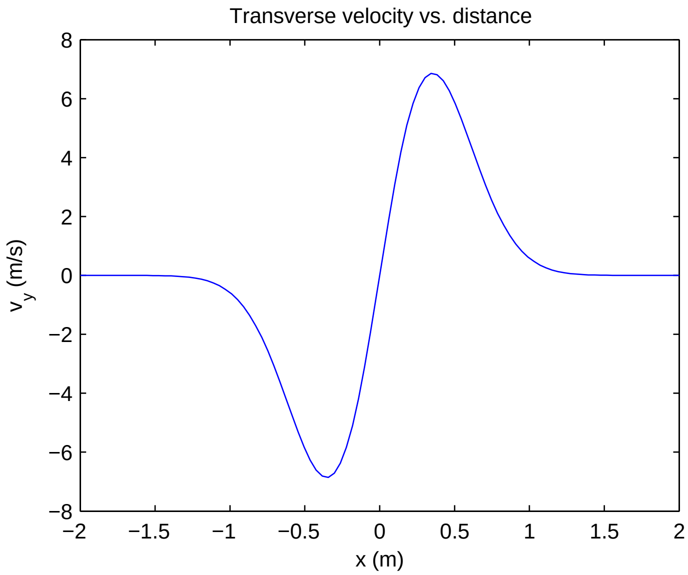
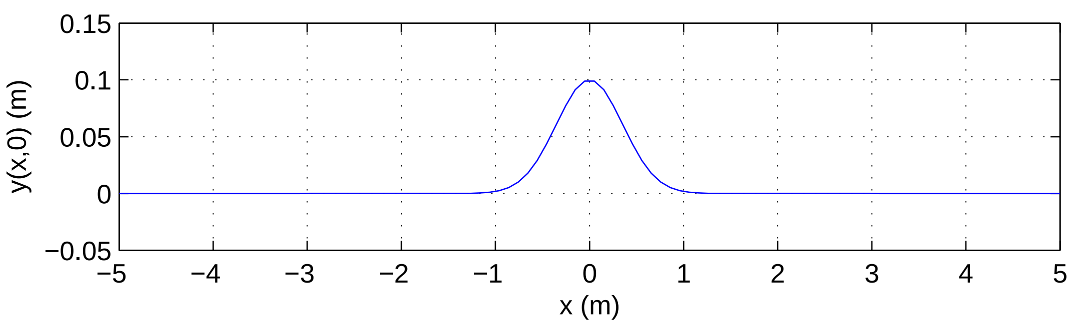
MIT 8.03 Fall 2004 – Solutions to Problem Set 4

Problem 4.1 (French 7-12) – Travelling pulse

(a) Since the pulse is travelling to the right, the piece of string on the right side of the peak is "rising" and the piece on the left is "falling." The transverse velocity of the peak is zero but it has the maximum acceleration (see the figure below).



(b) The pulse shape is



We can model the pulse with a Gaussian function. That is, the pulse resembles

$$y(\eta) = Ae^{-\alpha\eta^2},$$

where $\eta = x - vt$, $A = 0.1$ m and $\alpha = 4$ m^{-2}. The above graph of the pulse shape is actually this function. The transverse velocity is then

$$\begin{aligned} \frac{\partial y}{\partial t} &= -2A\alpha\eta e^{-\alpha\eta^2}\frac{\partial \eta}{\partial t} \\ &= 2A\alpha v\eta e^{-\alpha\eta^2} \end{aligned}$$

We can find the maximum transverse velocity at $t = 0$ by requiring that

$$\begin{aligned}
\left.\frac{\partial^2 y}{\partial t^2}\right|_{t=0} &= 0 \\
2A\alpha v^2 e^{-\alpha\eta^2}\left(2\alpha x_{\max}^2 - 1\right) &= 0 \\
\Rightarrow 2\alpha x_{\max}^2 - 1 &= 0 \\
x_{\max} &= \sqrt{\frac{1}{2\alpha}}.
\end{aligned}$$

Hence, the maximum transverse velocity at $t = 0$ is

$$\begin{aligned}
v_y\big|_{\max} &= \left.\frac{\partial y}{\partial t}\right|_{x=x_{\max}} \\
&= 2Av\sqrt{\frac{\alpha}{2}}e^{-1/2} \\
&\approx 6.86\text{m/s}.
\end{aligned}$$

(c) The mass density of the string is $\mu = 1/50$ kg/m. Then, the tension in the string is $T = \mu v^2 \approx 32$ N.

