MIT 8.03 Fall 2004 – Solutions to Problem Set 11

Problem 11.1 — Take-home experiment #9 – Transmission grating

Using a mini-maglite I (Igor Sylvester) was able to see the 0th, 1st and 2nd order spectra clearly. I observed that for both the 1st and 2nd order spectra, blue diffracted the least and red diffracted the most—as expected. For the maglite, I observed a continuum spectrum of colors; I did not observe any sharp lines (due to atomic transitions).

As I rotated the grating about the line of sight, the direction of the spectra rotated with the grating. From this we can determine the direction in which the grooves in the grating are cut. Rotating the maglite did not change the orientation of the spectra. Rotating the grating about an axis parallel to the grooves in the slide made the spectra wider and more spread. The reason is that the effective groove separation (in projection) becomes smaller. Tilting the grating about an axis parallel to the direction of the spectra bent the direction of the spectra into a "V", with the vertex at the 0th order spectrum.

I used the grating to observe the diffraction from a fluorescent light. Instead of observing a continuum of colors, I saw four distinct colors (red, yellow, blue and violet). I was able to see up to the 4th order spectra but only in the wavelength corresponding to violet; for wavelengths longer than approximately 470 nm the 4th order maxima do not exist. The discrete number of frequencies in the spectrum of the fluorescent light indicates that it does not radiate as a black body. Instead, the fluorescent light produces photons which are emitted by the discrete change in energy of electrons in atoms.

I looked at the light of the 0th, 1st, 2nd and 3rd order through a linear polarizer, and I rotated the polarizer in its own place. I could not notice any change in light intensity. Therefore, I suspect that the plastic gratings are probably simple phase gratings. Thus, only phase differences between the waves (Huygens' Principle) matter, and not the direction of polarization. The situation for metallic gratings is very different and rather complex as a result of the special boundary conditions on the surface of conductors[1]. There are metallic transmission gratings (your plastic grating is a transmission grating, but not metallic) and there are metallic reflection gratings (one was demonstrated in lectures on December 2). I copy here from a website that discusses metallic gratings:

> "However, the gratings efficiency is generally a rather complex function of wavelength and polarization of the incident light, and depends on the groove frequency, the shape of the grooves and the grating material. Especially for polarization when the electric vector is perpendicular to the grating grooves, one may observe rapid changes in efficiency for a small change in wavelength. This phenomenon was first discovered by R.W. Wood in 1902, and the rapid variations are usually called Wood's anomalies."

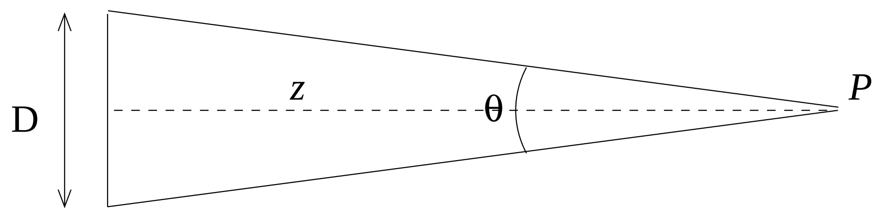
It seems safe to conclude that in metallic gratings, the intensity of the diffracted light depends on polarization in a rather complicated way.

Problem 11.2 — Think big

(a) An EM plane wave of wavelength λ passes through a circular aperture of diameter D and diffracts. If we measure the intensity of the diffracted light at a distance z away from the aperture, such that $z \geq 2D^2/\lambda$, then we will observe Fraunhofer diffraction.

(b) If we are located at P, a distance z from the aperture, $\sin\theta \approx D/z$ (see figure). For Fraunhofer diffraction, the first zero in intensity next to the prime maximum is observed when $\sin\theta \approx 1.2\lambda/D$. We therefore require that $D/z \ll 1.2\lambda/D$. If this were not the case, the zero in intensity would be blurred due to the extended size of the aperture. Thus $z \gg D^2/1.2\lambda$. Perhaps somewhat arbitrarily, $z > 2D^2/\lambda$ is generally adopted.

