Chapter 13

Maxwell's Equations and Electromagnetic Waves

Maxwell's Equations and Electromagnetic Waves

13.1 The Displacement Current

In Chapter 9, we learned that if a current-carrying wire possesses certain symmetry, the magnetic field can be obtained by using Ampere's law:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{enc} \qquad (13.1.1)$$

The equation states that the line integral of a magnetic field around an arbitrary closed loop is equal to $\mu_0 I_{enc}$, where I_{enc} is the conduction current passing through the surface bound by the closed path. In addition, we also learned in Chapter 10 that, as a consequence of the Faraday's law of induction, a changing magnetic field can produce an electric field, according to

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d}{dt} \iint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} \qquad (13.1.2)$$

One might then wonder whether or not the converse could be true, namely, a changing electric field produces a magnetic field. If so, then the right-hand side of Eq. (13.1.1) will have to be modified to reflect such "symmetry" between $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$.

To see how magnetic fields can be created by a time-varying electric field, consider a capacitor which is being charged. During the charging process, the electric field strength increases with time as more charge is accumulated on the plates. The conduction current that carries the charges also produces a magnetic field. In order to apply Ampere's law to calculate this field, let us choose curve *C* shown in Figure 13.1.1 to be the Amperian loop.
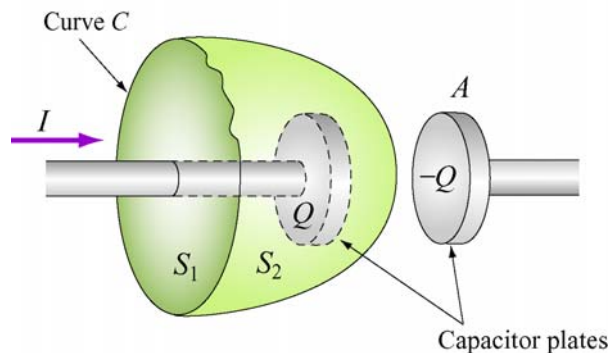

Figure 13.1.1 Surfaces S_1 and S_2 bound by curve *C*.

If the surface bounded by the path is the flat surface S_1, then the enclosed current is $I_{enc} = I$. On the other hand, if we choose S_2 to be the surface bounded by the curve, then $I_{enc} = 0$ since no current passes through S_2. Thus, we see that there exists an ambiguity in choosing the appropriate surface bounded by the curve C.

Maxwell showed that the ambiguity can be resolved by adding to the right-hand side of the Ampere's law an extra term

$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt}$$

(13.1.3)

which he called the "*displacement current*." The term involves a change in electric flux. The generalized Ampere's (or the Ampere-Maxwell) law now reads

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 (I + I_d)$$

(13.1.4)

The origin of the displacement current can be understood as follows:
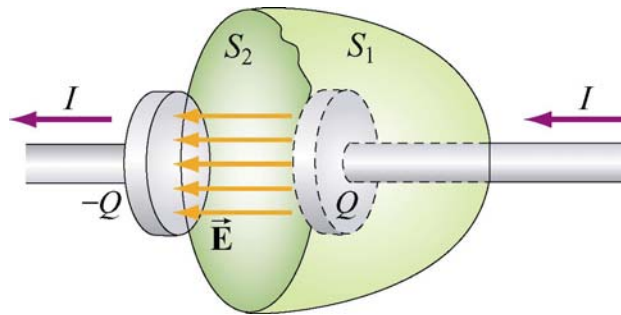
Figure 13.1.2 Displacement through S_2

In Figure 13.1.2, the electric flux which passes through S_2 is given by

$$\Phi_E = \oint\!\!\!\oint_S \vec{E} \cdot d\vec{A} = EA = \frac{Q}{\varepsilon_0}$$

(13.1.5)

where A is the area of the capacitor plates. From Eq. (13.1.3), we readily see that the displacement current I_d is related to the rate of increase of charge on the plate by

$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt} = \frac{dQ}{dt}$$

(13.1.6)

However, the right-hand-side of the expression, dQ/dt, is simply equal to the conduction current, I. Thus, we conclude that the conduction current that passes through S_1 is

precisely equal to the displacement current that passes through S_2, namely $I = I_d$. With the Ampere-Maxwell law, the ambiguity in choosing the surface bound by the Amperian loop is removed.

13.2 Gauss's Law for Magnetism

We have seen that Gauss's law for electrostatics states that the electric flux through a closed surface is proportional to the charge enclosed (Figure 13.2.1a). The electric field lines originate from the positive charge (source) and terminate at the negative charge (sink). One would then be tempted to write down the magnetic equivalent as

$$\Phi_B = \oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = \frac{Q_m}{\mu_0} \qquad (13.2.1)$$

where Q_m is the magnetic charge (monopole) enclosed by the Gaussian surface. However, despite intense search effort, no isolated magnetic monopole has ever been observed. Hence, $Q_m = 0$ and Gauss's law for magnetism becomes

$$\boxed{\Phi_B = \oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0} \qquad (13.2.2)$$
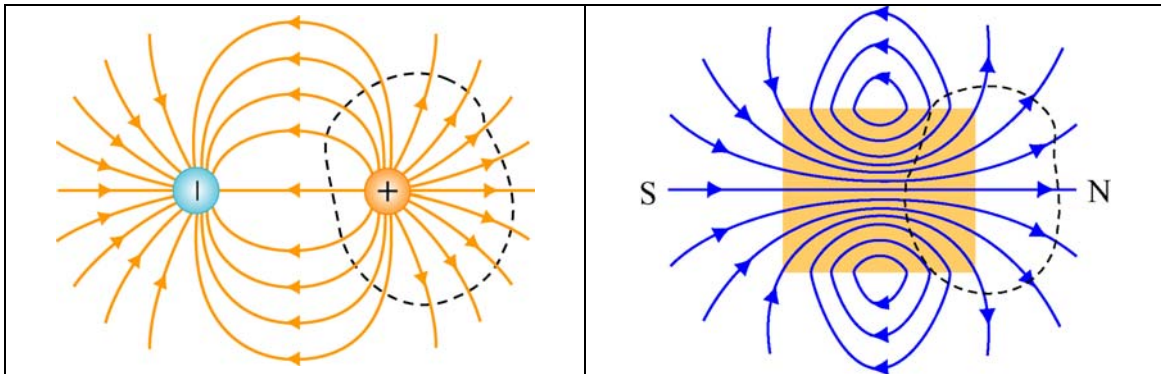
Figure 13.2.1 Gauss's law for (a) electrostatics, and (b) magnetism.

This implies that the number of magnetic field lines entering a closed surface is equal to the number of field lines leaving the surface. That is, there is no source or sink. In addition, the lines must be continuous with no starting or end points. In fact, as shown in Figure 13.2.1(b) for a bar magnet, the field lines that emanate from the north pole to the south pole outside the magnet return within the magnet and form a closed loop.

13.3 Maxwell's Equations

We now have four equations which form the foundation of electromagnetic phenomena:

Law	Equation	Physical Interpretation
Gauss's law for $\vec{\mathbf{E}}$	$\oiint\limits_{S} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \dfrac{Q}{\varepsilon_0}$	Electric flux through a closed surface is proportional to the charged enclosed
Faraday's law	$\oint \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{s}} = -\dfrac{d\Phi_B}{dt}$	Changing magnetic flux produces an electric field
Gauss's law for $\vec{\mathbf{B}}$	$\oiint\limits_{S} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$	The total magnetic flux through a closed surface is zero
Ampere $-$ Maxwell law	$\oint \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = \mu_0 I + \mu_0\varepsilon_0 \dfrac{d\Phi_E}{dt}$	Electric current and changing electric flux produces a magnetic field

Collectively they are known as Maxwell's equations. The above equations may also be written in differential forms as

$$\nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\nabla \cdot \vec{\mathbf{B}} = 0 \tag{13.3.1}$$

$$\nabla \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} + \mu_0\varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

where ρ and $\vec{\mathbf{J}}$ are the free charge and the conduction current densities, respectively. In the absence of sources where $Q = 0$, $I = 0$, the above equations become

$$\oiint\limits_{S} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 0$$

$$\oint \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt}$$

$$\oiint\limits_{S} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0 \tag{13.3.2}$$

$$\oint \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}$$

An important consequence of Maxwell's equations, as we shall see below, is the prediction of the existence of electromagnetic waves that travel with speed of light $c = 1/\sqrt{\mu_0\varepsilon_0}$. The reason is due to the fact that a changing electric field produces a magnetic field and vice versa, and the coupling between the two fields leads to the generation of electromagnetic waves. The prediction was confirmed by H. Hertz in 1887.

13.4 Plane Electromagnetic Waves

To examine the properties of the electromagnetic waves, let's consider for simplicity an electromagnetic wave propagating in the +x-direction, with the electric field $\vec{\mathbf{E}}$ pointing in the +y-direction and the magnetic field $\vec{\mathbf{B}}$ in the +z-direction, as shown in Figure 13.4.1 below.
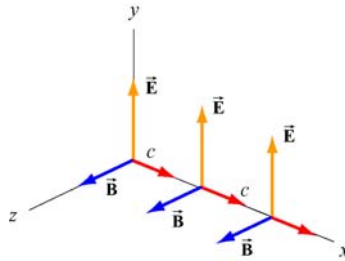
Figure 13.4.1 A plane electromagnetic wave

What we have here is an example of a plane wave since at any instant both $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are uniform over any plane perpendicular to the direction of propagation. In addition, the wave is *transverse* because both fields are perpendicular to the direction of propagation, which points in the direction of the cross product $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$.

Using Maxwell's equations, we may obtain the relationship between the magnitudes of the fields. To see this, consider a rectangular loop which lies in the *xy* plane, with the left side of the loop at x and the right at $x + \Delta x$. The bottom side of the loop is located at y, and the top side of the loop is located at $y + \Delta y$, as shown in Figure 13.4.2. Let the unit vector normal to the loop be in the positive *z*-direction, $\hat{\mathbf{n}} = \hat{\mathbf{k}}$.

Figure 13.4.2 Spatial variation of the electric field $\vec{\mathbf{E}}$

Using Faraday's law

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d}{dt} \iint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} \qquad (13.4.1)$$

the left-hand-side can be written as

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = E_y(x + \Delta x)\Delta y - E_y(x)\Delta y = [E_y(x + \Delta x) - E_y(x)]\Delta y = \frac{\partial E_y}{\partial x}\left(\Delta x\,\Delta y\right) \quad (13.4.2)$$
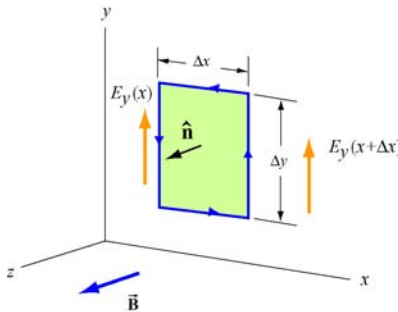
where we have made the expansion

$$E_y(x + \Delta x) = E_y(x) + \frac{\partial E_y}{\partial x}\Delta x + \dots \qquad (13.4.3)$$

On the other hand, the rate of change of magnetic flux on the right-hand-side is given by

$$-\frac{d}{dt}\iint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = -\left(\frac{\partial B_z}{\partial t}\right)\left(\Delta x\,\Delta y\right) \qquad (13.4.4)$$

Equating the two expressions and dividing through by the area $\Delta x \Delta y$ yields

$$\boxed{\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}} \qquad (13.4.5)$$

The second condition on the relationship between the electric and magnetic fields may be deduced by using the Ampere-Maxwell equation:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 \varepsilon_0 \frac{d}{dt}\iint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} \qquad (13.4.6)$$

Consider a rectangular loop in the xz plane depicted in Figure 13.4.3, with a unit normal $\hat{\mathbf{n}} = \hat{\mathbf{j}}$.
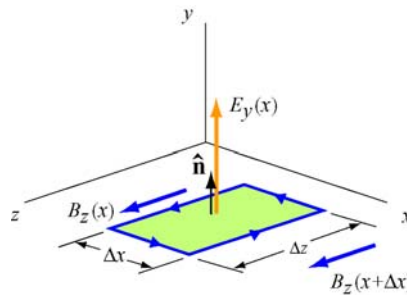

Figure 13.4.3 Spatial variation of the magnetic field $\vec{\mathbf{B}}$

The line integral of the magnetic field is

$$\oint \vec{B} \cdot d\vec{s} = B_z(x)\Delta z - B_z(x+\Delta x)\Delta z = [B_z(x) - B_z(x+\Delta x)]\Delta z$$

$$= -\left(\frac{\partial B_z}{\partial x}\right)(\Delta x \Delta z) \tag{13.4.7}$$

On the other hand, the time derivative of the electric flux is

$$\mu_0\varepsilon_0 \frac{d}{dt}\iint \vec{E} \cdot d\vec{A} = \mu_0\varepsilon_0\left(\frac{\partial E_y}{\partial t}\right)(\Delta x \Delta z) \tag{13.4.8}$$

Equating the two equations and dividing by $\Delta x \Delta z$, we have

$$-\frac{\partial B_z}{\partial x} = \mu_0\varepsilon_0\left(\frac{\partial E_y}{\partial t}\right) \tag{13.4.9}$$

The result indicates that a time-varying electric field is generated by a spatially varying magnetic field.

