8.03 Fall 2005
Problem Set 10 Solutions

Solution 10.1: (Bekefi & Barrett 8.1) Thin film interference

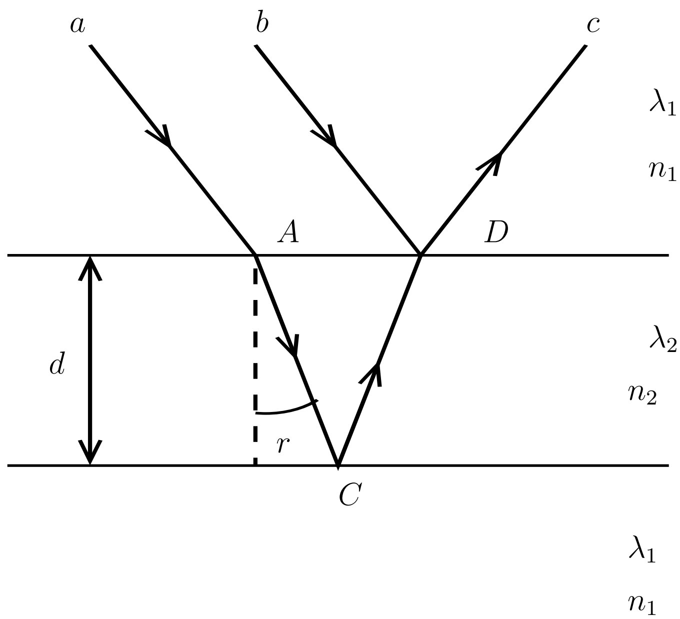

FIG. 1: Problem 10.1 Thin film interference

The glass is too thick to produce "thin film interference". We will therefore only concentrate on the air gap. The phase difference between a and b after 'joining' in c is:

$$\delta = \frac{4\pi d}{\lambda_1}\frac{n_2}{n_1}\cos r + \pi \tag{1}$$

For constructive interference the condition is that $\delta = 2m\pi \quad (m = 1, 2, ...)$. In case of normal incidence ($\cos r = 1$) the equation can be modified to:

$$\lambda_{1m} = \frac{4dn_2}{(2m-1)n_1} \qquad\qquad \lambda_2 = \frac{n_1}{n_2}\lambda_1$$

$$\begin{aligned}
\Rightarrow \qquad \lambda_2 &= 4d && for\ m = 1 \\
d &= \frac{\lambda_2}{4} = 1 \times 10^{-7} m && where\ \lambda_2 = 4 \times 10^{-7} m \\
&= 100\ nm
\end{aligned}$$

This answer could have been 'guessed' without this elaborate calculation. If $d = \lambda_2/4$ the extra distance traveled is $\lambda_2/2$. In addition, there will be a π phase difference! At D light reflects off a less dense medium (namely air) ($n_2 < n_1$) so there is no sign change in $\vec{E}$. However, at C, there is a reflection off a denser medium (namely glass), thus the phase of $\vec{E}$ changes by π. Thus the extra distance traveled in the air gap plus the sign change in E add up to a 2π 'change' in phase if $d = \lambda_2/4$.

Solution 10.2: (Bekefi & Barrett 8.4) Newton rings

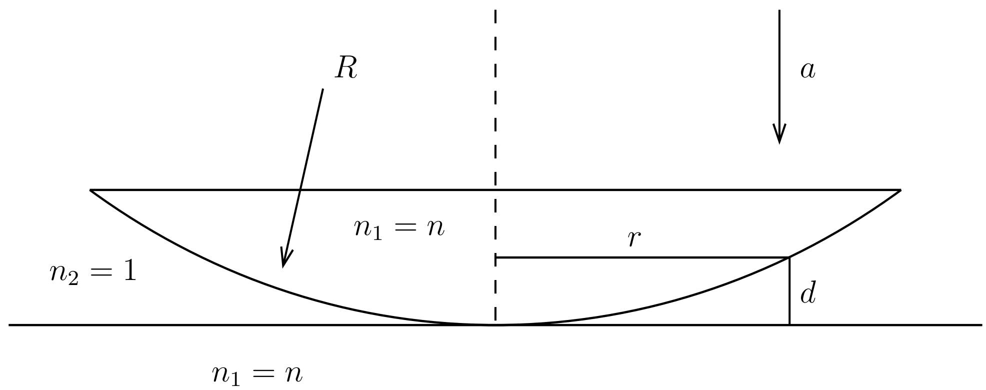

FIG. 2: Problem 10.2 Newton Rings

Part (a)

