

Massachusetts Institute of Technology

Physics 8.03

Exam 2

Tuesday, November 23, 2004

- You have 85 minutes
- There are FOUR problems
- You may use calculators
- This is a closed-book exam; no notes are allowed.

Useful Formulae

General differential equation for oscillators

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

has solutions

$$\begin{aligned} x(t) &= A e^{-\frac{\gamma t}{2}} \cos\left(\sqrt{\omega_0^2 - \frac{\gamma^2}{4}} t + \alpha\right) + x_{ss}(t) & \omega_0 > \frac{\gamma}{2} \\ x(t) &= (A + B t) e^{-\frac{\gamma t}{2}} + x_{ss}(t) & \omega_0 = \frac{\gamma}{2} \\ x(t) &= A e^{-\Gamma_1 t} + B e^{-\Gamma_2 t} + x_{ss}(t) & \omega_0 < \frac{\gamma}{2} \end{aligned}$$

where

$$\Gamma_{1,2} = \frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

and the steady-state solution is

$$x_{ss}(t) = A(\omega) \cos(\omega t - \delta(\omega))$$

$$A(\omega) = \frac{f_0}{[(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2]^{1/2}} \quad \tan \delta(\omega) = \frac{\gamma \omega}{\omega_0^2 - \omega^2}$$

Non-dispersive wave equation

$$\frac{\partial^2}{\partial x^2} y(x, t) = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} y(x, t)$$

where $v = \sqrt{T/\mu}$ for a string or $v = \sqrt{\kappa/\rho}$ for a gas.

Kinetic, potential energy and power

$$\frac{dK}{dx} = \frac{1}{2} \mu \left(\frac{\partial y}{\partial t} \right)^2 \quad \frac{dU}{dx} = \frac{1}{2} T \left(\frac{\partial y}{\partial x} \right)^2 \quad P(t) = -T \left(\frac{\partial y}{\partial t} \right) \left(\frac{\partial y}{\partial x} \right)$$

Reflection and transmission coefficients

$$R = \frac{v_2 - v_1}{v_2 + v_1} \quad T = \frac{2v_2}{v_2 + v_1}$$

Fourier series for a function $f(\theta) = f(\theta + 2\pi)$

$$f(\theta) = \sum_{m=1}^{\infty} \left[\frac{A_0}{2} + A_m \cos(m\theta) + B_m \sin(m\theta) \right]$$

$$A_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(m\theta) d\theta \quad m = 0, 1, 2, \dots$$

$$B_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(m\theta) d\theta \quad m = 1, 2, 3, \dots$$

Dispersion

$$v_{phase} = \frac{\omega}{k} \quad \text{and} \quad v_{group} = \frac{d\omega}{dk}$$

Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$

EM force, flux, energy, intensity

$$\vec{F} = q \left(\vec{E} + \vec{v} \times \vec{B} \right) \quad \vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

$$U_E = \frac{\epsilon_0}{2} |\vec{E}|^2 \quad U_M = \frac{1}{2\mu_0} |\vec{B}|^2$$

Dipole approximation

$$\vec{E}_{rad}(\vec{r}, t) = \frac{-q\vec{a}_{\perp}(t - r/c)}{4\pi\epsilon_0 c^2 r} \quad \text{Volt/m}$$

$$\vec{B}_{rad}(\vec{r}, t) = \frac{1}{c} \hat{r} \times \vec{E}_{rad}(t) \quad \text{Tesla}$$

$$\vec{S}_{rad}(\vec{r}, t) = \frac{1}{\mu_0} \vec{E}_{rad} \times \vec{B}_{rad} \quad \text{Watt/m}^2$$

$$P = \frac{q^2 a^2}{6\pi\epsilon_0 c^3} \quad \text{Watt}$$

Boundary conditions at the surface of a perfect conductor

$$\begin{aligned} E_{//} &= 0 & |B_{//}| &= \mu_0 |J_S| \\ E_{\perp} &= \frac{\rho_S}{\epsilon_0} & B_{\perp} &= 0 \end{aligned}$$

Transmission lines

$$\begin{aligned} \frac{\partial V}{\partial z} &= -L_0 \frac{\partial I}{\partial t} & v_p &= \frac{1}{\sqrt{L_0 C_0}} & \frac{V_r}{V_i} &= \frac{Z_L - Z_0}{Z_L + Z_0} \\ \frac{\partial I}{\partial z} &= -C_0 \frac{\partial V}{\partial t} & Z_0 &= \sqrt{\frac{L_0}{C_0}} & \frac{I_r}{I_i} &= \frac{Z_0 - Z_L}{Z_L + Z_0} \end{aligned}$$