(d) Any wave traveling in the negative x direction with a speed v can be described as

$$\begin{aligned}
y(x,t) &= f(\eta) \\
&= f(kx + \omega t),
\end{aligned}$$

where $f(\eta)$ is the shape of the wave, k is the wave number and ω is the angular frequency. For sinusoidal waves:

$$y(x,t) = A\sin(kx + \omega t + \phi),$$

where A is the amplitude of the wave and ϕ is the phase of the sinusoid. Furthermore, a wavelength of 5 m implies $k = 2\pi/\lambda = 0.4\pi$ m^{-1}. Since this wave is traveling on a string, it must obey the relation $\omega = kv = 16\pi$ s^{-1}. Therefore, the equation describing the wave is

$$y(x,t) = (0.2 \text{ m})\sin\left(\left(0.4\pi \text{ m}^{-1}\right)x + \left(16\pi \text{ s}^{-1}\right)t + \phi\right),$$

where ϕ is unknown since the phase of the wave was unspecified.

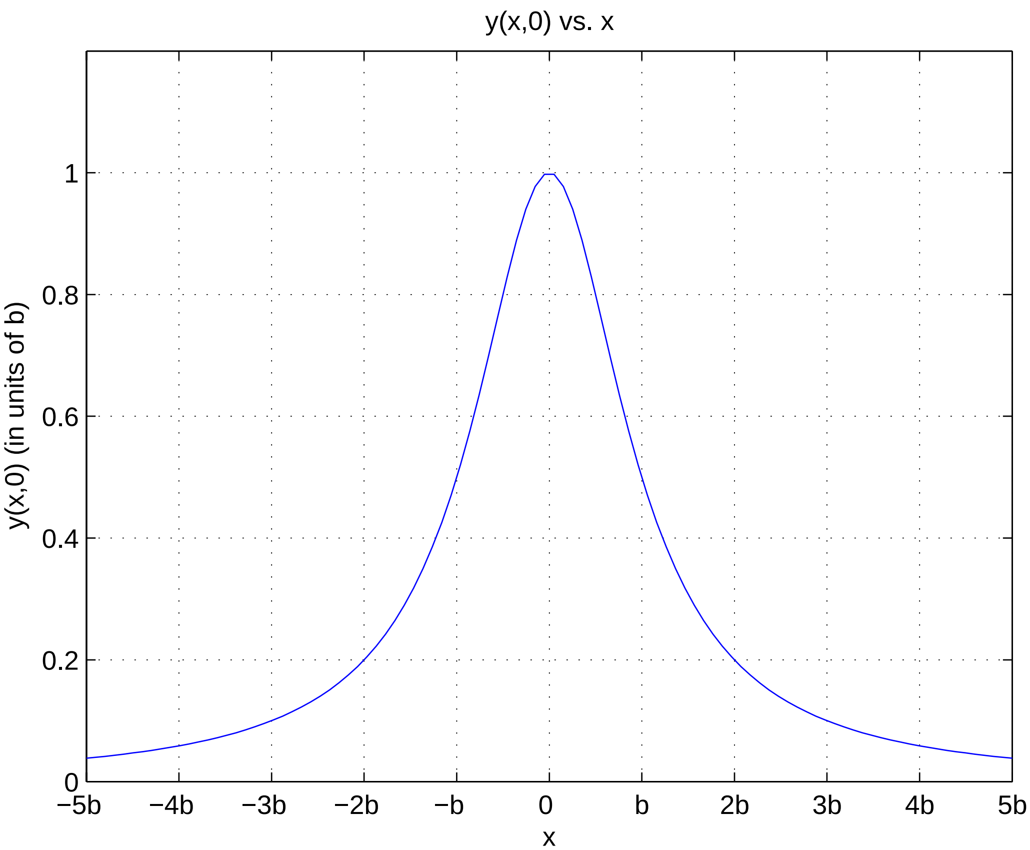
Problem 4.2 (French 7-13) – Travelling pulse