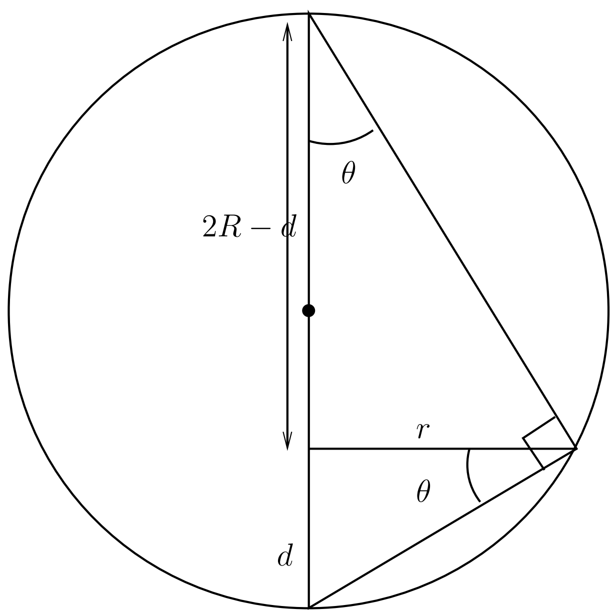

FIG. 3: Problem 10.2(a) Newton Rings

From Fig. 3:

$$\begin{aligned} \sin\theta = \frac{r}{2R-d} &= \frac{d}{r} \\ \Rightarrow \qquad r^2 &= d(2R-d) \\ \Rightarrow \qquad r^2 &\simeq 2dR \qquad\qquad (d^2 \ll dR) \\ d &\simeq \frac{r^2}{2R} \end{aligned} \tag{2}$$

Part (b)

For rays a as shown in Fig. 3, the phase difference between those that travel back and forth in the gap d and those that reflect off the curved surface of the glass is:

$$\delta = \frac{4\pi d}{\lambda_2} + \pi$$

where $n_2 = 1, n_1 = n$ and $\lambda_1 = \lambda_2/n$.

For constructive interference the condition is that $\delta = 2m\pi \quad (m = 1, 2, ...)$.

Thus

$$2m\pi = \pi\left(\frac{4d}{\lambda_2} + 1\right) \tag{3}$$

Combining Eqs. 2 and 3 and replacing λ_2 with λ (it is the wavelength in air):

$$r_m = \left[\frac{(2m-1)\lambda R}{2}\right]^{1/2} \tag{4}$$

a. *Example* For λ=500 nm and R=10 m, some of the ring radii are:

$$r_1 = 1.58\ mm \quad r_2 = \sqrt{3} \times 1.58\ mm \quad r_{13} = \sqrt{25} \times 1.58\ mm \simeq 7.9\ mm$$

The ring spacing decreases with increasing radius r. The ratio of ring radius $(m+1)$ to that of the m^{th} ring is $\sqrt{(2m+1)/(2m-1)}$. For $m = 1$ the ratio is $\simeq 1.7$, for $m = 13$ it is $\simeq 1.04$.