[1] Remember the demo wit a metal comb? (The comb was a transmission grating for radar!) Depending on the angle of the "teeth" of the comb relative to the direction of polarization, linearly polarized radar could be reflected by the comb or it could pass the comb (this even happened when Professor Lewin used his fingers instead of the comb—fingers are reasonably good conductors!)



(c) An approximation of the minimum distance between the photographic plate and the slit is given by the Fraunhofer diffraction relation,

$$z_{\text{min}} \approx \frac{2D^2}{\lambda} \approx 5.8 \times 10^8 \text{ m}.$$

This is farther than the distance to the Moon!

(d) Our guess is that the central maximum will be about the same as the size of the aperture, since the Fraunhofer condition is just met; thus ≈ 12 m. A more formal calculation supports that. For small ϕ, $\sin\phi \approx \phi$. Using $\sin\phi = \lambda/D$, $\phi \approx \lambda/D \approx x/z$. Hence, $x = z\lambda/D$. Using $z = z_{\text{min}}$, the width of the central maximum is approximately 24 m.

(e) We are now told that the new slit width is $D' = 2$ m. Hence, the width of the central maximum is now 6 times bigger, i.e. about $6 \times 24 \text{ m} = 144$ m.

(f) The aperture is now way too large to meet the Fraunhofer condition. Thus the bright maximum will be about 96 m wide.

(g) Alignment of the earth-star would be hopeless and contamination of your pattern by neighboring stars (after all, one does not simply aim a 12 m slit at a star) would make your task of differentiation impossible—not to mention problems of intensity.

Problem 11.3 (Bekefi & Barrett 8.7) — Angular resolution

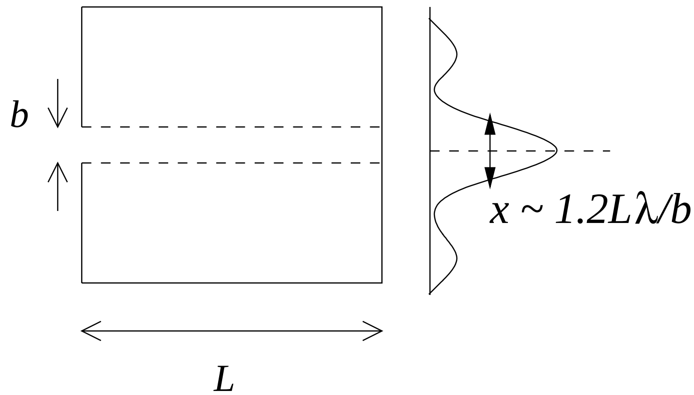
In order to resolve the two light sources, we must be able to differentiate the two diffraction patterns on the objective of the telescope. Let's first see whether the 5 cm lens is capable of resolving the two lights. Its angular resolution is

$$\Delta\theta \approx \frac{1.2\lambda}{D} \approx 1.4 \times 10^{-5} \text{ rad}.$$

The angular separation of 1 ft at a distance of 10 miles is about 1.9×10^{-5} rad. Thus the telescope will be able to resolve the two sources of light. If we now place a slit in front of the lens, whose width is less than 5 cm, it will become more difficult to resolve the two lights. It's not clear now whether we should use as angular resolution λ/D or $1.2\lambda/D$. For a narrow long slit we should use λ/D, but the length of the slit will NOT become much larger than its width, D. Therefore, we will stick (conservatively) with an angular resolution of $1.2\lambda/D$. Thus we require $1.9 \times 10^{-5} > 1.2\lambda/D$. Hence, $D > 3.8$ cm.

Problem 11.4 (Bekefi & Barrett 8.8) — Pinhole camera

The next figure shows the setup of the camera. The diffraction pattern is shown on the right.