(a) The sketch of $y(x, 0)$ is shown on the next page.

(b) Remember that any pulse or wave traveling in the positive x-direction can be expressed as $y(\omega t - kx)$, for $k \geq 0$ and that its speed of propagation is $v = \omega/k$. Then, letting $z = \omega t - kx$ and expressing $y(x,t)$ as a function of z,

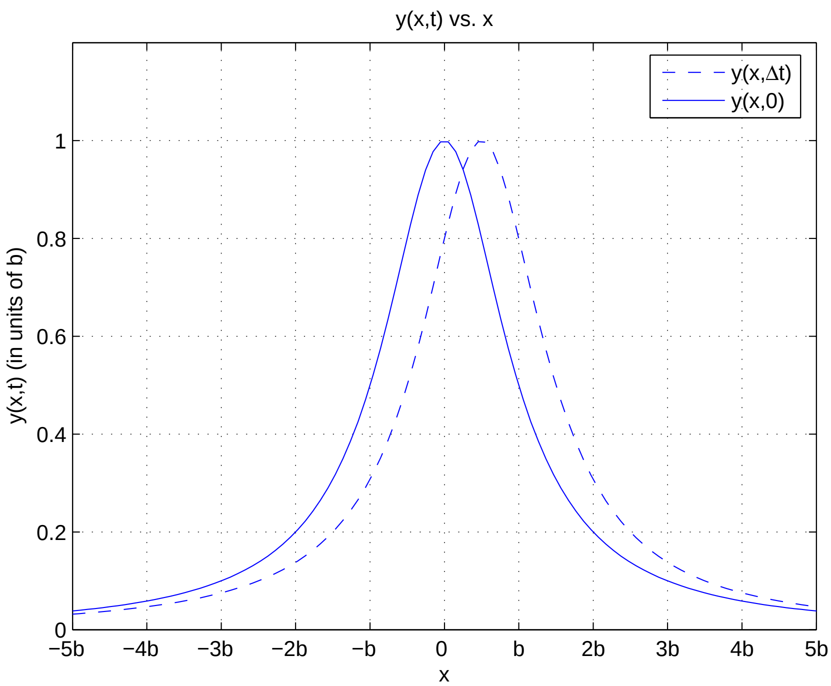
$$y(z) = \frac{b^3}{b^2 + z^2}.$$

Hence, $z = 2x - ut$. Therefore, for positive values of u, the pulse travels in the positive x direction with a speed $v = u/2$.



(c)

$$
\begin{aligned}
v_y(t=0) &= \left.\frac{\partial y}{\partial t}\right|_{t=0} \\
&= \left.\frac{2b^3u(2x-ut)}{\left(b^2+(2x-ut)^2\right)^2}\right|_{t=0} \\
&= \frac{4b^3xu}{\left(\, b^2+4x^2\right)^2}
\end{aligned}
$$