Part (c)

For destructive interference the condition is that $\delta = (2m+1)\pi \quad (m = 1, 2, ...)$.

$$(2m+1)\pi = \pi\left(\frac{4d}{\lambda} + 1\right) \tag{5}$$

Combining Eqs. 2 and 5

$$r_m = [m\lambda R]^{1/2} \tag{6}$$

Substituting values of $R = 2$ m and $\lambda = 640$ nm, the values of r_m are:

$$r_m = 1.13 m^{1/2} \times 10^{-3}\ m \tag{7}$$

Part (d)

$$r_1 = 1.13\ mm \quad r_2 = 1.60\ mm \quad \Delta r_{1,2} = r_2 - r_1 = 0.47\ mm$$

$$r_{25} = 5.66\ mm \quad r_{26} = 5.77\ mm \quad \Delta r_{25,26} = r_{26} - r_{25} = 0.11\ mm \tag{8}$$

where r_i is the i^{th} dark ring.

Solution 10.3: Take-home experiment #6-Thin film interference

Part (a) Microscope slide thin film interference

I (Tarun Agarwal) first viewed a set of clean microscope slides under a filament lamp. The thin film interference patterns were clearly visible when I pressed the two slides together near the edge. The pattern was visible for a few millimeters around my fingers and then disappeared near the center of the slide. I noticed that the minima of the pattern, the dark lines, were further apart when I pressed the slides harder together.

When I performed the same experiment under a fluorescent lamp, the thin film interference pattern was visible all over the slides without any need for pressing the two slides together.

Part (b) Thin sheet film interference

A piece of kitchen wrap was stretched over the mouth of a cup and spread so that its surface was as uniform and kink-free as possible. I saw the reflection of a fluorescent lamp on the thin sheet of plastic film. I was unable to clearly discern interference patterns, indicating thickness contours of the film. However, I did notice that the fluorescent light seemed to be of different shades of color, with regions being red, green or blue. This indicates that some of the colors constructively interfered. It shows us that the film had different thicknesses at different places.

When performing the same experiment with a single microscope cover slip, interference patterns with alternate bright and dark contour lines showed up. The variations in intensity were small and were much more noticeable when viewed using one of the lenses provided in the kit as a magnifier.

Part (c) Newton Rings

I tried to view the Newton rings using the longest focal length lens and putting it on a microscope slide. I then viewed the assembly under a fluorescent lamp. At the point where the lens was resting on the slide there was indeed a dark spot as expected from our calculations. Around this spot very very fine rings were visible especially near the boundary of the central dark spot. Using the other lenses to magnify this region produced only limited results, the rings were still not completely clear.

Solution 10.4: Rainbows

Use the Lecture Notes of Nov 16, 2004 (they are available on the web).

