

(d) Observer at $+r_0\hat{z}$, $t' = t - \frac{r_0}{c}$

$$\vec{a}_\perp = -\frac{v_0 q B_0}{m}\Big[\sin(\omega t')\hat{x} + \cos(\omega t')\hat{y}\Big] \tag{11}$$

Thus

$$\vec{E}(t) = \frac{q^2 v_0 B_0}{4\pi\epsilon_0 c^2 m r_0}\Big[\sin(\omega t')\hat{x} + \cos(\omega t')\hat{y}\Big] \tag{12}$$

It is circularly polarized radiation.

(e) The radiation is elliptically polarized.

Solution to Problem 3

(a) At $t = 0$, all energy is potential.
Potential energy density (see French 7-32)

$$\frac{dU}{dx} \cong \frac{1}{2}T\Big(\frac{\partial y}{\partial x}\Big)^2 \tag{13}$$

Energy conservation gives

$$E_{total} = U(t=0) = \frac{1}{2}T\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx \tag{14}$$

where

$$\left(\frac{\partial y}{\partial x}\right)^2 = \Big[\frac{4\pi}{L}\cos\left(\frac{2\pi x}{L}\right) + \frac{3\pi}{L}\cos\left(\frac{\pi x}{L}\right)\Big]^2 \tag{15}$$

The cross term in $\left(\frac{\partial y}{\partial x}\right)^2$ will NOT contribute to the integral $\int_0^L$, thus

$$\begin{aligned} E_{total} &= \frac{1}{2}T\frac{\pi^2}{L^2}\Big[16\int_0^L \cos^2\left(\frac{2\pi x}{L}\right)dx + 9\int_0^L \cos^2\left(\frac{\pi x}{L}\right)dx\Big] \\ &= \frac{25T\pi^2}{4L} \end{aligned} \tag{16}$$

(b) The displacement at time t is obtained as following:

$$\lambda = \frac{v2\pi}{\omega}, \quad \omega = \frac{2\pi v}{\lambda}, \quad v = \sqrt{\frac{T}{\mu}} \tag{17}$$

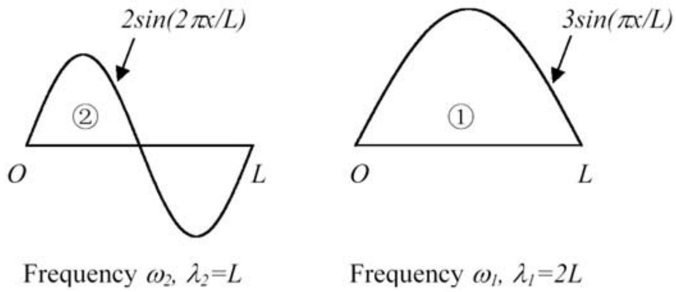
$$\omega_1 = \frac{2\pi}{2L}v = \frac{\pi}{L}v, \quad \omega_2 = 2\omega_1 = \frac{2\pi}{L}v \tag{18}$$

at $t = 0$, the string stands still, thus

$$y(x,t) = 3\sin\left(\frac{\pi x}{L}\right)\cos(\omega_1 t) + 2\sin\left(\frac{2\pi x}{L}\right)\cos(\omega_2 t) \tag{19}$$

2 standing waves.

(c) After a time $\frac{2\pi}{\omega_1}$, the shape (1) has made one complete oscillation. In that same time, shape (2) has made two complete oscillations. Thus, after $\frac{2\pi}{\omega_1} = \frac{2\pi L}{\pi v} = \frac{2L}{v}$ seconds, the shape will be the same as at time $t = 0$.



Solution to Problem 4

(a) Reflection at normal incidence:

$$\frac{E_r}{E_i} = \frac{n_1 - n_2}{n_1 + n_2} = -0.2 \tag{20}$$

Thus 4% will be reflected. There is no difference between the $\perp$ and the $\|$ components. The ratio of light intensity (W/m^2) is $\left(\frac{E_r}{E_i}\right)^2 = 0.04 = 4\%$.

(b) The reflected and the transmitted light is still circularly polarized as r and t are the same for the $\perp$ and $\|$ components at normal incidence.

(c) Light intensity is the product of the Poynting vector and the cross-sectional area of the light beam. As the light enters the prism, the cross-sectional area is the same as that of the incident beam because $\theta_1 = \theta_2 = 0$.

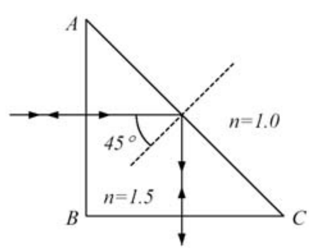
$$< S >= \frac{< \vec{E} \times \vec{B} >}{\mu_0}, \qquad |B| = \frac{|E|}{v} = \frac{|E|n}{c} \tag{21}$$

$$< S_i >= \frac{< E_i^2 > n_1}{\mu_0 c}, \qquad < S_t >= \frac{< E_t^2 > n_2}{\mu_0 c}, \qquad n_1 = 1, n_2 = 1.5 \tag{22}$$

Since $\frac{E_t}{E_i} = \frac{2n_1}{n_1+n_2} = 0.8$,

$$\frac{< S_t >}{< S_i >} = (0.8)^2 \times 1.5 = 0.96 \tag{23}$$

96% enters!



(d) At the surface AC, 100% of the light will be reflected. The angle of incidence, θ_1, is larger than the critical angle.

$$\theta_1 = 45^\circ \quad \theta_{crit} = 41.8^\circ \quad \sin\theta_{crit} = \frac{n_2}{n_1} = \frac{1.0}{1.5} \tag{24}$$

Remember, 1 is where you are, 2 is where you are going.

(e) The reflection $r = \frac{n_1 - n_2}{n_1 + n_2} = 0.2$, thus 4% will be reflected.

This is 4% of the 96% of I_0.
Thus 96% (of 96%) will emerge in air.