Problem 4.3 – Pulse reflection at a boundary

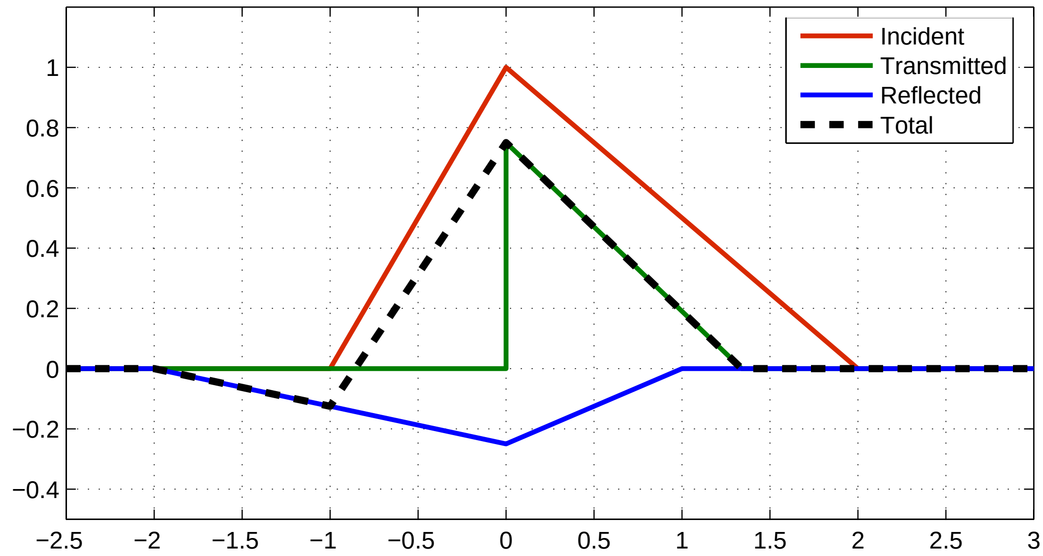
(a) The propagation speed in string 1 is $v_1 = \sqrt{T/\mu_1} = 10\sqrt{2}$ m/s $\approx$ 14 m/s and in string 2, $v_2 = \sqrt{T/\mu_2} = 10\sqrt{2/3}$ m/s $\approx$ 8 m/s. Then, the reflection coefficient is

$$R = \frac{v_2 - v_1}{v_1 + v_2} = \frac{\sqrt{3} - 3}{\sqrt{3} + 3} \approx -\frac{1}{4},$$

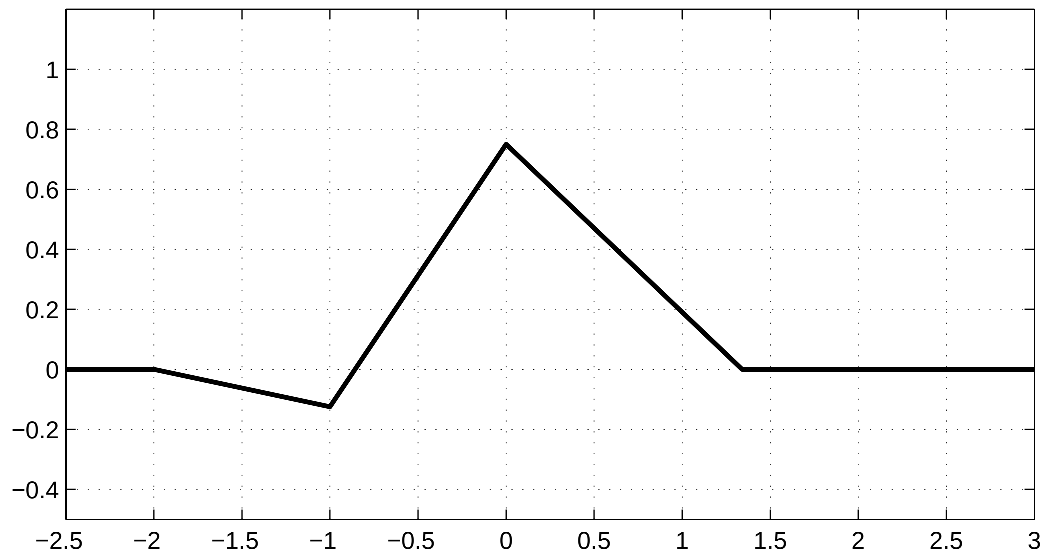
and the transmition coefficient is

$$T = \frac{2v_2}{v_1 + v_2} = \frac{2\sqrt{3}}{\sqrt{3} + 3} \approx \frac{3}{4}.$$