Using Eqs. (13.4.4) and (13.4.8), one may verify that both the electric and magnetic fields satisfy the one-dimensional wave equation.

To show this, we first take another partial derivative of Eq. (13.4.5) with respect to x, and then another partial derivative of Eq. (13.4.9) with respect to t:

$$\frac{\partial^2 E_y}{\partial x^2} = -\frac{\partial}{\partial x}\left(\frac{\partial B_z}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\frac{\partial B_z}{\partial x}\right) = -\frac{\partial}{\partial t}\left(-\mu_0\varepsilon_0\frac{\partial E_y}{\partial t}\right) = \mu_0\varepsilon_0\frac{\partial^2 E_y}{\partial t^2} \tag{13.4.10}$$

noting the interchangeability of the partial differentiations:

$$\frac{\partial}{\partial x}\left(\frac{\partial B_z}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial B_z}{\partial x}\right) \tag{13.4.11}$$

Similarly, taking another partial derivative of Eq. (13.4.9) with respect to x yields, and then another partial derivative of Eq. (13.4.5) with respect to t gives

$$\frac{\partial^2 B_z}{\partial x^2} = -\frac{\partial}{\partial x}\left(\mu_0\varepsilon_0\frac{\partial E_y}{\partial t}\right) = -\mu_0\varepsilon_0\frac{\partial}{\partial t}\left(\frac{\partial E_y}{\partial x}\right) = -\mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\frac{\partial B_z}{\partial t}\right) = \mu_0\varepsilon_0\frac{\partial^2 B_z}{\partial t^2} \tag{13.4.12}$$

The results may be summarized as:

$$\left(\frac{\partial^2}{\partial x^2} - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\right)\begin{Bmatrix} E_y(x,t) \\ B_z(x,t) \end{Bmatrix} = 0 \tag{13.4.13}$$

Recall that the general form of a one-dimensional wave equation is given by

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \right) \psi(x,t) = 0 \qquad (13.4.14)$$

where v is the speed of propagation and $\psi(x,t)$ is the wave function, we see clearly that both E_y and B_z satisfy the wave equation and propagate with the speed of light:

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{(4\pi \times 10^{-7}\,\mathrm{T \cdot m/A})(8.85 \times 10^{-12}\,\mathrm{C^2/N \cdot m^2})}} = 2.997 \times 10^8\,\mathrm{m/s} = c \quad (13.4.15)$$

Thus, we conclude that light is an electromagnetic wave. The spectrum of electromagnetic waves is shown in Figure 13.4.4.
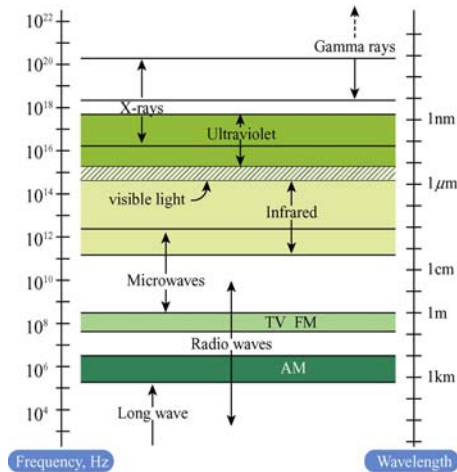
Figure 13.4.4 Electromagnetic spectrum

13.4.1 One-Dimensional Wave Equation

It is straightforward to verify that any function of the form $\psi(x \pm vt)$ satisfies the one-dimensional wave equation shown in Eq. (13.4.14). The proof proceeds as follows:

Let $x' = x \pm vt$ which yields $\partial x'/\partial x = 1$ and $\partial x'/\partial t = \pm v$. Using chain rule, the first two partial derivatives with respect to x are

$$\frac{\partial \psi(x')}{\partial x} = \frac{\partial \psi}{\partial x'} \frac{\partial x'}{\partial x} = \frac{\partial \psi}{\partial x'} \qquad (13.4.16)$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x'}\right) = \frac{\partial^2 \psi}{\partial x'^2}\frac{\partial x'}{\partial x} = \frac{\partial^2 \psi}{\partial x'^2} \tag{13.4.17}$$

Similarly, the partial derivatives in t are given by

$$\frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial x'}\frac{\partial x'}{\partial t} = \pm v\frac{\partial \psi}{\partial x'} \tag{13.4.18}$$

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial}{\partial t}\left(\pm v\frac{\partial \psi}{\partial x'}\right) = \pm v\frac{\partial^2 \psi}{\partial x'^2}\frac{\partial x'}{\partial t} = v^2\frac{\partial^2 \psi}{\partial x'^2} \tag{13.4.19}$$

Comparing Eq. (13.4.17) with Eq. (13.4.19), we have

$$\frac{\partial^2 \psi}{\partial x'^2} = \frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} \tag{13.4.20}$$

which shows that $\psi(x\pm vt)$ satisfies the one-dimensional wave equation. The wave equation is an example of a linear differential equation, which means that if $\psi_1(x,t)$ and $\psi_2(x,t)$ are solutions to the wave equation, then $\psi_1(x,t)\pm\psi_2(x,t)$ is also a solution. The implication is that electromagnetic waves obey the superposition principle.

One possible solution to the wave equations is

$$\begin{aligned}\vec{\mathbf{E}} &= E_y(x,t)\hat{\mathbf{j}} = E_0\cos k(x-vt)\hat{\mathbf{j}} = E_0\cos(kx-\omega t)\hat{\mathbf{j}}\\ \vec{\mathbf{B}} &= B_z(x,t)\hat{\mathbf{k}} = B_0\cos k(x-vt)\hat{\mathbf{k}} = B_0\cos(kx-\omega t)\hat{\mathbf{k}}\end{aligned} \tag{13.4.21}$$

where the fields are sinusoidal, with amplitudes E_0 and B_0. The angular wave number k is related to the wavelength λ by

$$k = \frac{2\pi}{\lambda} \tag{13.4.22}$$

and the angular frequency ω is

$$\omega = kv = 2\pi\frac{v}{\lambda} = 2\pi f \tag{13.4.23}$$

where f is the linear frequency. In empty space the wave propagates at the speed of light, $v = c$. The characteristic behavior of the sinusoidal electromagnetic wave is illustrated in Figure 13.4.5.
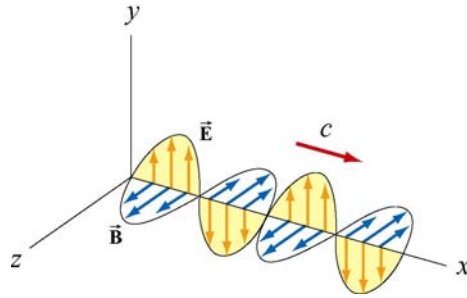
Figure 13.4.5 Plane electromagnetic wave propagating in the +*x* direction.

We see that the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields are always in phase (attaining maxima and minima at the same time.) To obtain the relationship between the field amplitudes E_0 and B_0, we make use of Eqs. (13.4.4) and (13.4.8). Taking the partial derivatives leads to

$$\frac{\partial E_y}{\partial x} = -kE_0 \sin(kx - \omega t) \tag{13.4.24}$$

and

$$\frac{\partial B_z}{\partial t} = \omega B_0 \sin(kx - \omega t) \tag{13.4.25}$$

which implies $E_0 k = \omega B_0$, or

$$\boxed{\frac{E_0}{B_0} = \frac{\omega}{k} = c} \tag{13.4.26}$$

From Eqs. (13.4.20) and (13.4.21), one may easily show that the magnitudes of the fields at any instant are related by

$$\frac{E}{B} = c \tag{13.4.27}$$

Let us summarize the important features of electromagnetic waves described in Eq. (13.4.21):

1. The wave is transverse since both $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields are perpendicular to the direction of propagation, which points in the direction of the cross product $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$.

2. The $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields are perpendicular to each other. Therefore, their dot product vanishes, $\vec{\mathbf{E}} \cdot \vec{\mathbf{B}} = 0$.

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3. The ratio of the magnitudes and the amplitudes of the fields is

$$\frac{E}{B} = \frac{E_0}{B_0} = \frac{\omega}{k} = c$$

4. The speed of propagation in vacuum is equal to the speed of light, $c = 1/\sqrt{\mu_0 \varepsilon_0}$.

5. Electromagnetic waves obey the superposition principle.

13.5 Standing Electromagnetic Waves

Let us examine the situation where there are two sinusoidal plane electromagnetic waves, one traveling in the $+x$-direction, with

$$E_{1y}(x,t) = E_{10}\cos(k_1 x - \omega_1 t), \qquad B_{1z}(x,t) = B_{10}\cos(k_1 x - \omega_1 t) \qquad (13.5.1)$$

and the other traveling in the $-x$-direction, with

$$E_{2y}(x,t) = -E_{20}\cos(k_2 x + \omega_2 t), \qquad B_{2z}(x,t) = B_{20}\cos(k_2 x + \omega_2 t) \qquad (13.5.2)$$

For simplicity, we assume that these electromagnetic waves have the same amplitudes ($E_{10} = E_{20} = E_0$, $B_{10} = B_{20} = B_0$) and wavelengths ($k_1 = k_2 = k$, $\omega_1 = \omega_2 = \omega$). Using the superposition principle, the electric field and the magnetic fields can be written as

$$E_y(x,t) = E_{1y}(x,t) + E_{2y}(x,t) = E_0\left[\cos(kx - \omega t) - \cos(kx + \omega t)\right] \qquad (13.5.3)$$

and

$$B_z(x,t) = B_{1z}(x,t) + B_{2z}(x,t) = B_0\left[\cos(kx - \omega t) + \cos(kx + \omega t)\right] \qquad (13.5.4)$$

Using the identities

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \qquad (13.5.5)$$

The above expressions may be rewritten as

$$\begin{aligned}
E_y(x,t) &= E_0\left[\cos kx \cos \omega t + \sin kx \sin \omega t - \cos kx \cos \omega t + \sin kx \sin \omega t\right] \\
&= 2E_0 \sin kx \sin \omega t
\end{aligned} \qquad (13.5.6)$$

and

$$B_z(x,t) = B_0 \left[\cos kx \cos \omega t + \sin kx \sin \omega t + \cos kx \cos \omega t - \sin kx \sin \omega t \right]$$
$$= 2B_0 \cos kx \cos \omega t \tag{13.5.7}$$

One may verify that the total fields $E_y(x,t)$ and $B_z(x,t)$ still satisfy the wave equation stated in Eq. (13.4.13), even though they no longer have the form of functions of $kx \pm \omega t$. The waves described by Eqs. (13.5.6) and (13.5.7) are *standing waves*, which do not propagate but simply oscillate in space and time.

Let's first examine the spatial dependence of the fields. Eq. (13.5.6) shows that the total electric field remains zero at all times if $\sin kx = 0$, or

$$x = \frac{n\pi}{k} = \frac{n\pi}{2\pi / \lambda} = \frac{n\lambda}{2}, \quad n = 0,1,2,\ldots \quad \text{(nodal planes of } \vec{E}) \tag{13.5.8}$$

The planes that contain these points are called the *nodal planes* of the electric field. On the other hand, when $\sin kx = \pm 1$, or

$$x = \left(n + \frac{1}{2} \right) \frac{\pi}{k} = \left(n + \frac{1}{2} \right) \frac{\pi}{2\pi / \lambda} = \left(\frac{n}{2} + \frac{1}{4} \right) \lambda, \quad n = 0,1,2,\ldots \quad \text{(anti-nodal planes of } \vec{E})$$
$$\tag{13.5.9}$$

the amplitude of the field is at its maximum $2E_0$. The planes that contain these points are the *anti-nodal planes* of the electric field. Note that in between two nodal planes, there is an anti-nodal plane, and vice versa.

For the magnetic field, the nodal planes must contain points which meets the condition $\cos kx = 0$. This yields

$$x = \left(n + \frac{1}{2} \right) \frac{\pi}{k} = \left(\frac{n}{2} + \frac{1}{4} \right) \lambda, \quad n = 0,1,2,\ldots \quad \text{(nodal planes of } \vec{B}) \tag{13.5.10}$$

Similarly, the anti-nodal planes for $\vec{B}$ contain points that satisfy $\cos kx = \pm 1$, or

$$x = \frac{n\pi}{k} = \frac{n\pi}{2\pi / \lambda} = \frac{n\lambda}{2}, \quad n = 0,1,2,\ldots \quad \text{(anti-nodal planes of } \vec{B}) \tag{13.5.11}$$

Thus, we see that a nodal plane of $\vec{E}$ corresponds to an anti-nodal plane of $\vec{B}$, and vice versa.