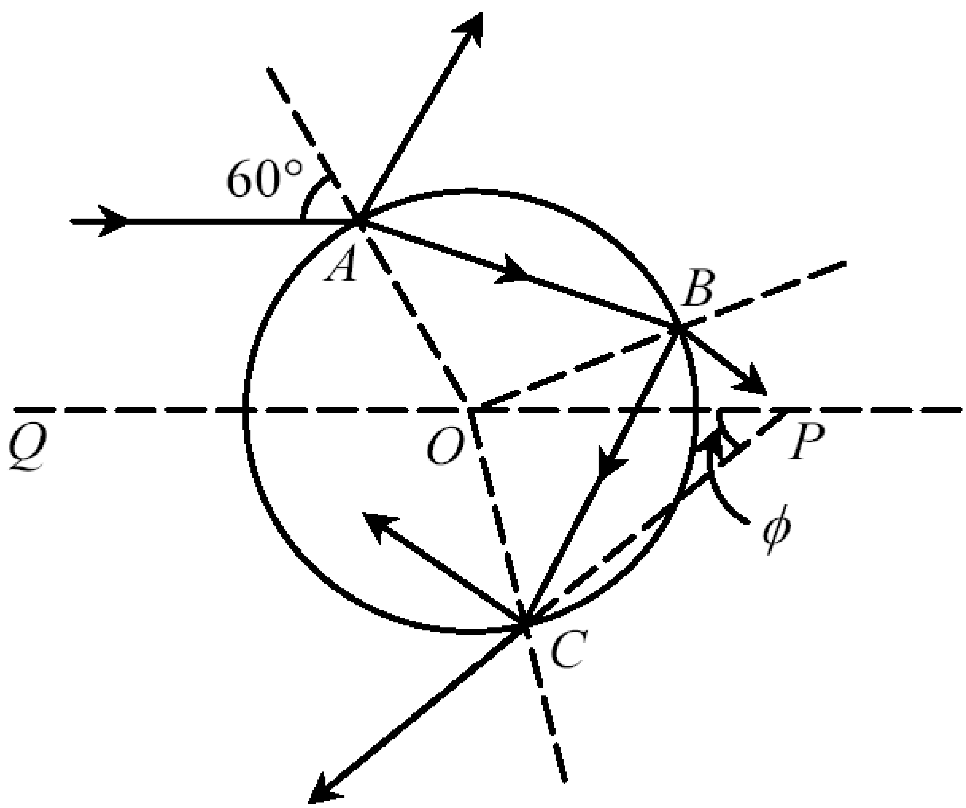

FIG. 4: Problem 10.4

Part (a)

Incident unpolarized light: $\parallel 0.5I_o, \perp 0.5I_o$

$$\begin{aligned}
\theta_1 &= 60.00^o & \theta_2 &= 40.59^o \ Snell's\ Law \\
n_1 &= 1.000 & n_2 &= 1.331 \\
r_\parallel &= 0.06587 & I_{r_\parallel} &= 0.06587^2 \times 0.5I_0 = 0.002170I_0 \\
& & I_{t_\parallel} &= 0.5I_0 - 0.002170I_0 = 0.4978I_0 \\
r_\perp &= -0.3381 & I_{r_\perp} &= 0.3381^2 \times 0.5I_0 = 0.05715I_0 \\
& & I_{t_\perp} &= 0.5I_0 - 0.05715I_0 = 0.4429I_0
\end{aligned}$$

Degree of linear polarization of the transmitted light:

$$V = \left|\frac{I_{t_\parallel} - I_{t_\perp}}{I_{t_\parallel} + I_{t_\perp}}\right| = \left|\frac{0.498 - 0.443}{0.498 + 0.443}\right| = 0.0584 \qquad 5.8\%\ linearly\ polarized \tag{9}$$

The parallel component dominates.

Part (b)

Reflection at B. Incoming radiation: $\perp 0.443I_o$, $\parallel 0.498I_o$

$$\begin{aligned}
\theta_1 = angle\ OAB = angle\ OBA &= 40.59^o & \theta_2 &= 60.00^o \\
n_1 &= 1.331 & n_2 &= 1.000 \\
r_\parallel &= -0.06587 & I_{r_\parallel} &= 0.06587^2 \times 0.498I_0 = 0.00216I_0 \\
r_\perp &= 0.3381 & I_{r_\perp} &= 0.3381^2 \times 0.443I_0 = 0.0506I_0
\end{aligned}$$

Degree of linear polarization of the reflected light:

$$V = \left|\frac{I_{r_\parallel} - I_{r_\perp}}{I_{r_\parallel} + I_{r_\perp}}\right| = \left|\frac{0.00216 - 0.0506}{0.0506 + 0.00216}\right| = 0.918 \qquad 92\%\ linearly\ polarized \tag{10}$$

It should not surprise you that the reflected light at B is so highly polarized. The Brewster angle for the transition water→air is 36.9^o. The angle of incidence, $\theta_1 = 40.6^o$. The angle of incidence is only $\sim 3.7^o$ away from the Brewster angle.

Part (c)

Radiation that arrives at C. Incoming radiation: $\parallel 0.00216I_o$, $\perp 0.0506I_o$

$$\begin{aligned}
\theta_1 &= 40.59^o & \theta_2 &= 60.00^o \\
n_1 &= 1.331 & n_2 &= 1.000 \\
r_\parallel &= -0.06587 & I_{r_\parallel} &= 0.06587^2 \times 0.00216I_0 = 9.37 \times 10^{-6}I_0 \\
& & I_{t_\parallel} &= 0.00216I_0 - 0.00000937I_0 = 0.00215I_0 \\
r_\perp &= 0.3381 & I_{r_\perp} &= 0.3381^2 \times 0.0506I_0 = 0.00578I_0 \\
& & I_{t_\perp} &= 0.0506I_0 - 0.00578I_0 = 0.04482I_0
\end{aligned}$$