A distant source produces a Fraunhofer diffraction pattern on the screen with a central maximum width

$$w \approx 1.2L\lambda/b. \tag{1}$$

This holds if, and only if, the coherence relation

$$\frac{b}{L} < \frac{\lambda}{b}$$

is satisfied. Otherwise, the pattern will be washed out and we are dealing with Fresnel diffraction. The "blur" that appears on the screen will then have a width of about b. This is clear if you imagine moving the screen closer to the slit (imagine $b = 1$ cm). Then, we expect to see a light spot of width b on the screen. Hence, the diffraction pattern width given by Equation 1 does not hold.

For example, let $\lambda = 500$ nm, $L = 1$ m and $b = 1$ cm. Then the Fraunhofer diffraction width (Equation 1) is $w \approx 60$ μm. However, Equation 1 does not apply. Notice that the coherence relation $L > b^2/\lambda$ dictates that $L > 10^{-4}/5 \times 10^{-7} \approx 200$ m, which is substantially larger than 1 m. The thought that you might see a spot with a width of about 60 μm is absurd! Instead, you will see a spot with a width of about 1 cm (Fresnel diffraction).

You can see now that, for given L, starting at very small values of b, the diffraction pattern will have a width of about $1.2L\lambda/b$. For increasing values of b the spot width will decrease (non-intuitive!). Then a point is reached, for increasing b, when the coherence relation is no longer satisfied and the spot size will have a width of about b and increases as b increases. Thus, the spot size as a function of b has a local minimum. At this minimum, you have approximately achieved the best resolution possible with a pinhole camera. Thus the smallest spot width will appear when

$$\frac{1.2L\lambda}{b} \approx b.$$

Thus, $b = \sqrt{1.2\lambda L} \approx \sqrt{\lambda L}$. Using $\lambda = 500$ nm, $L = 1$ m, the optimal size of the hole $b \approx 0.8$ mm.

Can you think of a way to do an experiment at home to demonstrate this phenomenon?

Problem 11.5 (Bekefi & Barrett 8.9) — Double slit interference

(a) The dielectric slab in the slit effectively changes the optical path length, i.e. it adds an extra phase to the waves that pass through it.

We can imagine an "air plate" of thickness d over slit A and a dielectric plate with the same thickness over slit B. When waves emerge from the plate at A, they have traveled a distance d which is equivalent to a phase angle $2\pi d/\lambda_0$.

The waves that emerge from plate B have traveled a distance d which is equivalent to a phase difference of $2\pi nd/\lambda_0$. Here n is the index of refraction of the dielectric slab. We are given

$$\frac{d}{\lambda_0}(\sqrt{\kappa} - 1) = 5/2.$$

So,

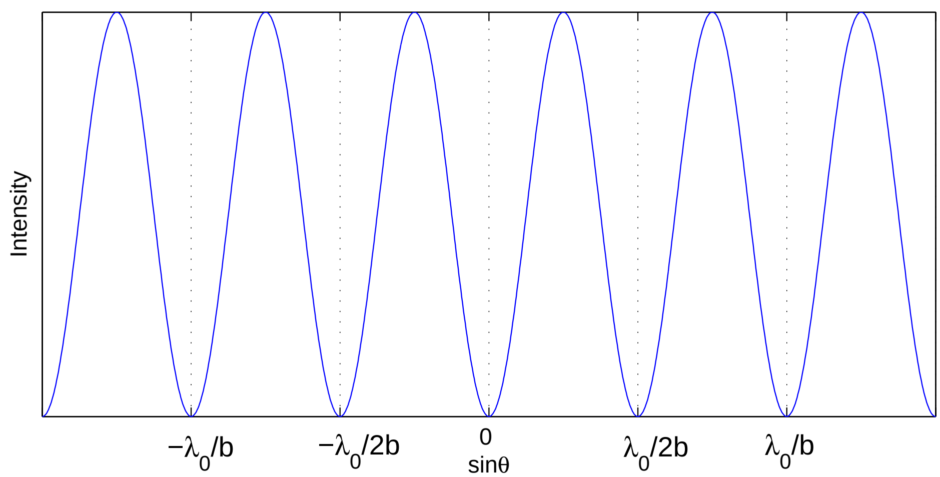
$$d = \frac{5/2\lambda_0}{n-1}, \tag{2}$$

where the index of refraction $n = \sqrt{\kappa}$.

