8.03 Fall 2004
Problem Set 9 Solutions

Solution 9.1: Nature is in a hurry - Fermat's Principle

Part (a)

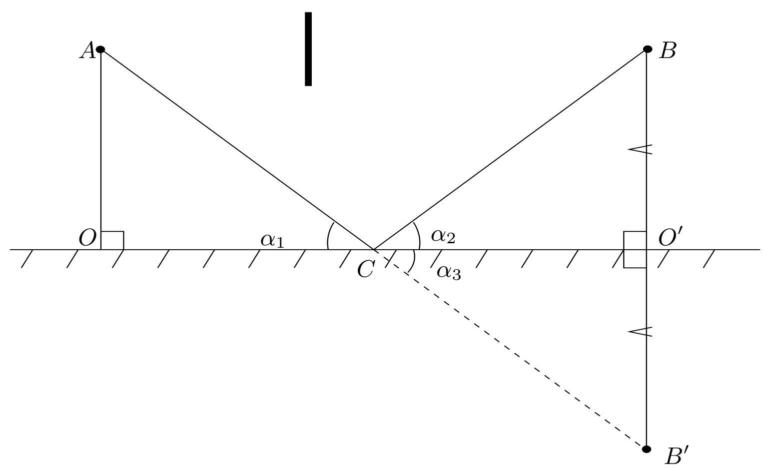

FIG. 1: Problem 9.1(a)

Shown in Fig. 1.

Part (b)

$BO' = B'O'$ and angle $BO'C$= angle $B'O'C$. So $\triangle BO'C$ is congruent to $\triangle B'O'C$. Hence

$$\begin{aligned} \alpha_2 &= \alpha_3 \quad \textit{Congruency} \\ \alpha_1 &= \alpha_3 \quad \textit{Opposite angles} \\ \Rightarrow \quad \alpha_1 &= \alpha_2 \end{aligned}$$

The angle of incidence is same as the angle of reflection.

Part (c)

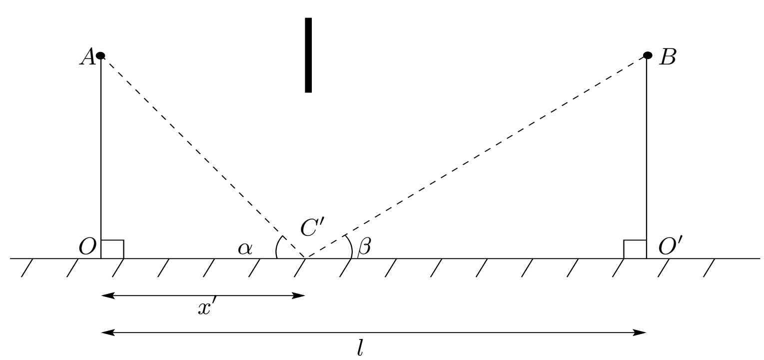

FIG. 2: Problem 9.1(c)

Fig. 2 shows the setup when the hypothetical light ray reflects off at a random point C'. The distance S the light ray has to travel from A to B is:

$$
\begin{aligned}
S &= AC' + C'B \\
&= \sqrt{d^2 + x'^2} + \sqrt{d^2 + (l - x')^2} \\
\frac{dS}{dx'} &= 0 \quad \textit{for minimum time of travel} \\
\frac{dS}{dx'} &= \frac{1}{2}\frac{1}{\sqrt{d^2 + x'^2}}(2x') + \frac{1}{2}\frac{1}{\sqrt{d^2 + (l - x')^2}}(2(l - x'))(-1) = 0 \\
&\Rightarrow \frac{x'}{\sqrt{d^2 + x'^2}} - \frac{(l - x')}{\sqrt{d^2 + (l - x')^2}} = 0 \\
&\textit{or} \quad \cos\alpha - \cos\beta = 0
\end{aligned}
$$

Then $\cos\alpha = \cos\beta$ or $\alpha = \beta \Rightarrow\ x' = l/2$.