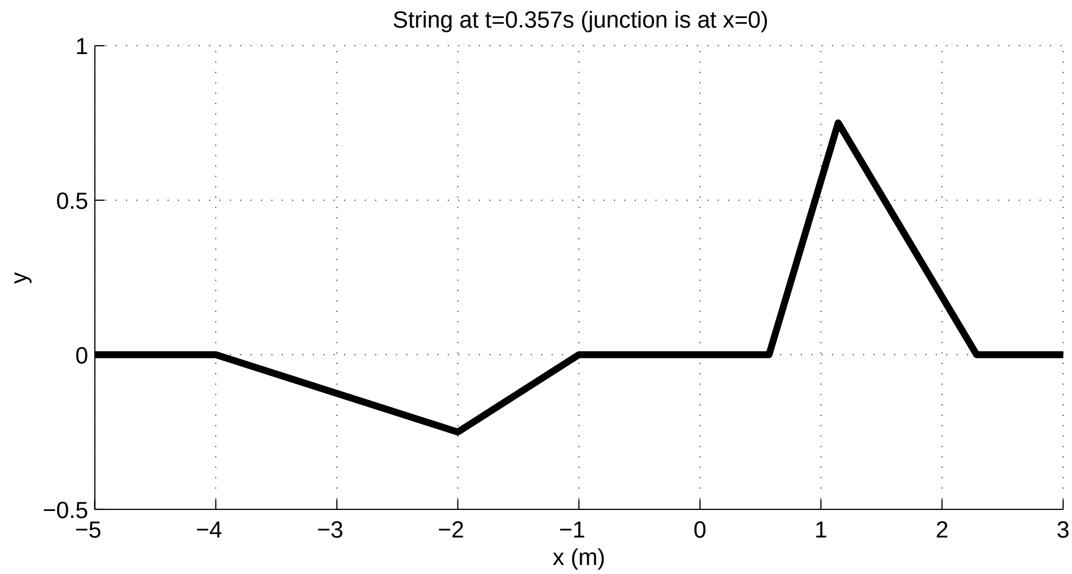
(b) The following graph shows the incident, reflected and transmitted waves when the pulse peak arrives at the junction ($x = 0$). Note that the reflected pulse is upside down and flipped right to left. Also, the transmitted pulse is narrower. Keep in mind that only the dashed black line is physical. The other lines (in red, green and blue) are there only for illustrative purposes.



The graph on the next page shows the total deformation of the string when the peak is at $x = 0$.



(c) The shape of the string at time $t = (5\text{ m})/v_1 = 0.357s$ is shown below.



(d) The sharp cusps of the pulse are unphysical because it leads to an infinite potential energy of the string. Recall that the potential energy density of a string is

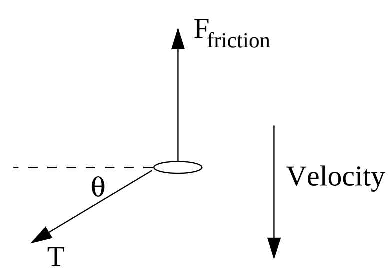
$$\frac{dU}{dx} = \frac{1}{2}T\frac{\partial y}{\partial x}.$$

Since the pulse is not smooth at the cusp the slope is infinite. Therefore, the potential energy of the string is infinite.

Alternatively, we could argue that, at any point in the string, the forces must cancel because each point has an infinitesimally small mass. We need vanishing forces in the presence of a vanishing mass so the acceleration remains finite. The cusps in the string cause an infinite acceleration since the forces at those points do not cancel.

Problem 4.4 – Boundary conditions on a string

(a) The following sketch shows the forces acting on the hoop.