Degree of linear polarization of the transmitted light:

$$V = \left|\frac{I_{t_\parallel} - I_{t_\perp}}{I_{t_\parallel} + I_{t_\perp}}\right| = \left|\frac{0.00215 - 0.04482}{0.04482 + 0.00215}\right| = 0.9085 \qquad 91\%\ linearly\ polarized \tag{11}$$

The perpendicular component dominates.

In conclusion: The intensity of light emerging into the air at C is 4.7% ($\perp$ 4.48% , $\|$ 0.22%) of I_0. This light is 91% linearly polarized in the $\perp$ direction.

Part (d)

Angle of incidence is θ_1 and the angle of refraction is θ_2.

$$\begin{aligned} angle\ AOB = angle\ BOC = 180^o - 2\theta_2 \quad &\Rightarrow \quad angle\ AOC = 4\theta_2 \\ angle\ QOC = 4\theta_2 - \theta_1 \quad &\Rightarrow \quad angle\ POC = 180^o - 4\theta_2 + \theta_1 \\ angle\ OCP = \theta_1 \quad &\Rightarrow \quad \phi = 180^o - angle\ POC - angle\ OCP \\ &\qquad = 180^o - 180^o + 4\theta_2 - 2\theta_1 \end{aligned}$$

$$\phi = 4\theta_2 - 2\theta_1$$

Part (e)

Red Light $n = 1.331$ $\theta_1 = 60^o$ $\theta_2 = 40.59^o$
$\phi_{red} = 4\theta_2 - 2\theta_1 = 42.4^o$

Blue/violet Light $n = 1.343$ $\theta_1 = 60^o$ $\theta_2 = 40.15^o$
$\phi_{red} = 4\theta_2 - 2\theta_1 = 40.6^o$

Part (f)

$$\begin{aligned} \phi &= 4\theta_2 - 2\theta_1 \\ \frac{d\phi}{d\theta_1} = 4\frac{d\theta_2}{d\theta_1} - 2 = 0 \quad &\Rightarrow \quad \frac{d\theta_2}{d\theta_1} = \frac{1}{2} \\ \sin\theta_1 &= n\sin\theta_2 \qquad\qquad Snell's\ Law \\ \cos\theta_1 d\theta_1 &= n\cos\theta_2 d\theta_2 \\ \frac{d\theta_2}{d\theta_1} &= \frac{\cos\theta_1}{n\cos\theta_2} = \frac{1}{2} \\ \cos\theta_2 = \frac{2}{n}\cos\theta_1 &= (1-\sin^2\theta_2)^{1/2} = (1 - \frac{1}{n^2}\sin^2\theta_1)^{1/2} \\ &= (1 - \frac{1}{n^2} + \frac{1}{n^2}\cos^2\theta_1)^{1/2} \\ \frac{4}{n^2}\cos^2\theta_1 &= 1 - \frac{1}{n^2} + \frac{1}{n^2}\cos^2\theta_1 \\ \frac{3}{n^2}\cos^2\theta_1 &= 1 - \frac{1}{n^2} = \frac{n^2-1}{n^2} \\ \cos^2\theta_1 &= \frac{n^2-1}{3} \end{aligned}$$

QED

Part (g)

Red Light:
$\cos^2\theta_1 = (1.331^2 - 1)/3 = 0.257$ $\theta_1 = 59.5^o$ $\theta_2 = 40.3^o$
$\phi_{max} = 4\theta_2 - 2\theta_1 = 42.4^o$

Blue/violet Light:
$\cos^2\theta_1 = (1.343^2 - 1)/3 = 0.2678$ $\theta_1 = 58.8^o$ $\theta_2 = 39.6^o$
$\phi_{max} = 4\theta_2 - 2\theta_1 = 40.6^o$