Part (d)
If A is distance d above the horizontal and B' is distance d below the horizontal, the line AB' will intersect the mirror at midpoint C of OO' such that $x_c = l/2$, same as for C' from Part(c).

Solution 9.2: (Bekefi & Barrett 7.4) Fiber optics

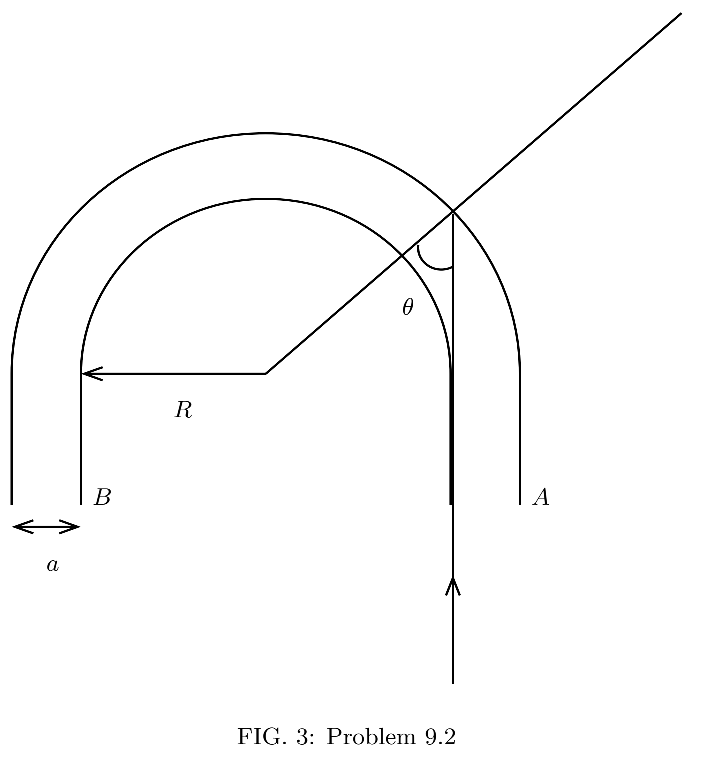

FIG. 3: Problem 9.2

To ensure that all light entering at A emerges at B, it is sufficient that the smallest incidence angle θ of the incoming beam be greater than the critical angle θ_c for the medium. The first bounce has the smallest angle of incidence. It is larger for the following bounces. Therefore, the light will arrive at the end of the light pipe, when the angle of incidence for the first bounce is larger than the critical angle for total internal reflection. This minimum incidence angle occurs when the incoming ray is grazing the inner curved surface. Since all beams which are incident at A have incidence angles which are greater than the incidence angle of this ray, they will all be totally internally reflected if this ray is totally internally reflected. That is true for the following condition:

$$\begin{aligned} \sin\theta &\geq \sin\theta_c \qquad\quad \textit{where} \quad \sin\theta = \frac{R}{R+a} \quad \sin\theta_c = \frac{1}{\eta} \\ \frac{R}{R+a} &\geq \frac{1}{\eta} \\ \Rightarrow \qquad a &\leq (\eta-1)R \qquad \textit{where} \quad \eta = 1.5 \\ a_{max} &= \frac{R}{2} \end{aligned}$$

The maximum value for which all the light entering at A will emerge at B is $a_{max} = R/2$.

Solution 9.3: (Bekefi & Barrett 7.5) Total reflection

Part (a)

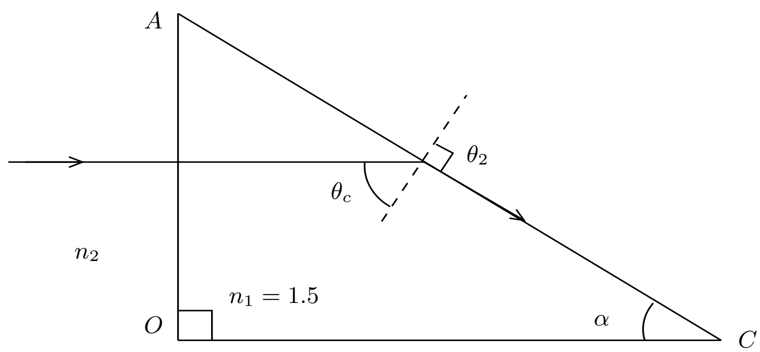

FIG. 4: Problem 9.3

The index of refraction of the material of prism $n_1 = 1.5$. For total reflection at side AC, the critical incidence angle θ_c is such that $\theta_2 = 90^o$