Applying Newton's second law gives

$$\begin{aligned} F &= ma \\ \Delta m\ddot{y} &= -T\sin\theta + F_{friction}. \end{aligned}$$

Assuming that oscillations are small,

$$\Delta m\ddot{y} = -T\frac{\partial y}{\partial x} - b\frac{\partial y}{\partial t}.$$

Since the mass of the hoop is negligible,

$$\begin{aligned} -T\frac{\partial y}{\partial x} - b\frac{\partial y}{\partial t} &= 0 \\ \Rightarrow \frac{\partial y}{\partial x} &= -\frac{b}{T}\frac{\partial y}{\partial t} \quad \text{At the hoop } \forall t. \end{aligned}$$

(b) Let's take the superposition of an incident wave and a reflected wave

$$y(x,t) = \underbrace{f(x-vt)}_{\text{Incident, known.}} + \underbrace{g(x+vt)}_{\text{Reflected, unknown.}}.$$

We now use the boundary condition at the hoop to solve for $g(x+vt)$. The respective derivatives are

$$\begin{aligned} \frac{\partial y}{\partial x} &= f'(x-vt) + g'(x+vt) \\ \frac{\partial y}{\partial t} &= v\left(-f'(x-vt) + g'(x+vt)\right). \end{aligned}$$

If the hoop is at $x = 0$, then

$$\begin{aligned} f'(-vt) + g'(vt) &= \frac{bv}{T}\left(f'(-vt) - g'(+vt)\right) \\ g'(vt) &= \frac{bv/T - 1}{bv/T + 1} f'(-vt). \end{aligned}$$

Letting $\eta = vt$ and integrating with respect to η,

$$\begin{aligned} \int g'(\eta)\, d\eta &= \int \frac{bv/T - 1}{bv/T + 1} f'(-\eta)\, d\eta \\ g(\eta) &= \frac{bv - T}{bv + T}(-1) f(-\eta) \\ g(\eta) &= \frac{T - bv}{T + bv} f(-\eta). \end{aligned}$$

Note that the integration constant must equal zero for the limiting cases discussed in the next part to hold.

(c) For $b = 0$, the hoop behaves as a free end. Our result gives $g(\eta) = f(-\eta)$, which is correct since the wave is reflected without flipping.

For $b \to \infty$, the hoop behaves as a clamped end. Our result gives $g(\eta) = -f(-\eta)$, which is correct since the wave is reflected flipped over.

Note that for the special case when $b = T/v$, $g(\eta) = 0$. Hence, there is no reflected wave. This is known as a matched load.

Problem 4.5 – Boundary conditions in a pipe

The wave equation for the over-pressure $p(z,t)$ inside a pipe is

$$\frac{\partial^2 p}{\partial z^2} = \frac{\rho_0}{\kappa}\frac{\partial^2 p}{\partial t^2}.$$

The solution to this equation is

$$p(z,t) = [A\cos kz + B\sin kz]\cos\omega t.$$

Since the pipe is open at both ends (remember, p is over-pressure),

$$\begin{aligned}
p(0,t) &= 0\\
A\cos\omega t &= 0\\
\Rightarrow A &= 0\\
&\text{and}\\
p(L,t) &= 0\\
B\sin kz\cos\omega t &= 0\\
\sin kz &= 0\\
\Rightarrow k &= \frac{n\pi}{L} \qquad \text{where } n = 1,2,3\ldots.
\end{aligned}$$

We can obtain the dispersion relation by inserting $p(z,t)$ into the wave equation for the system. The relevant derivatives are

$$\begin{aligned}
\frac{\partial p}{\partial z} &= kB\cos kz\cos\omega t,\\
\frac{\partial p}{\partial t} &= -\omega B\sin kz\sin\omega t,\\
\frac{\partial^2 p}{\partial z^2} &= -k^2 B\sin kz\cos\omega t,\\
\frac{\partial^2 p}{\partial t^2} &= -\omega^2 B\sin kz\cos\omega t.
\end{aligned}$$