The angular width of the colorful rainbow is therefore about $42.4^o - 40.6^o + 0.5^o = 2.3^o$
The reason why 0.5^o were added is due to the fact that the sun has a diameter of 0.5^o. Thus the width of the rainbow is about 5-6% of its radius

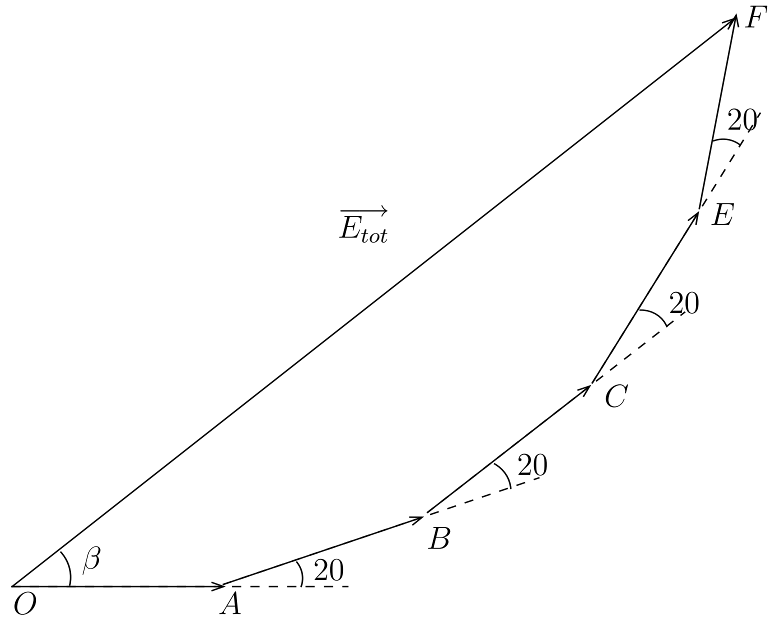

FIG. 5: Problem 10.5(a) Addition of 5 vectors

Part (h) $n = 1.5$ $\theta_1 = \sin^{-1}[1 - (n^2 - 1)/3]^{1/2} = 49.8^o$ $\theta_2 = 30.6^o$
$\phi_{max} = 4\theta_2 - 2\theta_1 = 22.8^o$

Notice: the radius of the glass bow is almost half that of the rainbow! The glass bow is also highly polarized as the Brewster angle (glass to air) is $\sim 33.7^o$; the angle of incidence at the reflection at B is only $\sim 3^o$ smaller.

Solution 10.5: (Bekefi & Barrett 8.5) Superposition of N oscillators

Part (a)
Let the points after addition of each successive phasor be A,B,C,D and F. The points are: A(3,0), B(3+3cos20^o,3sin20^o), C(B$_x$+3cos40^o,B$_y$+3sin40^o), D(C$_x$+3cos60^o,C$_y$+3sin60^o) and F(D$_x$+3cos80^o,D$_y$+3sin80^o).
F($\sim$10.138,$\sim$ 8.506)
The length OF=$(F_x^2+F_y^2)^{1/2} \simeq 13.234$ $\tan\beta = F_y/F_x$ $\Rightarrow$ $\beta \simeq 40.0^o$ $(2\pi/9)$
Thus $E(t) \simeq 13.234\cos(\omega t + 2\pi/9)$
Fig. 5 shows a graphical representation of the addition.

Part (b)
In Fig. 6, let MO=R. Adding the N vectors, we end in Q. All the tips lie on a circle with center at M.
It follows from $\triangle$ MOP: OP/2=Rsin(α/2)
It follows from $\triangle$ MOQ: OQ/2=Rsin($N\alpha$/2)
Eliminate R from both equations to give:

$$OQ = OP\frac{\sin(\frac{1}{2}N\alpha)}{\sin\frac{1}{2}\alpha}$$

OQ is ahead of OP by phase angle QOP. Angle QOP=angle QOT-angle POT=(Nα-α)/2.
If you do not immediately see that angle QOT=Nα/2, draw the circle with center at M through O,P and Q; OT$\perp$MO thus angle QOT=angle OMQ/2=Nα/2.
So Q is ahead by phase angle α/2(N-1). Since |OP|=A in this problem, we find:

$$E(t) = A\frac{\sin(N\alpha/2)}{\sin(\alpha/2)}\cos\left[\omega t + \frac{1}{2}\alpha(N-1)\right] \quad (12)$$