$$\begin{aligned} n_1\sin\theta_c &= n_2\sin\theta_2 \quad \textit{where } \theta_2 = 90^o \\ n_1\sin\theta_c &= 1 \\ \sin\theta_c &= 1/n_1 \\ \theta &= \pi/2-\alpha \\ \Rightarrow \quad \cos\alpha_c &= 1/n_1 \qquad \textit{where} \quad n_1 = 1.5 \\ \alpha_c &= 48.2^o \end{aligned}$$

Part (b)

So for the light to be totally internally reflected $\theta > \theta_c \ \Rightarrow \ \alpha < \alpha_c$. It is a maxima and the light can undergo total internal reflection only for prism angles $\alpha < \alpha_c = 48.2^o$.

Solution 9.4: Light under water

Part (a)

The red light ($\lambda \approx 650$ nm) is incident on the surface of water at a swimming pool. The frequency of the light is unchanged after refraction at a medium interface. Hence

$$\begin{aligned} \nu &= \nu' \\ \frac{v}{\lambda} &= \frac{v'}{\lambda'} \qquad\qquad \textit{where} \quad v = \frac{c}{n} \\ n\lambda &= n'\lambda' \qquad\qquad \textit{where} \quad n_1 = 1 \quad n_2 = 1.33 \\ \lambda' &= \frac{n}{n'}\lambda \approx \frac{650\ nm}{1.33} \qquad \textit{where} \quad \lambda \approx 650\ nm \\ &\simeq 488\ nm \end{aligned}$$

If we swim under water and look up at the refracted light coming from the surface, we will still see red light. One way of looking at this is that the frequency remains the same, thus our brains process the signal the same way. Another way of looking is to calculate the wavelength as the radiation reaches our retina. Convince yourself that this is the same whether you are under water or above water.

Part (b)

From the Lecture Notes of Nov 16, 2004 (they are available on the web), we know that the Fresnel equations reduce to the following reflection and transmission coefficients for normal incidence ($\theta_{incidence} = 0$) :

$$r_\perp = \frac{n_1 - n_2}{n_1 + n_2} \qquad t_\perp = \frac{2n_1}{n_1 + n_2} \tag{1}$$

Now the power of electromagnetic waves in a medium of refractive index n is given by $n|E|^2/c$. Let the power in the incident, reflected and transmitted light be P_i, P_r and P_t respectively. Then the sum of the total power in the reflected and the transmitted light is:

$$\begin{aligned}
P_r + P_t &= \frac{n_r}{c}|E_r|^2 + \frac{n_t}{c}|E_t|^2 \\
&= \frac{n_1}{c}{r_\perp}^2|E_i|^2 + \frac{n_2}{c}t_\perp^2|E_i|^2 && n_r = n_1 \quad n_t = n_2 \\
&= \frac{|E_i|^2}{c}\left[n_1\left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 + n_2\left(\frac{2n_1}{n_1 + n_2}\right)^2\right] && \textit{From Eq. 1} \\
&= \frac{|E_i|^2}{c}\left[\frac{n_1(n_1^2 + n_2^2 - 2n_2n_1 + 4n_2n_1)}{(n_1 + n_2)^2}\right] \\
&= \frac{|E_i|^2}{c}\left[\frac{n_1(n_1 + n_2)^2}{(n_1 + n_2)^2}\right] \\
&= \frac{n_1}{c}|E_i|^2 \\
\Rightarrow \qquad P_r + P_t &= P_i
\end{aligned}$$

QED