The wave equation then reduces to

$$\begin{aligned}
-k^2 B\sin kz\cos\omega t &= -\omega^2 B\sin kz\cos\omega t\\
\Rightarrow \omega &= \sqrt{\frac{\kappa}{\rho_0}}k\\
\omega_n &= n\frac{\pi}{L}\sqrt{\frac{\kappa}{\rho_0}}.
\end{aligned}$$

Finally, the initial condition determines k_n and B. The initial condition is

$$\begin{aligned}
p(L/2,0) &= p_0\\
B\sin k\frac{L}{2} &= p_0\\
B\sin\frac{n\pi}{2} &= p_0\\
\Rightarrow B &= \pm p_0 \qquad \text{if } n = 1,3,5,7\ldots.
\end{aligned}$$

Hence, n must be an odd integer. Otherwise, B would equal zero and $p(z,t) = 0$ which is indeed a—trivial—solution. Finally, the wave number is

$$k_n = \frac{n\pi}{L} \qquad \text{where } n = 1,3,5,7\ldots.$$

And,

$$\begin{aligned}
B &= +p_0 \qquad \text{for } n = 1,5,9\ldots\\
B &= -p_0 \qquad \text{for } n = 3,7,11,\ldots.
\end{aligned}$$

Problem 4.6 – Normal modes of discrete vs. continuous systems

(a) The most general solution for a standing wave in a string is

$$y(x,t) = A\cos(kx+\phi_x)\cos(wt+\phi_t)\,.$$

The boundary conditions are

$$\begin{aligned} y(0,t) &= 0 \\ A\cos\phi_x &= 0 \\ \Rightarrow \phi_x &= \frac{\pi}{2} \\ &\text{and} \\ y(L,t) &= 0 \\ A\sin kL &= 0 \\ \Rightarrow kL &= n\pi. \end{aligned}$$

Hence, the n-th normal mode of the string is

$$y_n(x,t) = A_n \sin\left(\frac{n\pi}{L}x\right)\cos(\omega_n t+\phi_t)\,,$$

where

$$\begin{aligned} \omega_n &= n\omega_1 \\ &= \frac{n\pi v}{L} \\ &= \frac{n\pi}{L}\sqrt{\frac{T}{\mu}} \\ &= n\pi\sqrt{\frac{T}{ML}}. \end{aligned}$$

(b) The general formula for the frequency of the n-th mode is

$$\begin{aligned} \nu_n &= \frac{\omega_n}{2\pi} \\ &= \frac{n}{2}\sqrt{\frac{T}{ML}}. \end{aligned}$$

The five lowest normal modes are

$$\begin{aligned} \nu_1 &= \tfrac{1}{2}\sqrt{\tfrac{T}{ML}} \equiv \nu_0, & \nu_2 &= \sqrt{\tfrac{T}{ML}} = 2\nu_0, \\ \nu_3 &= \tfrac{3}{2}\sqrt{\tfrac{T}{ML}} = 3\nu_0, & \nu_4 &= 2\sqrt{\tfrac{T}{ML}} = 4\nu_0, \\ \nu_5 &= \tfrac{5}{2}\sqrt{\tfrac{T}{ML}} = 5\nu_0. \end{aligned}$$

(c) From French 5-25,

$$\begin{aligned} \omega_n &= 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right) \\ \Rightarrow \nu_n &= \frac{\omega_0}{\pi}\sin\left(\frac{n\pi}{2(N+1)}\right). \end{aligned}$$