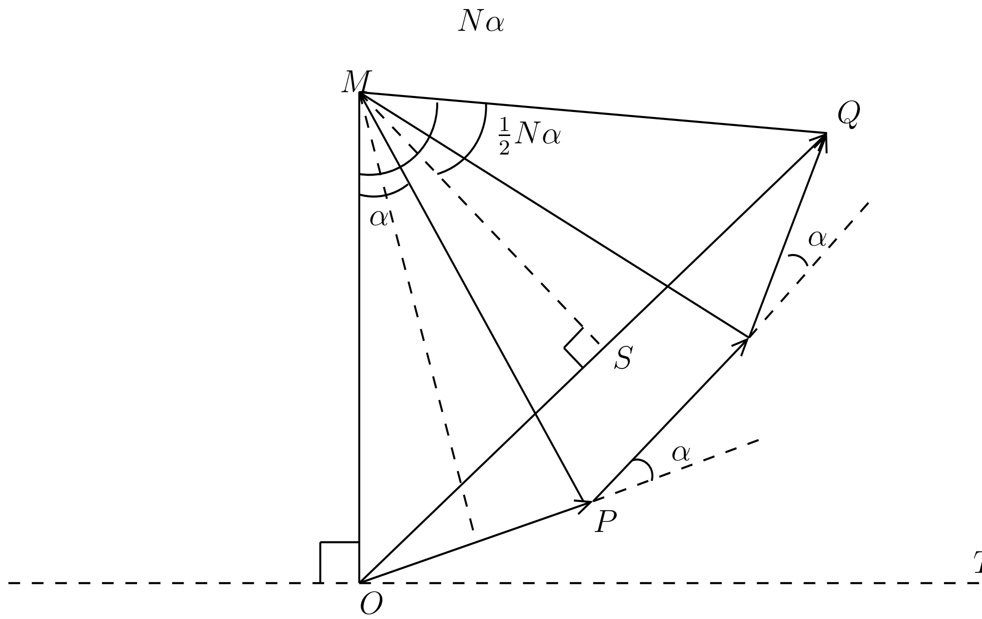

FIG. 6: Problem 10.5(b) Addition of N vectors

Let us now test our result of Part (a): N=5 and $\alpha = \pi/9$. $N(\alpha - 1)/2 = 2\alpha$ $(= 40^o)$. The amplitude of the vector is (A=3) $3\sin(2.5\pi/9)/\sin(\pi/18) \simeq$ 13.234. On the button!

By adding vectors, point Q 'marches' on the circumference of the circle and will reach O (amplitude E=0), then it traces its old route; a maximum is reached when Q is on the line OM. The amplitude $2R$ is OP/$sin(\alpha/2)$ and thus depends of α.

Part (c)

Let us now plot the vector amplitude as function of α. When $\alpha = 0$ the vectors all line up (they are in phase) and we obtain the largest amplitude possible. This amplitude then should be N times the individual value of A; thus NA. Indeed this can be found from Eq. 12 as well. For $\alpha = 0, 2\pi, 4\pi, 6\pi$ etc the "upstairs" and "downstairs" of the amplitude in Eq. 12 is zero. Applying l'Hôpital's rule:

$$\lim_{\beta\to\pi} \frac{\sin N\beta}{\sin\beta} = \lim_{\beta\to\pi} \frac{N\cos\beta}{\cos\beta} = N$$

Thus we see that our function makes a maximum amplitude of NA when $\alpha = 0, 2\pi, 4\pi, ...$ The amplitude is zero whenever $\sin(N\alpha/2) = 0$ thus for $\alpha = 2\pi/N, 4\pi/N, 6\pi/N$ etc. *However, when at the same time* $\sin(\alpha/2) = 0$*, the amplitude is a maximum!*

Thus there are minima when $\alpha = 2n\pi/N$ (n is an integer) except when $n = N, 2N, 3N, ...$

In Fig. 7, $N = 7$. Notice that there are $N - 1 = 6$ minima in between the main maxima. I have plotted the $\vec{E}$ amplitude. The light intensity is proportional to $(E_o)^2$ thus at maximum proportional with N^2.