For the time dependence, Eq. (13.5.6) shows that the electric field is zero everywhere when $\sin \omega t = 0$, or

$$t = \frac{n\pi}{\omega} = \frac{n\pi}{2\pi/T} = \frac{nT}{2}, \quad n = 0, 1, 2, \ldots \tag{13.5.12}$$

where $T = 1/f = 2\pi/\omega$ is the period. However, this is precisely the maximum condition for the magnetic field. Thus, unlike the traveling electromagnetic wave in which the electric and the magnetic fields are always in phase, in standing electromagnetic waves, the two fields are $90°$ out of phase.

Standing electromagnetic waves can be formed by confining the electromagnetic waves within two perfectly reflecting conductors, as shown in Figure 13.4.6.
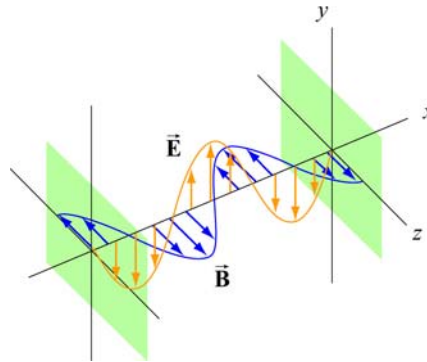
Figure 13.4.6 Formation of standing electromagnetic waves using two perfectly reflecting conductors.

13.6 Poynting Vector

In Chapters 5 and 11 we had seen that electric and magnetic fields store energy. Thus, energy can also be carried by the electromagnetic waves which consist of both fields. Consider a plane electromagnetic wave passing through a small volume element of area A and thickness dx, as shown in Figure 13.6.1.
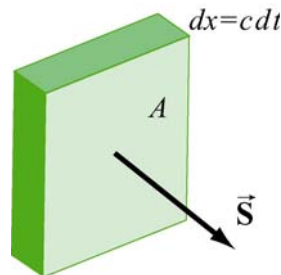
Figure 13.6.1 Electromagnetic wave passing through a volume element

The total energy in the volume element is given by

$$dU = uA\,dx = (u_E + u_B)A\,dx = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)A\,dx \qquad (13.6.1)$$

where

$$u_E = \frac{1}{2}\varepsilon_0 E^2, \qquad u_B = \frac{B^2}{2\mu_0} \qquad (13.6.2)$$

are the energy densities associated with the electric and magnetic fields. Since the electromagnetic wave propagates with the speed of light c, the amount of time it takes for the wave to move through the volume element is $dt = dx/c$. Thus, one may obtain the rate of change of energy per unit area, denoted with the symbol S, as

$$S = \frac{dU}{A\,dt} = \frac{c}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) \qquad (13.6.3)$$

The SI unit of S is W/m^2. Noting that $E = cB$ and $c = 1/\sqrt{\mu_0 \varepsilon_0}$, the above expression may be rewritten as

$$S = \frac{c}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) = \frac{cB^2}{\mu_0} = c\varepsilon_0 E^2 = \frac{EB}{\mu_0} \qquad (13.6.4)$$

In general, the rate of the energy flow per unit area may be described by the Poynting vector $\vec{\mathbf{S}}$ (after the British physicist John Poynting), which is defined as

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}} \qquad (13.6.5)$$

with $\vec{\mathbf{S}}$ pointing in the direction of propagation. Since the fields $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are perpendicular, we may readily verify that the magnitude of $\vec{\mathbf{S}}$ is

$$|\vec{\mathbf{S}}| = \frac{\left|\vec{\mathbf{E}} \times \vec{\mathbf{B}}\right|}{\mu_0} = \frac{EB}{\mu_0} = S \qquad (13.6.6)$$

As an example, suppose the electric component of the plane electromagnetic wave is $\vec{\mathbf{E}} = E_0 \cos(kx - \omega t)\hat{\mathbf{j}}$. The corresponding magnetic component is $\vec{\mathbf{B}} = B_0 \cos(kx - \omega t)\hat{\mathbf{k}}$, and the direction of propagation is $+x$. The Poynting vector can be obtained as

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\left(E_0 \cos(kx - \omega t)\hat{\mathbf{j}}\right) \times \left(B_0 \cos(kx - \omega t)\hat{\mathbf{k}}\right) = \frac{E_0 B_0}{\mu_0}\cos^2(kx - \omega t)\hat{\mathbf{i}} \qquad (13.6.7)$$
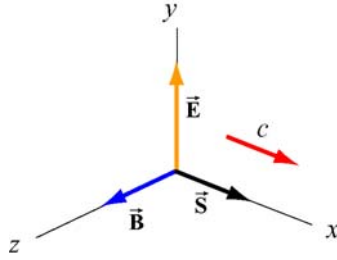
Figure 13.6.2 Poynting vector for a plane wave

As expected, $\vec{\mathbf{S}}$ points in the direction of wave propagation (see Figure 13.6.2).

The intensity of the wave, I, defined as the time average of S, is given by

$$I = \langle S \rangle = \frac{E_0 B_0}{\mu_0} \langle \cos^2(kx - \omega t) \rangle = \frac{E_0 B_0}{2\mu_0} = \frac{E_0^2}{2c\mu_0} = \frac{cB_0^2}{2\mu_0} \qquad (13.6.8)$$

where we have used

$$\langle \cos^2(kx - \omega t) \rangle = \frac{1}{2} \qquad (13.6.9)$$

To relate intensity to the energy density, we first note the equality between the electric and the magnetic energy densities:

$$u_B = \frac{B^2}{2\mu_0} = \frac{(E/c)^2}{2\mu_0} = \frac{E^2}{2c^2\mu_0} = \frac{\varepsilon_0 E^2}{2} = u_E \qquad (13.6.10)$$

The average total energy density then becomes

$$\langle u \rangle = \langle u_E + u_B \rangle = \varepsilon_0 \langle E^2 \rangle = \frac{\varepsilon_0}{2} E_0^2$$

$$= \frac{1}{\mu_0} \langle B^2 \rangle = \frac{B_0^2}{2\mu_0} \qquad (13.6.11)$$

Thus, the intensity is related to the average energy density by

$$I = \langle S \rangle = c \langle u \rangle \qquad (13.6.12)$$

Example 13.1: Solar Constant

At the upper surface of the Earth's atmosphere, the time-averaged magnitude of the Poynting vector, $\langle S \rangle = 1.35 \times 10^3 \ \mathrm{W/m^2}$, is referred to as the *solar constant*.

(a) Assuming that the Sun's electromagnetic radiation is a plane sinusoidal wave, what are the magnitudes of the electric and magnetic fields?

(b) What is the total time-averaged power radiated by the Sun? The mean Sun-Earth distance is $R = 1.50 \times 10^{11}\,\text{m}$.

Solution:

(a) The time-averaged Poynting vector is related to the amplitude of the electric field by

$$\langle S \rangle = \frac{c}{2} \varepsilon_0 E_0^2 \,.$$

Thus, the amplitude of the electric field is

$$E_0 = \sqrt{\frac{2\langle S \rangle}{c\varepsilon_0}} = \sqrt{\frac{2\left(1.35 \times 10^3\,\text{W}/\text{m}^2\right)}{\left(3.0 \times 10^8\,\text{m}/\text{s}\right)\left(8.85 \times 10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)}} = 1.01 \times 10^3\,\text{V}/\text{m} \,.$$

The corresponding amplitude of the magnetic field is

$$B_0 = \frac{E_0}{c} = \frac{1.01 \times 10^3\,\text{V}/\text{m}}{3.0 \times 10^8\,\text{m}/\text{s}} = 3.4 \times 10^{-6}\,\text{T} \,.$$

Note that the associated magnetic field is less than one-tenth the Earth's magnetic field.

(b) The total time averaged power radiated by the Sun at the distance R is

$$\langle P \rangle = \langle S \rangle A = \langle S \rangle 4\pi R^2 = \left(1.35 \times 10^3\,\text{W}/\text{m}^2\right) 4\pi \left(1.50 \times 10^{11}\,\text{m}\right)^2 = 3.8 \times 10^{26}\,\text{W}$$

The type of wave discussed in the example above is a spherical wave (Figure 13.6.3a), which originates from a "point-like" source. The intensity at a distance r from the source is

$$I = \langle S \rangle = \frac{\langle P \rangle}{4\pi r^2} \qquad (13.6.13)$$

which decreases as $1/r^2$. On the other hand, the intensity of a plane wave (Figure 13.6.3b) remains constant and there is no spreading in its energy.
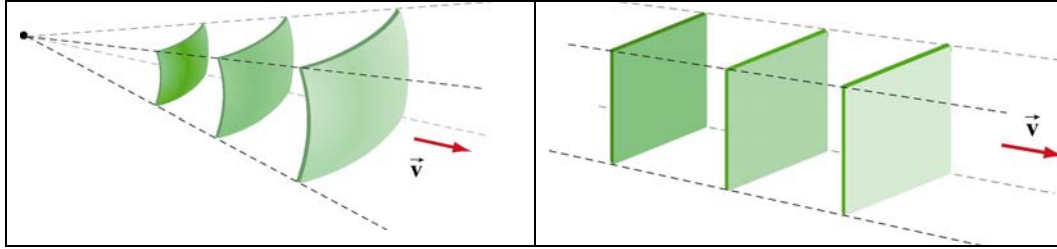
Figure 13.6.3 (a) a spherical wave, and (b) plane wave.

Example 13.2: Intensity of a Standing Wave

Compute the intensity of the standing electromagnetic wave given by

$$E_y(x,t) = 2E_0 \cos kx \cos \omega t, \quad B_z(x,t) = 2B_0 \sin kx \sin \omega t$$

Solution:

The Poynting vector for the standing wave is

$$
\begin{aligned}
\vec{\mathbf{S}} &= \frac{\vec{\mathbf{E}} \times \vec{\mathbf{B}}}{\mu_0} = \frac{1}{\mu_0}(2E_0 \cos kx \cos \omega t\, \hat{\mathbf{j}}) \times (2B_0 \sin kx \sin \omega t\, \hat{\mathbf{k}}) \\
&= \frac{4E_0 B_0}{\mu_0}(\sin kx \cos kx \sin \omega t \cos \omega t)\hat{\mathbf{i}} \\
&= \frac{E_0 B_0}{\mu_0}(\sin 2kx \sin 2\omega t)\hat{\mathbf{i}}
\end{aligned}
\tag{13.6.14}
$$

The time average of S is

$$\langle S \rangle = \frac{E_0 B_0}{\mu_0} \sin 2kx \langle \sin 2\omega t \rangle = 0 \tag{13.6.15}$$

The result is to be expected since the standing wave does not propagate. Alternatively, we may say that the energy carried by the two waves traveling in the opposite directions to form the standing wave exactly cancel each other, with no net energy transfer.

13.6.1 Energy Transport

Since the Poynting vector $\vec{\mathbf{S}}$ represents the rate of the energy flow per unit area, the rate of change of energy in a system can be written as

$$\boxed{\frac{dU}{dt} = -\oiint \vec{\mathbf{S}} \cdot d\vec{\mathbf{A}}} \tag{13.6.16}$$

where $d\vec{\mathbf{A}} = dA\,\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector in the *outward* normal direction. The above expression allows us to interpret $\vec{\mathbf{S}}$ as the energy flux density, in analogy to the current density $\vec{\mathbf{J}}$ in

$$I = \frac{dQ}{dt} = \iint \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} \tag{13.6.17}$$

If energy flows out of the system, then $\vec{\mathbf{S}} = S\,\hat{\mathbf{n}}$ and $dU/dt < 0$, showing an overall decrease of energy in the system. On the other hand, if energy flows into the system, then $\vec{\mathbf{S}} = S(-\hat{\mathbf{n}})$ and $dU/dt > 0$, indicating an overall increase of energy.