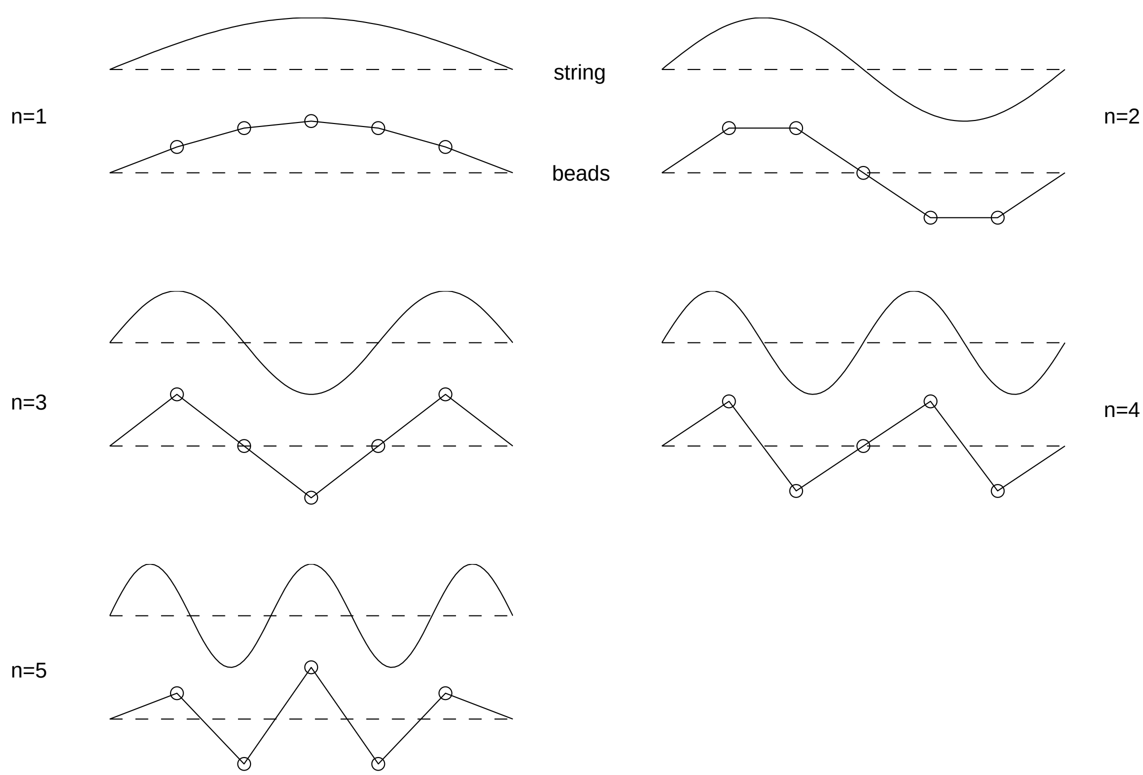
The fundamental frequency is

$$
\begin{aligned}
\omega_0 &= \sqrt{\frac{T}{\frac{M}{5}\frac{L}{6}}} \\
&= \sqrt{\frac{30T}{ML}} \\
&= \sqrt{120}\nu_0.
\end{aligned}
$$

The first five frequencies are then $(N = 5)$

$$
\begin{aligned}
\nu_1 &= \frac{\sqrt{120}}{\pi}\sin\left(\frac{\pi}{12}\nu_0\right) = 0.9\nu_0, & \nu_2 &= \frac{\sqrt{120}}{\pi}\sin\left(\frac{\pi}{6}\nu_0\right) = 1.7\nu_0, \\
\nu_3 &= \frac{\sqrt{120}}{\pi}\sin\left(\frac{\pi}{4}\nu_0\right) = 2.5\nu_0, & \nu_4 &= \frac{\sqrt{120}}{\pi}\sin\left(\frac{\pi}{3}\nu_0\right) = 3.0\nu_0, \\
\nu_5 &= \frac{\sqrt{120}}{\pi}\sin\left(\frac{5\pi}{12}\nu_0\right) = 3.5\nu_0.
\end{aligned}
$$

(d) The following figures show the first 5 normal modes for the string and the beads.



(e) Since $N = 5$ is still not $N \gg 1$, the normal mode frequencies and shapes are not identical.