By going from a radio interferometer with five dishes to one of ten dishes, the signal strength received increases by $10^2/5^2 = 4$ (assuming the dishes are the same).

Fig. 7 shows the sketch of amplitude of $E(t)$ as function of α for $N = 7$.

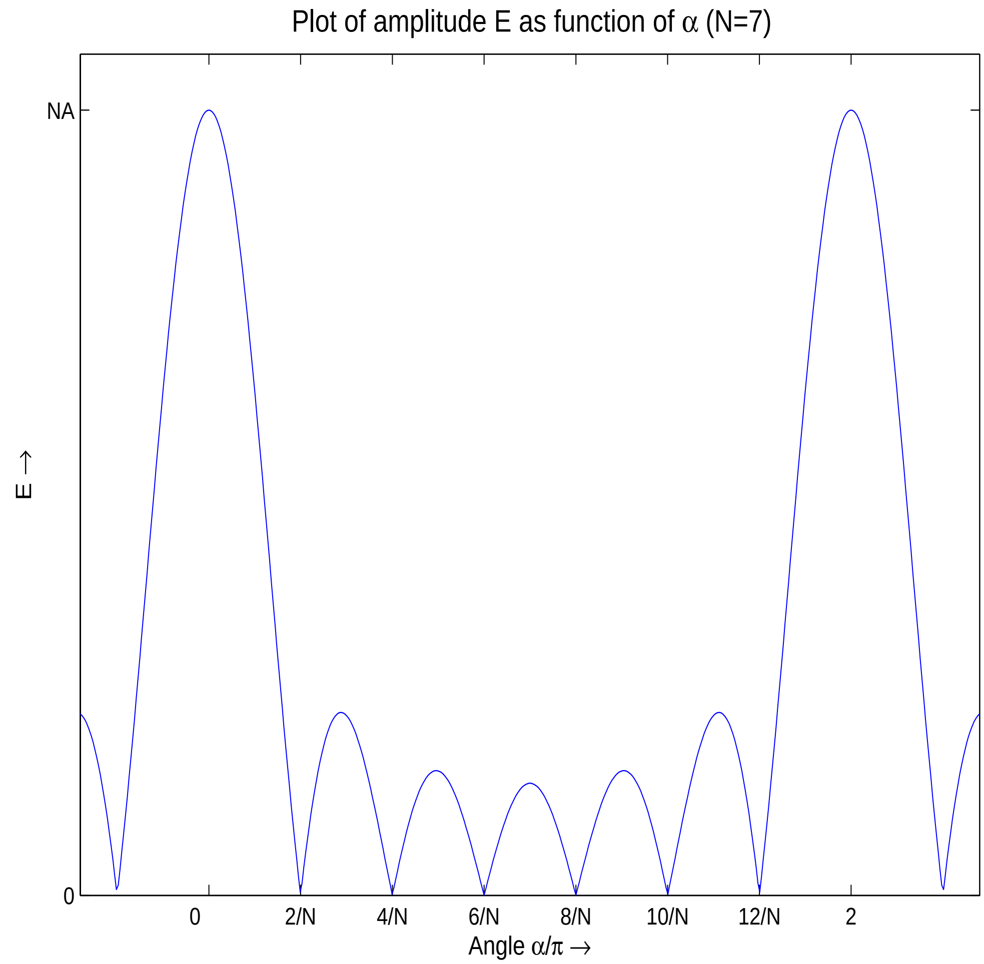

FIG. 7: Problem 10.5(c)