As an example to elucidate the physical meaning of the above equation, let's consider an inductor made up of a section of a very long air-core solenoid of length l, radius r and n turns per unit length. Suppose at some instant the current is changing at a rate $dI/dt > 0$. Using Ampere's law, the magnetic field in the solenoid is

$$\oint_C \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = Bl = \mu_0(NI)$$

or

$$\vec{\mathbf{B}} = \mu_0 nI\,\hat{\mathbf{k}} \tag{13.6.18}$$

Thus, the rate of increase of the magnetic field is

$$\frac{dB}{dt} = \mu_0 n \frac{dI}{dt} \tag{13.6.19}$$

According to Faraday's law:

$$\varepsilon = \oint_C \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt} \tag{13.6.20}$$

changing magnetic flux results in an induced electric field., which is given by

$$E(2\pi r) = -\mu_0 n\left(\frac{dI}{dt}\right)\pi r^2$$

or

$$\vec{\mathbf{E}} = -\frac{\mu_0 nr}{2}\left(\frac{dI}{dt}\right)\hat{\boldsymbol{\varphi}} \tag{13.6.21}$$

The direction of $\vec{E}$ is clockwise, the same as the induced current, as shown in Figure 13.6.4.
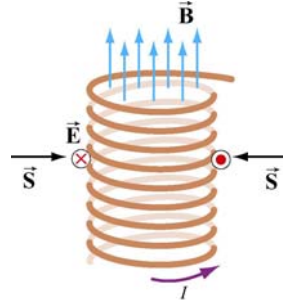

Figure 13.6.4 Poynting vector for a solenoid with $dI / dt > 0$

The corresponding Poynting vector can then be obtained as

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} = \frac{1}{\mu_0} \left[-\frac{\mu_0 nr}{2} \left(\frac{dI}{dt} \right) \hat{\boldsymbol{\varphi}} \right] \times \left(\mu_0 nI \, \hat{\mathbf{k}} \right) = -\frac{\mu_0 n^2 rI}{2} \left(\frac{dI}{dt} \right) \hat{\mathbf{r}} \qquad (13.6.22)$$

which points radially inward, i.e., along the $-\hat{\mathbf{r}}$ direction. The directions of the fields and the Poynting vector are shown in Figure 13.6.4.

Since the magnetic energy stored in the inductor is

$$U_B = \left(\frac{B^2}{2\mu_0} \right) (\pi r^2 l) = \frac{1}{2} \mu_0 \pi n^2 I^2 r^2 l \qquad (13.6.23)$$

the rate of change of U_B is

$$P = \frac{dU_B}{dt} = \mu_0 \pi n^2 I r^2 l \left(\frac{dI}{dt} \right) = I \, |\varepsilon| \qquad (13.6.24)$$

where

$$\varepsilon = -N \frac{d\Phi_B}{dt} = -(nl) \left(\frac{dB}{dt} \right) \pi r^2 = -\mu_0 n^2 l \pi r^2 \left(\frac{dI}{dt} \right) \qquad (13.6.25)$$

is the induced emf. One may readily verify that this is the same as

$$-\oint \vec{S} \cdot d\vec{A} = \frac{\mu_0 n^2 rI}{2} \left(\frac{dI}{dt} \right) \cdot (2\pi rl) = \mu_0 \pi n^2 I r^2 l \left(\frac{dI}{dt} \right) \qquad (13.6.26)$$

Thus, we have

$$\frac{dU_B}{dt} = -\oint \vec{\mathbf{S}} \cdot d\vec{\mathbf{A}} > 0 \tag{13.6.27}$$

The energy in the system is increased, as expected when $dI/dt > 0$. On the other hand, if $dI/dt < 0$, the energy of the system would decrease, with $dU_B/dt < 0$.

13.7 Momentum and Radiation Pressure

The electromagnetic wave transports not only energy but also momentum, and hence can exert a *radiation pressure* on a surface due to the absorption and reflection of the momentum. Maxwell showed that if the plane electromagnetic wave is completely absorbed by a surface, the momentum transferred is related to the energy absorbed by

$$\Delta p = \frac{\Delta U}{c} \quad \text{(complete absorption)} \tag{13.7.1}$$

On the other hand, if the electromagnetic wave is completely reflected by a surface such as a mirror, the result becomes

$$\Delta p = \frac{2\Delta U}{c} \quad \text{(complete reflection)} \tag{13.7.2}$$

For the complete absorption case, the average radiation pressure (force per unit area) is given by

$$P = \frac{\langle F \rangle}{A} = \frac{1}{A}\left\langle \frac{dp}{dt} \right\rangle = \frac{1}{Ac}\left\langle \frac{dU}{dt} \right\rangle \tag{13.7.3}$$

Since the rate of energy delivered to the surface is

$$\left\langle \frac{dU}{dt} \right\rangle = \langle S \rangle A = IA$$

we arrive at

$$P = \frac{I}{c} \quad \text{(complete absorption)} \tag{13.7.4}$$

Similarly, if the radiation is completely reflected, the radiation pressure is twice as great as the case of complete absorption:

$$P = \frac{2I}{c} \quad \text{(complete reflection)} \tag{13.7.5}$$

13.8 Production of Electromagnetic Waves

Electromagnetic waves are produced when electric charges are accelerated. In other words, a charge must radiate energy when it undergoes acceleration. Radiation cannot be produced by stationary charges or steady currents. Figure 13.8.1 depicts the electric field lines produced by an oscillating charge at some instant.
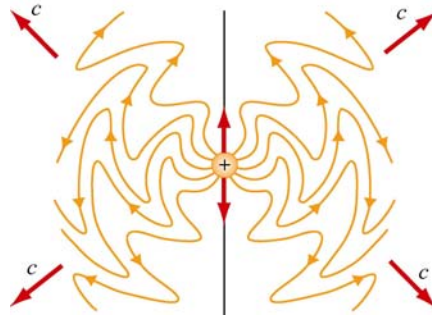
Figure 13.8.1 Electric field lines of an oscillating point charge

A common way of producing electromagnetic waves is to apply a sinusoidal voltage source to an antenna, causing the charges to accumulate near the tips of the antenna. The effect is to produce an oscillating electric dipole. The production of electric-dipole radiation is depicted in Figure 13.8.2.
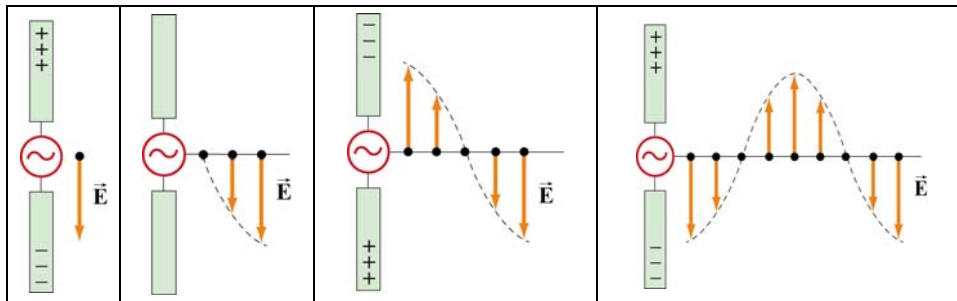
Figure 13.8.2 Electric fields produced by an electric-dipole antenna.

At time $t=0$ the ends of the rods are charged so that the upper rod has a maximum positive charge and the lower rod has an equal amount of negative charge. At this instant the electric field near the antenna points downward. The charges then begin to decrease. After one-fourth period, $t=T/4$, the charges vanish momentarily and the electric field strength is zero. Subsequently, the polarities of the rods are reversed with negative charges continuing to accumulate on the upper rod and positive charges on the lower until $t=T/2$, when the maximum is attained. At this moment, the electric field near the rod points upward. As the charges continue to oscillate between the rods, electric fields are produced and move away with speed of light. The motion of the charges also produces a current which in turn sets up a magnetic field encircling the rods. However, the behavior

of the fields near the antenna is expected to be very different from that far away from the antenna.

Let us consider a half-wavelength antenna, in which the length of each rod is equal to one quarter of the wavelength of the emitted radiation. Since charges are driven to oscillate back and forth between the rods by the alternating voltage, the antenna may be approximated as an oscillating electric dipole. Figure 13.8.3 depicts the electric and the magnetic field lines at the instant the current is upward. Notice that the Poynting vectors at the positions shown are directed outward.
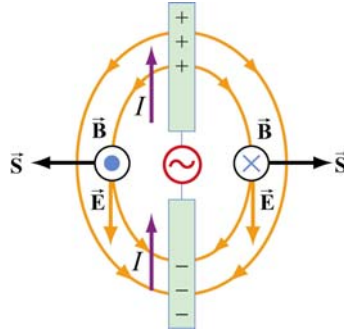

Figure 13.8.3 Electric and magnetic field lines produced by an electric-dipole antenna.

In general, the radiation pattern produced is very complex. However, at a distance which is much greater than the dimensions of the system and the wavelength of the radiation, the fields exhibit a very different behavior. In this "far region," the radiation is caused by the continuous induction of a magnetic field due to a time-varying electric field and vice versa. Both fields oscillate in phase and vary in amplitude as $1/r$.

The intensity of the variation can be shown to vary as $\sin^2\theta/r^2$, where θ is the angle measured from the axis of the antenna. The angular dependence of the intensity $I(\theta)$ is shown in Figure 13.8.4. From the figure, we see that the intensity is a maximum in a plane which passes through the midpoint of the antenna and is perpendicular to it.
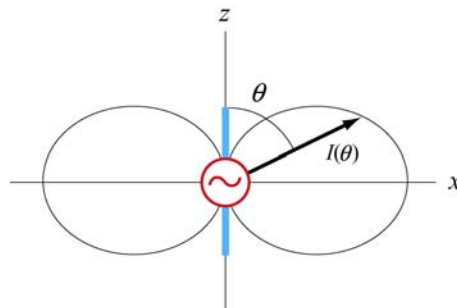

Figure 13.8.4 Angular dependence of the radiation intensity.

Animation 13.1: Electric Dipole Radiation 1

Consider an electric dipole whose dipole moment varies in time according to

$$\vec{\mathbf{p}}(t) = p_0 \left[1 + \frac{1}{10} \cos\left(\frac{2\pi t}{T} \right) \right] \hat{\mathbf{k}} \qquad (13.8.1)$$

Figure 13.8.5 shows one frame of an animation of these fields. Close to the dipole, the field line motion and thus the Poynting vector is first outward and then inward, corresponding to energy flow outward as the quasi-static dipolar electric field energy is being built up, and energy flow inward as the quasi-static dipole electric field energy is being destroyed.
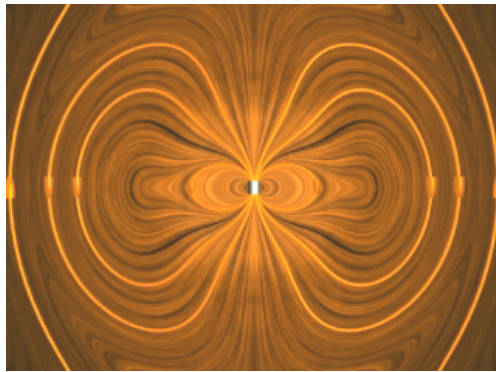
Figure 13.8.5 Radiation from an electric dipole whose dipole moment varies by 10%.

Even though the energy flow direction changes sign in these regions, there is still a small time-averaged energy flow outward. This small energy flow outward represents the small amount of energy radiated away to infinity. Outside of the point at which the outer field lines detach from the dipole and move off to infinity, the velocity of the field lines, and thus the direction of the electromagnetic energy flow, is always outward. This is the region dominated by radiation fields, which consistently carry energy outward to infinity.

Animation 13.2: Electric Dipole Radiation 2

Figure 13.8.6 shows one frame of an animation of an electric dipole characterized by

$$\vec{\mathbf{p}}(t) = p_0 \cos\left(\frac{2\pi t}{T} \right) \hat{\mathbf{k}} \qquad (13.8.2)$$

The equation shows that the direction of the dipole moment varies between $+\hat{\mathbf{k}}$ and $-\hat{\mathbf{k}}$.

Figure 13.8.6 Radiation from an electric dipole whose dipole moment completely reverses with time.

Figure 13.8.7(a) shows the radiation pattern at one instant of time from a quarter-wave antenna. Figure 13.8.7(b) shows this radiation pattern in a plane over the full period of the radiation. A quarter-wave antenna produces radiation whose wavelength is twice the tip to tip length of the antenna. This is evident in the animation of Figure 13.8.7(b).
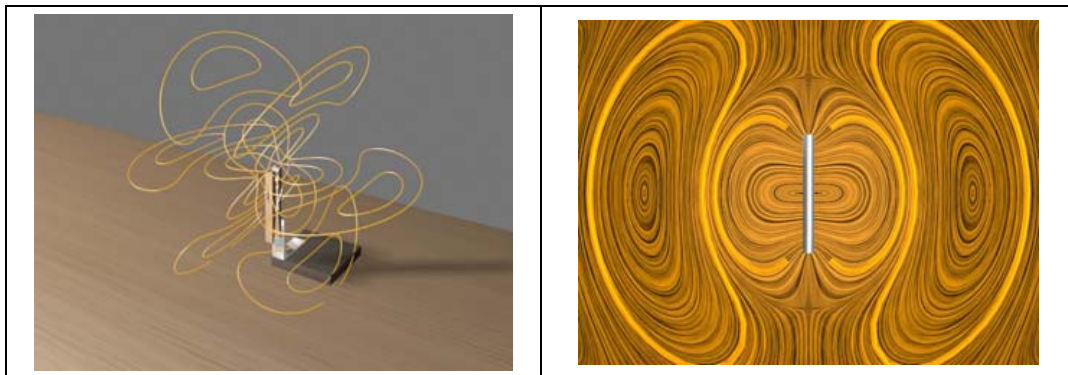
Figure 13.8.7 Radiation pattern from a quarter-wave antenna: (a) The azimuthal pattern at one instant of time, and (b) the radiation pattern in one plane over the full period.