Boundary conditions at the surface of a perfect dielectric

$$\begin{aligned} E_{//}^{(1)} &= E_{//}^{(2)} & \frac{B_{//}^{(1)}}{\mu_1} &= \frac{B_{//}^{(2)}}{\mu_2} \\ \kappa_{e1} E_{\perp}^{(1)} - \kappa_{e2} E_{\perp}^{(2)} &= \frac{\rho_s}{\epsilon_0} & B_{\perp}^{(1)} &= B_{\perp}^{(2)} \end{aligned}$$

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

Fresnel equations

$$\begin{aligned} r_{//} &= E_{0r//}/E_{0i//} = \frac{n_1 \cos \theta_2 - n_2 \cos \theta_1}{n_1 \cos \theta_2 + n_2 \cos \theta_1} = -\frac{\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)} \\ r_{\perp} &= E_{0r\perp}/E_{0i\perp} = \frac{n_1 \cos \theta_1 - n_2 \cos \theta_2}{n_1 \cos \theta_1 + n_2 \cos \theta_2} = -\frac{\sin(\theta_1 - \theta_2)}{\sin(\theta_1 + \theta_2)} \\ t_{//} &= E_{0t//}/E_{0i//} = \frac{2n_1 \cos \theta_1}{n_1 \cos \theta_2 + n_2 \cos \theta_1} = \frac{2 \sin \theta_2 \cos \theta_1}{\sin(\theta_1 + \theta_2) \cos(\theta_1 - \theta_2)} \\ t_{\perp} &= E_{0t\perp}/E_{0i\perp} = \frac{2n_1 \cos \theta_1}{n_1 \cos \theta_1 + n_2 \cos \theta_2} = \frac{2 \sin \theta_2 \cos \theta_1}{\sin(\theta_1 + \theta_2)} \end{aligned}$$

Special case of normal incidence ($\theta_1 = \theta_2 = 0$)

$$r_{//,\perp} = \frac{E_r}{E_i} = \frac{n_1 - n_2}{n_1 + n_2} \quad t_{//,\perp} = \frac{E_t}{E_i} = \frac{2n_1}{n_1 + n_2}$$

Doppler Effect

$$\begin{aligned} \frac{\lambda'}{\lambda} &= \frac{1 - \beta \cos \theta}{\sqrt{1 - \beta^2}} & & \text{for EM waves} \\ \frac{f'}{f} &= \frac{v_s + v_r \cos \theta_r}{v_s - v_t \cos \theta_t} & & \text{for sound waves} \end{aligned}$$

Physical constants

Speed of light	c	3×10^8	m s^{-1}
Vacuum permeability	μ_0	1.26×10^{-6}	$(\text{V m}^{-1})/\text{A}$
Vacuum permittivity	ϵ_0	8.85×10^{-12}	$\text{C}/(\text{V m}^{-1})$
Electron rest mass	m	9.1×10^{-31}	kg
Elementary charge	e	1.6×10^{-19}	C
Gravitational constant	G	6.7×10^{-11}	$\text{N m}^2/\text{kg}^2$

Trigonometric Formulae

$$\begin{aligned}\sin(a+b) &= \sin a \cos b + \cos a \sin b \\ \cos(a+b) &= \cos a \cos b - \sin a \sin b \\ \sin a + \sin b &= 2 \sin\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right) \\ \sin a - \sin b &= 2 \cos\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right) \\ \cos a + \cos b &= 2 \cos\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right) \\ \cos a - \cos b &= -2 \sin\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right)\end{aligned}$$

$$\begin{aligned}\sin\left(\theta \pm \frac{\pi}{2}\right) &= \pm \cos \theta \\ \cos\left(\theta \pm \frac{\pi}{2}\right) &= \mp \sin \theta \\ \sin(\theta \pm \pi) &= \mp \sin \theta \\ \cos(\theta \pm \pi) &= -\cos \theta\end{aligned}$$

Complex exponentials

$$e^{j\theta} = \cos \theta + j \sin \theta \quad \sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad \cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Problem 1 (25 pts): Boundary conditions at conductors

Two nearly infinite, perfectly conducting planes are placed at $x = 0$ and $x = L$. The electric field of a plane wave, linearly polarized in the y -direction with angular frequency ω and amplitude E_0 , is propagating in the negative z -direction between the two planes. Since the conducting planes are very large in the y and z directions, assume that the \vec{E} and \vec{B} fields do not depend on y .