The diffraction pattern depends on the phase difference δ between the waves emerging from the two slits.

$$\delta = 2\pi \frac{d}{\lambda_0}(n-1),$$

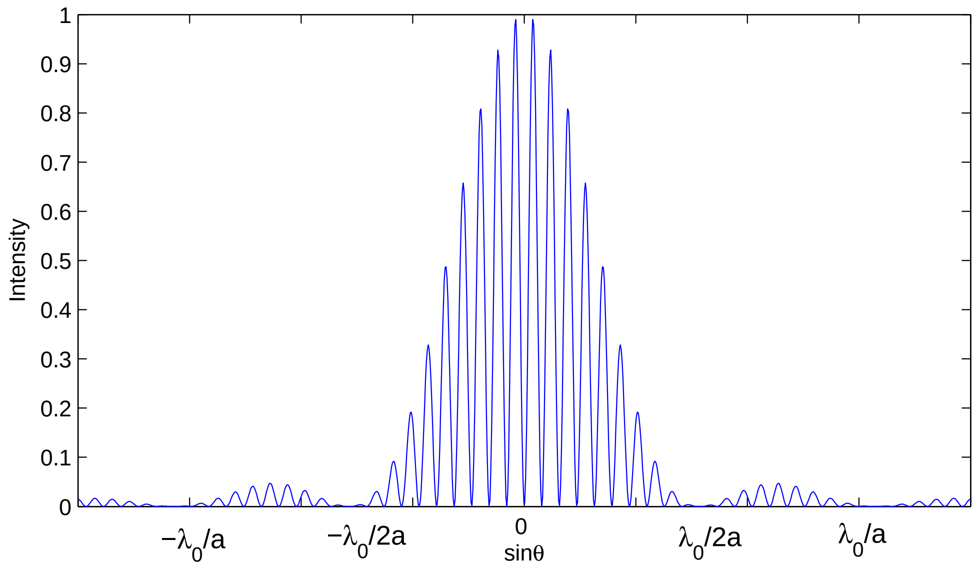
using Equation 2, $\delta = 5\pi$. Thus, the Huygens sources at the 2 slits after traveling the distance d are out of phase by π. Hence, the diffraction pattern has been shifted by π. So there will be a minimum at $\theta = 0$ and there will be maxima at angles θ_m such that $2b\sin\theta_m = (2m+1)\lambda_0/2$. A plot of intensity vs $\sin\theta$ is shown below.



(b) Diffraction causes the interference pattern to be modulated with a term $\sin^2(\beta)/\beta^2$, where $\beta = (2\pi a/\lambda_0)\sin\theta$. Hence, considering interference and diffraction, the intensity pattern is given by

$$I = 4I_0 \left(\frac{\sin\left(\frac{2\pi a}{\lambda_0}\sin\theta\right)}{\frac{2\pi a}{\lambda_0}\sin\theta} \right)^2 \cos^2\left(\frac{2\pi b}{\lambda_0}\sin\theta - \frac{5}{2}\pi\right),$$

where I_0 is the maximum intensity (W/m^2), i.e. the intensity of light if there were only one slit. Note the $5\pi/2$ phase shift in the cosine term due to the dielectric slab. The modulation due to the slit width term produces a first minimum when $(2\pi a/\lambda_0)\sin\theta = \pi$, or $\sin\theta = \lambda_0/2a$. Since $b/a = 10$, $\sin\theta \approx \theta \approx 10\lambda_0/2b = 5\lambda_0/b$. A plot of intensity vs $\sin\theta$ is on the next page.


Intensity
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
$-\lambda_0/a$
$-\lambda_0/2a$
0
$\lambda_0/2a$
λ_0/a
sinθ