13.8.1 Plane Waves

We have seen that electromagnetic plane waves propagate in empty space at the speed of light. Below we demonstrate how one would create such waves in a particularly simple planar geometry. Although physically this is not particularly applicable to the real world, it is reasonably easy to treat, and we can see directly how electromagnetic plane waves are generated, *why it takes work to make them*, and how much energy they carry away with them.

To make an electromagnetic plane wave, we do much the same thing we do when we make waves on a string. We grab the string somewhere and shake it, and thereby

generate a wave on the string. We do work against the tension in the string when we shake it, and that work is carried off as an energy flux in the wave. Electromagnetic waves are much the same proposition. The electric field line serves as the "string." As we will see below, there is a tension associated with an electric field line, in that when we shake it (try to displace it from its initial position), there is a restoring force that resists the shake, and a wave propagates along the field line as a result of the shake. To understand in detail what happens in this process will involve using most of the electromagnetism we have learned thus far, from Gauss's law to Ampere's law plus the reasonable assumption that electromagnetic information propagates at speed c in a vacuum.

How do we shake an electric field line, and what do we grab on to? What we do is shake the electric charges that the field lines are attached to. After all, it is these charges that produce the electric field, and in a very real sense the electric field is "rooted" in the electric charges that produce them. Knowing this, and assuming that in a vacuum, electromagnetic signals propagate at the speed of light, we can pretty much puzzle out how to make a plane electromagnetic wave by shaking charges. Let's first figure out how to make a *kink* in an electric field line, and then we'll go on to make sinusoidal waves.

Suppose we have an infinite sheet of charge located in the yz-plane, initially at rest, with surface charge density σ, as shown in Figure 13.8.8.
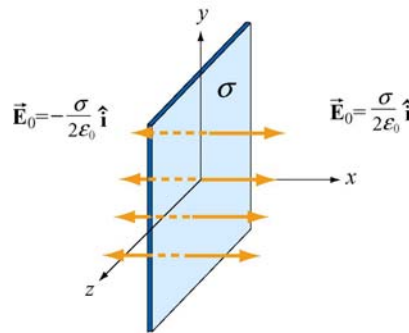
Figure 13.8.8 Electric field due to an infinite sheet with charge density σ.

From Gauss's law discussed in Chapter 4, we know that this surface charge will give rise to a static electric field $\vec{\mathbf{E}}_0$:

$$\vec{\mathbf{E}}_0 = \begin{cases} +(\sigma/2\varepsilon_0)\hat{\mathbf{i}}, & x > 0 \\ -(\sigma/2\varepsilon_0)\hat{\mathbf{i}}, & x < 0 \end{cases} \tag{13.8.3}$$

Now, at $t = 0$, we grab the sheet of charge and start pulling it *downward* with *constant* velocity $\vec{\mathbf{v}} = -v\hat{\mathbf{j}}$. Let's examine how things will then appear at a later time $t = T$. In particular, before the sheet starts moving, let's look at the field line that goes through $y = 0$ for $t < 0$, as shown in Figure 13.8.9(a).

Figure 13.8.9 Electric field lines (a) through $y = 0$ at $t < 0$, and (b) at $t = T$.

The "foot" of this electric field line, that is, where it is anchored, is rooted in the electric charge that generates it, and that "foot" must move downward with the sheet of charge, at the same speed as the charges move downward. Thus the "foot" of our electric field line, which was initially at $y = 0$ at $t = 0$, will have moved a distance $y = -vT$ down the y-axis at time $t = T$.

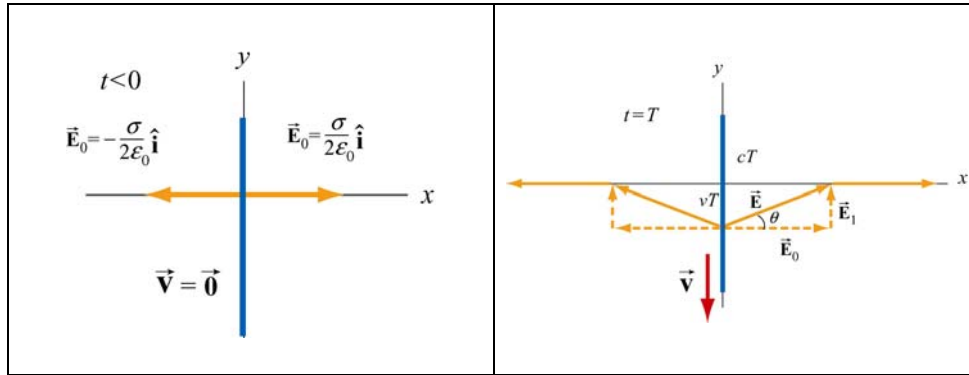We have assumed that the information that this field line is being dragged downward will propagate outward from $x = 0$ at the speed of light c. Thus the portion of our field line located a distance $x > cT$ along the x-axis from the origin doesn't know the charges are moving, and thus has not yet begun to move downward. Our field line therefore must appear at time $t = T$ as shown in Figure 13.8.9(b). Nothing has happened outside of $|x| > cT$; the foot of the field line at $x = 0$ is a distance $y = -vT$ down the y-axis, and we have guessed about what the field line must look like for $0 < |x| < cT$ by simply connecting the two positions on the field line that we know about at time T ($x = 0$ and $|x| = cT$) by a straight line. This is exactly the guess we would make if we were dealing with a string instead of an electric field. This is a reasonable thing to do, and it turns out to be the right guess.

What we have done by pulling down on the charged sheet is to generate a perturbation in the electric field, $\vec{\mathbf{E}}_1$ in addition to the static field $\vec{\mathbf{E}}_0$. Thus, the total field $\vec{\mathbf{E}}$ for $0 < |x| < cT$ is

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 + \vec{\mathbf{E}}_1 \tag{13.8.4}$$

As shown in Figure 13.8.9(b), the field vector $\vec{\mathbf{E}}$ must be parallel to the line connecting the foot of the field line and the position of the field line at $|x| = cT$. This implies

$$\tan\theta = \frac{E_1}{E_0} = \frac{vT}{cT} = \frac{v}{c} \tag{13.8.5}$$

27

where $E_1 = |\vec{\mathbf{E}}_1|$ and $E_0 = |\vec{\mathbf{E}}_0|$ are the magnitudes of the fields, and θ is the angle with the x-axis. Using Eq. (13.8.5), the perturbation field can be written as

$$\vec{\mathbf{E}}_1 = \left(\frac{v}{c}E_0\right)\hat{\mathbf{j}} = \left(\frac{v\sigma}{2\varepsilon_0 c}\right)\hat{\mathbf{j}} \tag{13.8.6}$$

where we have used $E_0 = \sigma/2\varepsilon_0$. We have generated an electric field perturbation, and this expression tells us how large the perturbation field $\vec{\mathbf{E}}_1$ is for a given speed of the sheet of charge, v.

This explains why the electric field line has a *tension* associated with it, just as a string does. The direction of $\vec{\mathbf{E}}_1$ is such that the forces it exerts on the charges in the sheet *resist* the motion of the sheet. That is, there is an *upward* electric force on the sheet when we try to move it *downward*. For an infinitesimal area dA of the sheet containing charge $dq = \sigma\, dA$, the upward "tension" associated with the electric field is

$$d\vec{\mathbf{F}}_e = dq\vec{\mathbf{E}}_1 = (\sigma dA)\left(\frac{v\sigma}{2\varepsilon_0 c}\right)\hat{\mathbf{j}} = \left(\frac{v\sigma^2 dA}{2\varepsilon_0 c}\right)\hat{\mathbf{j}} \tag{13.8.7}$$

Therefore, to overcome the tension, the external agent must apply an equal but opposite (*downward*) force

$$d\vec{\mathbf{F}}_{ext} = -d\vec{\mathbf{F}}_e = -\left(\frac{v\sigma^2 dA}{2\varepsilon_0 c}\right)\hat{\mathbf{j}} \tag{13.8.8}$$

Since the amount of work done is $dW_{ext} = \vec{\mathbf{F}}_{ext} \cdot d\vec{\mathbf{s}}$, the work done per unit time per unit area by the external agent is

$$\frac{d^2 W_{ext}}{dA\, dt} = \frac{d\vec{\mathbf{F}}_{ext}}{dA} \cdot \frac{d\vec{\mathbf{s}}}{dt} = \left(-\frac{v\sigma^2}{2\varepsilon_0 c}\hat{\mathbf{j}}\right)\cdot\left(-v\hat{\mathbf{j}}\right) = \frac{v^2\sigma^2}{2\varepsilon_0 c} \tag{13.8.9}$$

What else has happened in this process of moving the charged sheet down? Well, once the charged sheet is in motion, we have created a sheet of current with surface current density (current per unit length) $\vec{\mathbf{K}} = -\sigma v\hat{\mathbf{j}}$. From Ampere's law, we know that a *magnetic field* has been created, in addition to $\vec{\mathbf{E}}_1$. The current sheet will produce a magnetic field (see Example 9.4)

$$\vec{\mathbf{B}}_1 = \begin{cases} +(\mu_0\sigma v/2)\hat{\mathbf{k}}, & x > 0 \\ -(\mu_0\sigma v/2)\hat{\mathbf{k}}, & x < 0 \end{cases} \tag{13.8.10}$$

This magnetic field changes direction as we move from negative to positive values of x, (across the current sheet). The configuration of the field due to a downward current is shown in Figure 13.8.10 for $|x| < cT$. Again, the information that the charged sheet has started moving, producing a current sheet and associated magnetic field, can only propagate outward from $x = 0$ at the speed of light c. Therefore the magnetic field is still zero, $\vec{\mathbf{B}} = \vec{\mathbf{0}}$ for $|x| > cT$. Note that

$$\frac{E_1}{B_1} = \frac{v\sigma / 2\varepsilon_0 c}{\mu_0 \sigma v / 2} = \frac{1}{c\mu_0 \varepsilon_0} = c \tag{13.8.11}$$
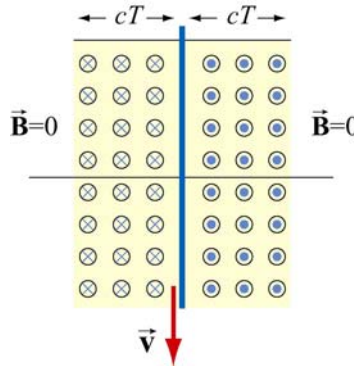

Figure 13.8.10 Magnetic field at $t = T$.

The magnetic field $\vec{\mathbf{B}}_1$ generated by the current sheet is perpendicular to $\vec{\mathbf{E}}_1$ with a magnitude $B_1 = E_1 / c$, as expected for a transverse electromagnetic wave.

Now, let's discuss the energy carried away by these perturbation fields. The energy flux associated with an electromagnetic field is given by the Poynting vector $\vec{\mathbf{S}}$. For $x > 0$, the energy flowing to the *right* is

$$\vec{\mathbf{S}} = \frac{1}{\mu_0} \vec{\mathbf{E}}_1 \times \vec{\mathbf{B}}_1 = \frac{1}{\mu_0} \left(\frac{v\sigma}{2\varepsilon_0 c} \hat{\mathbf{j}} \right) \times \left(\frac{\mu_0 \sigma v}{2} \hat{\mathbf{k}} \right) = \left(\frac{v^2 \sigma^2}{4\varepsilon_0 c} \right) \hat{\mathbf{i}} \tag{13.8.12}$$

This is only half of the work we do per unit time per unit area to pull the sheet down, as given by Eq. (13.8.9). Since the fields on the left carry exactly the same amount of energy flux *to the left*, (the magnetic field $\vec{\mathbf{B}}_1$ changes direction across the plane $x = 0$ whereas the electric field $\vec{\mathbf{E}}_1$ does not, so the Poynting flux also changes across $x = 0$). So the total energy flux carried off by the perturbation electric and magnetic fields we have generated is *exactly equal* to the rate of work per unit area to pull the charged sheet down against the tension in the electric field. Thus we have generated perturbation electromagnetic fields that carry off energy at exactly the rate that it takes to create them.

Where does the energy carried off by the electromagnetic wave come from? The external agent who originally "shook" the charge to produce the wave had to do work against the perturbation electric field the shaking produces, and that agent is the ultimate source of the energy carried by the wave. An exactly analogous situation exists when one asks where the energy carried by a wave on a string comes from. The agent who originally shook the string to produce the wave had to do work to shake it against the restoring tension in the string, and that agent is the ultimate source of energy carried by a wave on a string.