- a. (6 pts) Using boundary conditions for the electric field on the conducting planes, write an expression for $\vec{E}(x, y, z, t)$ in the n -th mode.
- b. (6 pts) Find the corresponding magnetic field. Give its magnitude and direction.
- c. (6 pts) What is the phase velocity in the z -direction (in the n -th mode) as a function of ω , and what is the group velocity in the n -th mode? Express your answers only in terms of n, L, c , and ω .
- d. (3 pts) What is the lowest angular frequency ω for which this wave will propagate in the z -direction?
- e. (4 pts) If the radiation were linearly polarized in the x -direction, what then would be the phase velocity and what would be the group velocity in the z -direction?

Problem 2 (25 pts): Cyclotron radiation from an accelerating charge

A positively charged particle with charge q and mass m , and with velocity $\vec{v} = v_0 \hat{x}$, is injected at time $t = 0$ into a region of the x - y plane where there is a uniform magnetic field $\vec{B} = B_0 \hat{z}$. Assume that $v \ll c$ and that any observer is far away. Consult the formula sheet for values of physical constants you may need.

- a. (6 pts) Find an expression for the radius R of the circular trajectory of the particle in terms of m , q , v_0 , and B_0 .
- b. (4 pts) What is the angular frequency ω of the radiation?
- c. (5 pts) What is the acceleration of the particle at time t ? Give the magnitude and direction.
- d. (6 pts) Give the magnitude and direction of the electric field \vec{E} detected by a distant observer at $+r_0 \hat{z}$. What is the nature of the polarization? By “nature” we mean is it polarized at all, and if so, is it linearly polarized or circularly or elliptically?
- e. (4 pts) What would be the nature of the polarization of the radiation detected by a distant observer at $+r_0(\hat{y} + \hat{z})$. We are not asking what the magnitude of the E-field is. We only want to know what the nature of the polarization is. Give your reasons; a simple sketch always helps!

Problem 3 (25 pts): Oscillating String

A string with tension T and mass per unit length μ is clamped down at $x = 0$ and at $x = L$. At $t = 0$, the string is at rest and displaced in the y -direction:

$$y(x, 0) = 2 \sin \frac{2\pi x}{L} + 3 \sin \frac{\pi x}{L}$$

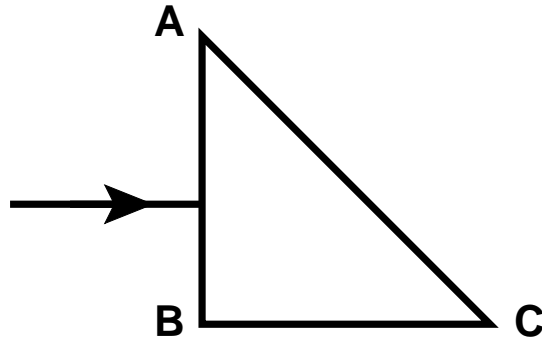
- a. (10 pts) What is the total energy at $t = 0$?

The string is released at $t = 0$, and it starts to oscillate.

- b. (10 pts) What is the displacement at time t ?
- c. (5 pts) At what time t will the string for the first time have exactly the same shape as it did at time $t = 0$? Or will this never happen? Give your reasons.

Problem 4 (25 pts): Glass Prism

Circularly polarized light with intensity I_0 is incident on a glass prism. The prism is completely surrounded on all sides by air. The surfaces indicated by AB, BC, and AC are perpendicular to the plane of the paper. Angle $ABC = 90^\circ$, the other two angles are both 45° . The index of refraction of the glass is 1.5. The light strikes the surface (AB) at right angles (normal incidence).



- (5 pts) What percentage of the light intensity is reflected off the surface AB?
- (5 pts) Is this light still circularly polarized? Give your reasons.
- (5 pts) Show that the sum of the intensities of this reflected light and that of the light that enters the prism is I_0 .

The light that penetrates the glass will now be incident on the surface AC.

- (5 pts) What percentage of this light is reflected and what percentage emerges into the air? Make a clear sketch, and calculate the relevant angles of reflection and refraction.

The reflected light (at surface AC) will now be incident on the surface BC.

- (5 pts) What percentage of this light is reflected and what percentage emerges into the air at the surface BC? Make a clear sketch, and calculate the relevant angles of reflection and refraction.