13.8.2 Sinusoidal Electromagnetic Wave

How about generating a sinusoidal wave with angular frequency ω? To do this, instead of pulling the charge sheet down at constant speed, we just shake it up and down with a velocity $\vec{\mathbf{v}}(t) = -v_0 \cos \omega t \, \hat{\mathbf{j}}$. The oscillating sheet of charge will generate fields which are given by:

$$\vec{\mathbf{E}}_1 = \frac{c\mu_0 \sigma v_0}{2} \cos \omega \left(t - \frac{x}{c} \right) \hat{\mathbf{j}}, \quad \vec{\mathbf{B}}_1 = \frac{\mu_0 \sigma v_0}{2} \cos \omega \left(t - \frac{x}{c} \right) \hat{\mathbf{k}} \qquad (13.8.13)$$

for $x > 0$ and, for $x < 0$,

$$\vec{\mathbf{E}}_1 = \frac{c\mu_0 \sigma v_0}{2} \cos \omega \left(t + \frac{x}{c} \right) \hat{\mathbf{j}}, \quad \vec{\mathbf{B}}_1 = -\frac{\mu_0 \sigma v_0}{2} \cos \omega \left(t + \frac{x}{c} \right) \hat{\mathbf{k}} \qquad (13.8.14)$$

In Eqs. (13.8.13) and (13.8.14) we have chosen the *amplitudes* of these terms to be the amplitudes of the kink generated above for constant speed of the sheet, with $E_1 / B_1 = c$, but now allowing for the fact that the speed is varying sinusoidally in time with frequency ω. But why have we put the $(t - x/c)$ and $(t + x/c)$ in the arguments for the cosine function in Eqs. (13.8.13) and (13.8.14)?

Consider first $x > 0$. If we are sitting at some $x > 0$ at time t, and are measuring an electric field there, the field we are observing should not depend on what the current sheet is doing *at that* observation time t. Information about what the current sheet is doing takes a time x/c to propagate out to the observer at $x > 0$. Thus what the observer at $x > 0$ sees at time t depends on what the current sheet was doing *at an earlier time*, namely $t - x/c$. The electric field as a function of time should reflect that time delay due to the finite speed of propagation from the origin to some $x > 0$, and this is the reason the $(t - x/c)$ appears in Eq. (13.8.13), and not t itself. For $x < 0$, the argument is exactly the same, except if $x < 0$, $t + x/c$ is the expression for the earlier time, and not $t - x/c$. This is exactly the time-delay effect one gets when one measures waves on a string. If we are measuring wave amplitudes on a string some distance away from the agent who is shaking the string to generate the waves, what we measure at time t depends on what the

agent was doing at an earlier time, allowing for the wave to propagate from the agent to the observer.

If we note that $\cos\omega(t - x/c) = \cos(\omega t - kx)$ where $k = \omega/c$ is the wave number, we see that Eqs. (13.8.13) and (13.8.14) are precisely the kinds of plane electromagnetic waves we have studied. Note that we can also easily arrange to get rid of our static field $\vec{\mathbf{E}}_0$ by simply putting a stationary charged sheet with charge per unit area $-\sigma$ at $x = 0$. That charged sheet will cancel out the static field due to the positive sheet of charge, but will not affect the perturbation field we have calculated, since the negatively-charged sheet is not moving. In reality, that is how electromagnetic waves are generated--with an overall neutral medium where charges of one sign (usually the electrons) are accelerated while an equal number of charges of the opposite sign essentially remain at rest. Thus an observer only sees the wave fields, and not the static fields. In the following, we will assume that we have set $\vec{\mathbf{E}}_0$ to zero in this way.
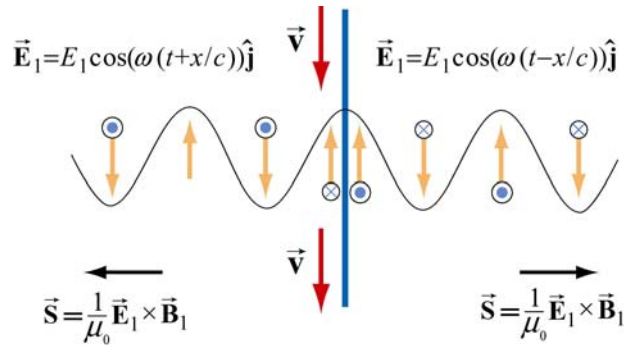

Figure 13.9.4 Electric field generated by the oscillation of a current sheet.

The electric field generated by the oscillation of the current sheet is shown in Figure 13.8.11, for the instant when the sheet is moving *down* and the perturbation electric field is *up*. The magnetic fields, which point into or out of the page, are also shown.

What we have accomplished in the construction here, which really only assumes that the feet of the electric field lines move with the charges, and that information propagates at c is to show we can generate such a wave by shaking a plane of charge sinusoidally. The wave we generate has electric and magnetic fields perpendicular to one another, and transverse to the direction of propagation, with the ratio of the electric field magnitude to the magnetic field magnitude equal to the speed of light. Moreover, we see directly where the energy flux $\vec{\mathbf{S}} = \vec{\mathbf{E}} \times \vec{\mathbf{B}}/\mu_0$ carried off by the wave comes from. The agent who shakes the charges, and thereby generates the electromagnetic wave puts the energy in. If we go to more complicated geometries, these statements become much more complicated in detail, but the overall picture remains as we have presented it.

Let us rewrite slightly the expressions given in Eqs. (13.8.13) and (13.8.14) for the fields generated by our oscillating charged sheet, in terms of the current per unit length in the sheet, $\vec{\mathbf{K}}(t) = \sigma v(t)\hat{\mathbf{j}}$. Since $\vec{\mathbf{v}}(t) = -v_0 \cos \omega t \,\hat{\mathbf{j}}$, it follows that $\vec{\mathbf{K}}(t) = -\sigma v_0 \cos \omega t \,\hat{\mathbf{j}}$. Thus,

$$\vec{\mathbf{E}}_1(x,t) = -\frac{c\mu_0}{2}\vec{\mathbf{K}}(t - x/c), \quad \vec{\mathbf{B}}_1(x,t) = \hat{\mathbf{i}} \times \frac{\vec{\mathbf{E}}_1(x,t)}{c} \tag{13.8.15}$$

for $x > 0$, and

$$\vec{\mathbf{E}}_1(x,t) = -\frac{c\mu_0}{2}\vec{\mathbf{K}}(t + x/c), \quad \vec{\mathbf{B}}_1(x,t) = -\hat{\mathbf{i}} \times \frac{\vec{\mathbf{E}}_1(x,t)}{c} \tag{13.8.16}$$

for $x < 0$. Note that $\vec{\mathbf{B}}_1(x,t)$ reverses direction across the current sheet, with a jump of $\mu_0 \left| \vec{\mathbf{K}}(t) \right|$ at the sheet, as it must from Ampere's law. *Any* oscillating sheet of current *must* generate the plane electromagnetic waves described by these equations, just as *any* stationary electric charge *must* generate a Coulomb electric field.

Note: To avoid possible future confusion, we point out that in a more advanced electromagnetism course, you will study the radiation fields generated by a *single* oscillating charge, and find that they are proportional to the *acceleration* of the charge. This is very different from the case here, where the radiation fields of our oscillating sheet of charge are proportional to the *velocity* of the charges. However, there is no contradiction, because when you add up the radiation fields due to all the single charges making up our sheet, you recover the same result we give in Eqs. (13.8.15) and (13.8.16) (see Chapter 30, Section 7, of Feynman, Leighton, and Sands, *The Feynman Lectures on Physics, Vol* 1, Addison-Wesley, 1963).

13.9   Summary

- The **Ampere-Maxwell law** reads

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 (I + I_d)$$

where

$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt}$$

is called the **displacement current**. The equation describes how changing electric flux can induce a magnetic field.

- Gauss's law for magnetism is

$$\Phi_B = \oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

  The law states that the magnetic flux through a closed surface must be zero, and implies the absence of magnetic monopoles.

- Electromagnetic phenomena are described by the **Maxwell's equations**:

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q}{\varepsilon_0} \qquad \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt}$$

$$\oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0 \qquad \oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

- In free space, the electric and magnetic components of the electromagnetic wave obey a wave equation:

$$\left(\frac{\partial^2}{\partial x^2} - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \right) \left\{ \begin{array}{c} E_y(x,t) \\ B_z(x,t) \end{array} \right\} = 0$$

- The magnitudes and the amplitudes of the electric and magnetic fields in an electromagnetic wave are related by

$$\frac{E}{B} = \frac{E_0}{B_0} = \frac{\omega}{k} = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \approx 3.00 \times 10^8 \text{ m/s}$$

- A **standing electromagnetic wave** does not propagate, but instead the electric and magnetic fields execute simple harmonic motion perpendicular to the would-be direction of propagation. An example of a standing wave is

$$E_y(x,t) = 2E_0 \sin kx \sin \omega t, \qquad B_z(x,t) = 2B_0 \cos kx \cos \omega t$$

- The energy flow rate of an electromagnetic wave through a closed surface is given by

$$\frac{dU}{dt} = -\oiint \vec{\mathbf{S}} \cdot d\vec{\mathbf{A}}$$

  where

$$\vec{\mathbf{S}} = \frac{1}{\mu_0} \vec{\mathbf{E}} \times \vec{\mathbf{B}}$$

is the **Poynting vector**, and $\vec{\mathbf{S}}$ points in the direction the wave propagates.

- The **intensity** of an electromagnetic wave is related to the average energy density by

$$I = \langle S \rangle = c \langle u \rangle$$

- The momentum transferred is related to the energy absorbed by

$$\Delta p = \begin{cases} \dfrac{\Delta U}{c} & \text{(complete absorption)} \\ 2\dfrac{\Delta U}{c} & \text{(complete reflection)} \end{cases}$$

- The average **radiation pressure** on a surface by a normally incident electromagnetic wave is

$$P = \begin{cases} \dfrac{I}{c} & \text{(complete absorption)} \\ \dfrac{2I}{c} & \text{(complete reflection)} \end{cases}$$

13.10 Appendix: Reflection of Electromagnetic Waves at Conducting Surfaces

How does a very good conductor reflect an electromagnetic wave falling on it? In words, what happens is the following. The time-varying electric field of the incoming wave drives an oscillating current on the surface of the conductor, following Ohm's law. That oscillating current sheet, of necessity, must generate waves propagating in both directions from the sheet. One of these waves is the reflected wave. The other wave cancels out the incoming wave inside the conductor. Let us make this qualitative description quantitative.
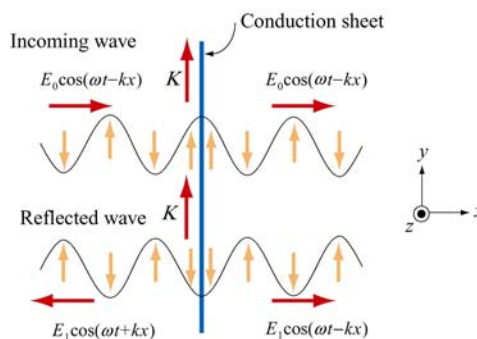

Figure 13.10.1 Reflection of electromagnetic waves at conducting surface

Suppose we have an infinite plane wave propagating in the $+x$-direction, with

$$\vec{\mathbf{E}}_0 = E_0 \cos(\omega t - kx)\hat{\mathbf{j}}, \qquad \vec{\mathbf{B}}_0 = B_0 \cos(\omega t - kx)\hat{\mathbf{k}} \tag{13.10.1}$$

as shown in the top portion of Figure 13.10.1. We put at the origin ($x = 0$) a conducting sheet with width D, which is much smaller than the wavelength of the incoming wave.

This conducting sheet will *reflect* our incoming wave. How? The electric field of the incoming wave will cause a current $\vec{\mathbf{J}} = \vec{\mathbf{E}}/\rho$ to flow in the sheet, where ρ is the resistivity (not to be confused with charge per unit volume), and is equal to the reciprocal of conductivity σ (not to be confused with charge per unit area). Moreover, the direction of $\vec{\mathbf{J}}$ *will be in the same direction as the electric field of the incoming wave*, as shown in the sketch. Thus our incoming wave sets up an oscillating sheet of current with current per unit length $\vec{\mathbf{K}} = \vec{\mathbf{J}}D$. As in our discussion of the generation of plane electromagnetic waves above, this current sheet *will also generate electromagnetic waves*, moving both to the right and to the left (see lower portion of Figure 13.10.1) away from the oscillating sheet of charge. Using Eq. (13.8.15) for $x > 0$ the wave generated by the current will be

$$\vec{\mathbf{E}}_1(x,t) = -\frac{c\mu_0 JD}{2}\cos(\omega t - kx)\hat{\mathbf{j}} \tag{13.10.2}$$

where $J = |\vec{\mathbf{J}}|$. For $x < 0$, we will have a similar expression, except that the argument will be $(\omega t + kx)$ (see Figure 13.10.1). Note the sign of this electric field $\vec{\mathbf{E}}_1$ at $x = 0$; it is *down* ($-\hat{\mathbf{j}}$) when the sheet of current is *up* (and $\vec{\mathbf{E}}_0$ is *up*, $+\hat{\mathbf{j}}$), and vice-versa, just as we saw before. Thus, for $x > 0$, the generated electric field $\vec{\mathbf{E}}_1$ will always be *opposite* the direction of the electric field of the incoming wave, and *it will tend to cancel out the incoming wave for $x > 0$*. For a very good conductor, we have (see next section)

$$K = |\vec{\mathbf{K}}| = JD = \frac{2E_0}{c\mu_0} \tag{13.10.3}$$

so that for $x > 0$ we will have

$$\vec{\mathbf{E}}_1(x,t) = -E_0 \cos(\omega t - kx)\hat{\mathbf{j}} \tag{13.10.4}$$

That is, for a very good conductor, the electric field of the wave generated by the current *will exactly cancel* the electric field of the incoming wave *for $x > 0$*! And that's what a very good conductor does. It supports exactly the amount of current per unit length $K = 2E_0/c\mu_0$ needed to cancel out the incoming wave for $x > 0$. For $x < 0$, *this same current* generates a "reflected" wave propagating back in the direction from which the

original incoming wave came, *with the same amplitude as the original incoming wave*. This is how a very good conductor *totally* reflects electromagnetic waves. Below we shall show that K will in fact approach the value needed to accomplish this in the limit the resistivity ρ approaches zero.

In the process of reflection, there is a force per unit area exerted on the conductor. This is just the $\vec{v} \times \vec{B}$ force due to the current $\vec{J}$ flowing in the presence of the magnetic field of the incoming wave, or a force per unit volume of $\vec{J} \times \vec{B}_0$. If we calculate the total force $d\vec{F}$ acting on a cylindrical volume with area dA and length D of the conductor, we find that it is in the $+x$ - direction, with magnitude

$$dF = D \,|\, \vec{J} \times \vec{B}_0 \,|\, dA = DJB_0 \, dA = \frac{2E_0 B_0 dA}{c\mu_0} \tag{13.10.5}$$

so that the force per unit area,

$$\frac{dF}{dA} = \frac{2E_0 B_0}{c\mu_0} = \frac{2S}{c} \tag{13.10.6}$$

or radiation pressure, is just twice the Poynting flux divided by the speed of light c.

We shall show that a perfect conductor will perfectly reflect an incident wave. To approach the limit of a perfect conductor, we first consider the finite resistivity case, and then let the resistivity go to zero.

For simplicity, we assume that the sheet is thin compared to a wavelength, so that the entire sheet sees essentially the same electric field. This implies that the current density $\vec{J}$ will be uniform across the thickness of the sheet, and outside of the sheet we will see fields appropriate to an equivalent surface current $\vec{K}(t) = D\vec{J}(t)$. This current sheet will generate additional electromagnetic waves, moving both to the right and to the left, away from the oscillating sheet of charge. The total electric field, $\vec{E}(x,t)$, will be the sum of the incident electric field and the electric field generated by the current sheet. Using Eqs. (13.8.15) and (13.8.16) above, we obtain the following expressions for the total electric field:

$$\vec{E}(x,t) = \vec{E}_0(x,t) + \vec{E}_1(x,t) = \begin{cases} \vec{E}_0(x,t) - \dfrac{c\mu_0}{2}\vec{K}(t - x/c), & x > 0 \\[2mm] \vec{E}_0(x,t) - \dfrac{c\mu_0}{2}\vec{K}(t + x/c), & x < 0 \end{cases} \tag{13.10.7}$$

We also have a relation between the current density $\vec{J}$ and $\vec{E}$ from the microscopic form of Ohm's law: $\vec{J}(t) = \vec{E}(0,t)/\rho$, where $\vec{E}(0,t)$ is the total electric field at the position of

the conducting sheet. Note that it is appropriate to use the total electric field in Ohm's law -- the currents arise from the total electric field, irrespective of the origin of that field. Thus, we have

$$\vec{\mathbf{K}}(t) = D\vec{\mathbf{J}}(t) = \frac{D\vec{\mathbf{E}}(0,t)}{\rho} \qquad (13.10.8)$$

At $x = 0$, either expression in Eq. (13.10.7) gives

$$\vec{\mathbf{E}}(0,t) = \vec{\mathbf{E}}_0(0,t) + \vec{\mathbf{E}}_1(0,t) = \vec{\mathbf{E}}_0(0,t) - \frac{c\mu_0}{2}\vec{\mathbf{K}}(t)$$

$$= \vec{\mathbf{E}}_0(0,t) - \frac{Dc\mu_0\vec{\mathbf{E}}(0,t)}{2\rho} \qquad (13.10.9)$$

where we have used Eq. (13.10.9) for the last step. Solving for $\vec{\mathbf{E}}(0,t)$, we obtain

$$\vec{\mathbf{E}}(0,t) = \frac{\vec{\mathbf{E}}_0(0,t)}{1 + Dc\mu_0/2\rho} \qquad (13.10.10)$$

Using the expression above, the surface current density in Eq. (13.10.8) can be rewritten as

$$\vec{\mathbf{K}}(t) = D\vec{\mathbf{J}}(t) = \frac{D\vec{\mathbf{E}}_0(0,t)}{\rho + Dc\mu_0/2} \qquad (13.10.11)$$

In the limit where $\rho \simeq 0$ (no resistance, a perfect conductor), $\vec{\mathbf{E}}(0,t) = \vec{\mathbf{0}}$, as can be seen from Eq. (13.10.8), and the surface current becomes

$$\vec{\mathbf{K}}(t) = \frac{2\vec{\mathbf{E}}_0(0,t)}{c\mu_0} = \frac{2E_0}{c\mu_0}\cos\omega t\,\hat{\mathbf{j}} = \frac{2B_0}{\mu_0}\cos\omega t\,\hat{\mathbf{j}} \qquad (13.10.12)$$

In this same limit, the total electric fields can be written as

$$\vec{\mathbf{E}}(x,t) = \begin{cases} (E_0 - E_0)\cos(\omega t - kx)\hat{\mathbf{j}} = \vec{\mathbf{0}}, & x > 0 \\ E_0[\cos(\omega t - kx) - \cos(\omega t + kx)]\hat{\mathbf{j}} = 2E_0\sin\omega t\sin kx\,\hat{\mathbf{j}}, & x < 0 \end{cases} \qquad (13.10.13)$$

Similarly, the total magnetic fields in this limit are given by

$$\vec{B}(x,t) = \vec{B}_0(x,t) + \vec{B}_1(x,t) = B_0 \cos(\omega t - kx)\hat{k} + \hat{i} \times \left(\frac{\vec{E}_1(x,t)}{c} \right)$$ (13.10.14)

$$= B_0 \cos(\omega t - kx)\hat{k} - B_0 \cos(\omega t - kx)\hat{k} = \vec{0}$$

for $x > 0$, and

$$\vec{B}(x,t) = B_0[\cos(\omega t - kx) + \cos(\omega t + kx)]\hat{k} = 2B_0 \cos\omega t \cos kx\, \hat{k}$$ (13.10.15)

for $x < 0$. Thus, from Eqs. (13.10.13) - (13.10.15) we see that we get no electromagnetic wave for $x > 0$, and standing electromagnetic waves for $x < 0$. Note that at $x = 0$, the total electric field vanishes. The current per unit length at $x = 0$,

$$\vec{K}(t) = \frac{2B_0}{\mu_0} \cos\omega t\, \hat{j}$$ (13.10.16)

is just the current per length we need to bring the magnetic field down from its value at $x < 0$ to zero for $x > 0$.

You may be perturbed by the fact that in the limit of a perfect conductor, the electric field vanishes at $x = 0$, since it is the electric field at $x = 0$ that is driving the current there! In the limit of very small resistance, the electric field required to drive any finite current is very small. In the limit where $\rho = 0$, the electric field is zero, but as we approach that limit, we can still have a perfectly finite and well determined value of $\vec{J} = \vec{E}/\rho$, as we found by taking this limit in Eqs. (13.10.8) and (13.10.12) above.

13.11 Problem-Solving Strategy: Traveling Electromagnetic Waves

This chapter explores various properties of the electromagnetic waves. The electric and the magnetic fields of the wave obey the wave equation. Once the functional form of either one of the fields is given, the other can be determined from Maxwell's equations. As an example, let's consider a sinusoidal electromagnetic wave with

$$\vec{E}(z,t) = E_0 \sin(kz - \omega t)\hat{i}$$

The equation above contains the complete information about the electromagnetic wave:

1. Direction of wave propagation: The argument of the sine form in the electric field can be rewritten as $(kz - \omega t) = k(z - vt)$, which indicates that the wave is propagating in the $+z$-direction.

2. Wavelength: The wavelength λ is related to the wave number k by $\lambda = 2\pi / k$.

3. Frequency: The frequency of the wave, f, is related to the angular frequency ω by $f = \omega / 2\pi$.

4. Speed of propagation: The speed of the wave is given by

$$v = \lambda f = \frac{2\pi}{k} \cdot \frac{\omega}{2\pi} = \frac{\omega}{k}$$

   In vacuum, the speed of the electromagnetic wave is equal to the speed of light, c.

5. Magnetic field $\vec{\mathbf{B}}$: The magnetic field $\vec{\mathbf{B}}$ is perpendicular to both $\vec{\mathbf{E}}$ which points in the +x-direction, and $+\hat{\mathbf{k}}$, the unit vector along the +z-axis, which is the direction of propagation, as we have found. In addition, since the wave propagates in the same direction as the cross product $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$, we conclude that $\vec{\mathbf{B}}$ must point in the +y-direction (since $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$).

   Since $\vec{\mathbf{B}}$ is always in phase with $\vec{\mathbf{E}}$, the two fields have the same functional form. Thus, we may write the magnetic field as

$$\vec{\mathbf{B}}(z,t) = B_0 \sin(kz - \omega t)\hat{\mathbf{j}}$$

   where B_0 is the amplitude. Using Maxwell's equations one may show that $B_0 = E_0 (k / \omega) = E_0 / c$ in vacuum.

6. The Poytning vector: Using Eq. (13.6.5), the Poynting vector can be obtained as

$$\vec{\mathbf{S}} = \frac{1}{\mu_0} \vec{\mathbf{E}} \times \vec{\mathbf{B}} = \frac{1}{\mu_0} \left[E_0 \sin(kz - \omega t)\hat{\mathbf{i}} \right] \times \left[B_0 \sin(kz - \omega t)\hat{\mathbf{j}} \right] = \frac{E_0 B_0 \sin^2(kz - \omega t)}{\mu_0} \hat{\mathbf{k}}$$

7. Intensity: The intensity of the wave is equal to the average of S:

$$I = \langle S \rangle = \frac{E_0 B_0}{\mu_0} \left\langle \sin^2(kz - \omega t) \right\rangle = \frac{E_0 B_0}{2\mu_0} = \frac{E_0^2}{2c\mu_0} = \frac{cB_0^2}{2\mu_0}$$

8. Radiation pressure: If the electromagnetic wave is normally incident on a surface and the radiation is completely *reflected*, the radiation pressure is

$$P = \frac{2I}{c} = \frac{E_0 B_0}{c\mu_0} = \frac{E_0^2}{c^2\mu_0} = \frac{B_0^2}{\mu_0}$$

13.12 Solved Problems

13.12.1 Plane Electromagnetic Wave

Suppose the electric field of a plane electromagnetic wave is given by

$$\vec{E}(z,t) = E_0 \cos\left(kz - \omega t\right)\hat{i} \qquad (13.12.1)$$

Find the following quantities:

(a) The direction of wave propagation.

(b) The corresponding magnetic field $\vec{B}$.

Solutions:

(a) By writing the argument of the cosine function as $kz - \omega t = k(z - ct)$ where $\omega = ck$, we see that the wave is traveling in the $+z$ direction.

(b) The direction of propagation of the electromagnetic waves coincides with the direction of the Poynting vector which is given by $\vec{S} = \vec{E} \times \vec{B}/\mu_0$. In addition, $\vec{E}$ and $\vec{B}$ are perpendicular to each other. Therefore, if $\vec{E} = E(z,t)\hat{i}$ and $\vec{S} = S\hat{k}$, then $\vec{B} = B(z,t)\hat{j}$. That is, $\vec{B}$ points in the +y-direction. Since $\vec{E}$ and $\vec{B}$ are in phase with each other, one may write

$$\vec{B}(z,t) = B_0 \cos(kz - \omega t)\hat{j} \qquad (13.12.2)$$

To find the magnitude of $\vec{B}$, we make use of Faraday's law:

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \qquad (13.12.3)$$

which implies

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \qquad (13.12.4)$$

From the above equations, we obtain

$$-E_0 k \sin(kz - \omega t) = -B_0 \omega \sin(kz - \omega t) \qquad (13.12.5)$$

or

$$\frac{E_0}{B_0} = \frac{\omega}{k} = c \qquad (13.12.6)$$

Thus, the magnetic field is given by

$$\vec{B}(z,t) = (E_0 / c)\cos(kz - \omega t)\,\hat{\mathbf{j}} \qquad (13.12.7)$$

13.12.2 One-Dimensional Wave Equation

Verify that, for $\omega = kc$,

$$E(x,t) = E_0 \cos(kx - \omega t)$$
$$B(x,t) = B_0 \cos(kx - \omega t) \qquad (13.12.8)$$

satisfy the one-dimensional wave equation:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right)\begin{Bmatrix} E(x,t) \\ B(x,t) \end{Bmatrix} = 0 \qquad (13.12.9)$$

Solution:

Differentiating $E = E_0 \cos(kx - \omega t)$ with respect to x gives

$$\frac{\partial E}{\partial x} = -kE_0 \sin(kx - \omega t), \quad \frac{\partial^2 E}{\partial x^2} = -k^2 E_0 \cos(kx - \omega t) \qquad (13.12.10)$$

Similarly, differentiating E with respect to t yields

$$\frac{\partial E}{\partial t} = \omega E_0 \sin(kx - \omega t), \quad \frac{\partial^2 E}{\partial t^2} = -\omega^2 E_0 \cos(kx - \omega t) \qquad (13.12.11)$$

Thus,

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \left(-k^2 + \frac{\omega^2}{c^2} \right) E_0 \cos(kx - \omega t) = 0 \qquad (13.12.12)$$

where we have made used of the relation $\omega = kc$. One may follow a similar procedure to verify the magnetic field.

13.12.3 Poynting Vector of a Charging Capacitor

A parallel-plate capacitor with circular plates of radius *R* and separated by a distance *h* is charged through a straight wire carrying current *I*, as shown in the Figure 13.12.1:
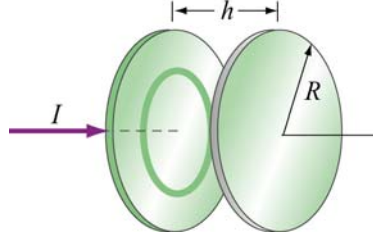

Figure 13.12.1 Parallel plate capacitor

(a) Show that as the capacitor is being charged, the Poynting vector $\vec{\mathbf{S}}$ points radially inward toward the center of the capacitor.

(b) By integrating $\vec{\mathbf{S}}$ over the cylindrical boundary, show that the rate at which energy enters the capacitor is equal to the rate at which electrostatic energy is being stored in the electric field.

Solutions:

(a) Let the axis of the circular plates be the *z*-axis, with current flowing in the +*z*-direction. Suppose at some instant the amount of charge accumulated on the positive plate is +*Q*. The electric field is

$$\vec{\mathbf{E}} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{k}} = \frac{Q}{\pi R^2 \varepsilon_0}\hat{\mathbf{k}} \qquad (13.12.13)$$

According to the Ampere-Maxwell's equation, a magnetic field is induced by changing electric flux:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} + \mu_0 \varepsilon_0 \frac{d}{dt}\iint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$
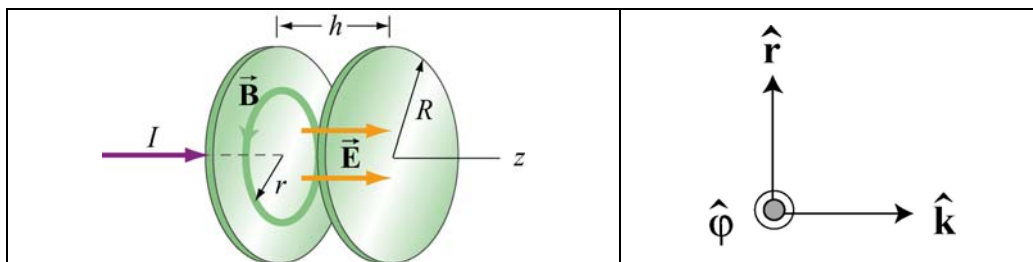

Figure 13.12.2

From the cylindrical symmetry of the system, we see that the magnetic field will be circular, centered on the z-axis, i.e., $\vec{\mathbf{B}} = B\hat{\boldsymbol{\varphi}}$ (see Figure 13.12.2.)

Consider a circular path of radius $r < R$ between the plates. Using the above formula, we obtain

$$B(2\pi r) = 0 + \mu_0 \varepsilon_0 \frac{d}{dt}\left(\frac{Q}{\pi R^2 \varepsilon_0}\pi r^2\right) = \frac{\mu_0 r^2}{R^2}\frac{dQ}{dt} \tag{13.12.14}$$

or

$$\vec{\mathbf{B}} = \frac{\mu_0 r}{2\pi R^2}\frac{dQ}{dt}\hat{\boldsymbol{\varphi}} \tag{13.12.15}$$

The Poynting $\vec{\mathbf{S}}$ vector can then be written as

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}}\times\vec{\mathbf{B}} = \frac{1}{\mu_0}\left(\frac{Q}{\pi R^2 \varepsilon_0}\hat{\mathbf{k}}\right)\times\left(\frac{\mu_0 r}{2\pi R^2}\frac{dQ}{dt}\hat{\boldsymbol{\varphi}}\right)$$
$$= -\left(\frac{Qr}{2\pi^2 R^4 \varepsilon_0}\right)\left(\frac{dQ}{dt}\right)\hat{\mathbf{r}} \tag{13.12.16}$$

Note that for $dQ/dt > 0$ $\vec{\mathbf{S}}$ points in the $-\hat{\mathbf{r}}$ direction, or radially inward toward the center of the capacitor.

(b) The energy per unit volume carried by the electric field is $u_E = \varepsilon_0 E^2 / 2$. The total energy stored in the electric field then becomes

$$U_E = u_E V = \frac{\varepsilon_0}{2}E^2\left(\pi R^2 h\right) = \frac{1}{2}\varepsilon_0\left(\frac{Q}{\pi R^2 \varepsilon_0}\right)^2 \pi R^2 h = \frac{Q^2 h}{2\pi R^2 \varepsilon_0} \tag{13.12.17}$$

Differentiating the above expression with respect to t, we obtain the rate at which this energy is being stored:

$$\frac{dU_E}{dt} = \frac{d}{dt}\left(\frac{Q^2 h}{2\pi R^2 \varepsilon_0}\right) = \frac{Qh}{\pi R^2 \varepsilon_0}\left(\frac{dQ}{dt}\right) \tag{13.12.18}$$

On the other hand, the rate at which energy flows into the capacitor through the cylinder at $r = R$ can be obtained by integrating $\vec{\mathbf{S}}$ over the surface area:

$$\oint \vec{S} \cdot d\vec{A} = SA_R = \left(\frac{Qr}{2\pi^2 \varepsilon_o R^4} \frac{dQ}{dt} \right)(2\pi Rh) = \frac{Qh}{\varepsilon_0 \pi R^2} \left(\frac{dQ}{dt} \right) \qquad (13.12.19)$$

which is equal to the rate at which energy stored in the electric field is changing.

13.12.4 Poynting Vector of a Conductor

A cylindrical conductor of radius a and conductivity σ carries a steady current I which is distributed uniformly over its cross-section, as shown in Figure 13.12.3.
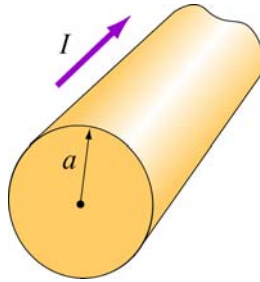

Figure 13.12.3

(a) Compute the electric field $\vec{E}$ inside the conductor.

(b) Compute the magnetic field $\vec{B}$ just outside the conductor.

(c) Compute the Poynting vector $\vec{S}$ at the surface of the conductor. In which direction does $\vec{S}$ point?

(d) By integrating $\vec{S}$ over the surface area of the conductor, show that the rate at which electromagnetic energy enters the surface of the conductor is equal to the rate at which energy is dissipated.

Solutions:

(a) Let the direction of the current be along the z-axis. The electric field is given by

$$\vec{E} = \frac{\vec{J}}{\sigma} = \frac{I}{\sigma \pi a^2} \hat{k} \qquad (13.12.20)$$

where R is the resistance and l is the length of the conductor.

(b) The magnetic field can be computed using Ampere's law:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{enc} \tag{13.12.21}$$

Choosing the Amperian loop to be a circle of radius r, we have $B(2\pi r) = \mu_0 I$, or

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2\pi r} \hat{\boldsymbol{\varphi}} \tag{13.12.22}$$

(c) The Poynting vector on the surface of the wire $(r = a)$ is

$$\vec{\mathbf{S}} = \frac{\vec{\mathbf{E}} \times \vec{\mathbf{B}}}{\mu_0} = \frac{1}{\mu_0} \left(\frac{I}{\sigma \pi a^2} \hat{\mathbf{k}} \right) \times \left(\frac{\mu_0 I}{2\pi a} \hat{\boldsymbol{\varphi}} \right) = - \left(\frac{I^2}{2\pi^2 \sigma a^3} \right) \hat{\mathbf{r}} \tag{13.12.23}$$

Notice that $\vec{\mathbf{S}}$ points radially inward toward the center of the conductor.

(d) The rate at which electromagnetic energy flows into the conductor is given by

$$P = \frac{dU}{dt} = \oiint_S \vec{\mathbf{S}} \cdot d\vec{\mathbf{A}} = \left(\frac{I^2}{2\sigma \pi^2 a^3} \right) 2\pi a l = \frac{I^2 l}{\sigma \pi a^2} \tag{13.12.24}$$

However, since the conductivity σ is related to the resistance R by

$$\sigma = \frac{1}{\rho} = \frac{l}{AR} = \frac{l}{\pi a^2 R} \tag{13.12.25}$$

The above expression becomes

$$P = I^2 R \tag{13.12.26}$$

which is equal to the rate of energy dissipation in a resistor with resistance R.

13.13 Conceptual Questions

1. In the Ampere-Maxwell's equation, is it possible that both a conduction current and a displacement are non-vanishing?

2. What causes electromagnetic radiation?

3. When you touch the indoor antenna on a TV, the reception usually improves. Why?

4. Explain why the reception for cellular phones often becomes poor when used inside a steel-framed building.

5. Compare sound waves with electromagnetic waves.

6. Can parallel electric and magnetic fields make up an electromagnetic wave in vacuum?

7. What happens to the intensity of an electromagnetic wave if the amplitude of the electric field is halved? Doubled?

13.14 Additional Problems

13.14.1 Solar Sailing

It has been proposed that a spaceship might be propelled in the solar system by radiation pressure, using a large sail made of foil. How large must the sail be if the radiation force is to be equal in magnitude to the Sun's gravitational attraction? Assume that the mass of the ship and sail is 1650 kg, that the sail is perfectly reflecting, and that the sail is oriented at right angles to the Sun's rays. Does your answer depend on where in the solar system the spaceship is located?

13.14.2 Reflections of True Love

(a) A light bulb puts out 100 W of electromagnetic radiation. What is the time-average intensity of radiation from this light bulb at a distance of one meter from the bulb?   What are the maximum values of electric and magnetic fields, E_0 and B_0, at this same distance from the bulb? Assume a plane wave.

(b) The face of your true love is one meter from this 100 W bulb. What maximum surface current must flow on your true love's face in order to reflect the light from the bulb into your adoring eyes? Assume that your true love's face is (what else?) perfect--perfectly smooth and perfectly reflecting--and that the incident light and reflected light are normal to the surface.

13.14.3 Coaxial Cable and Power Flow

A coaxial cable consists of two concentric long hollow cylinders of zero resistance; the inner has radius a, the outer has radius b, and the length of both is l, with $l >> b$. The cable transmits DC power from a battery to a load. The battery provides an electromotive force ε between the two conductors at one end of the cable, and the load is a resistance R connected between the two conductors at the other end of the cable. A

current I flows down the inner conductor and back up the outer one. The battery charges the inner conductor to a charge $-Q$ and the outer conductor to a charge $+Q$.
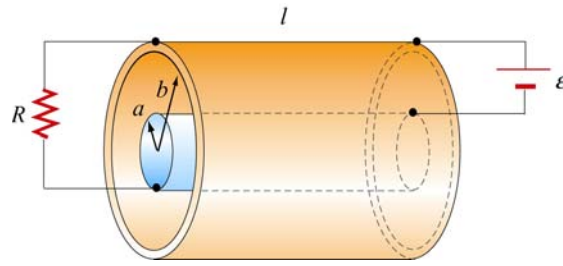

Figure 13.14.1

(a) Find the direction and magnitude of the electric field $\vec{E}$ everywhere.

(b) Find the direction and magnitude of the magnetic field $\vec{B}$ everywhere.

(c) Calculate the Poynting vector $\vec{S}$ in the cable.

(d) By integrating $\vec{S}$ over appropriate surface, find the power that flows into the coaxial cable.

(e) How does your result in (d) compare to the power dissipated in the resistor?

13.14.4 Superposition of Electromagnetic Waves

Electromagnetic wave are emitted from two different sources with

$$\vec{E}_1(x,t) = E_{10}\cos(kx - \omega t)\hat{\mathbf{j}}, \quad \vec{E}_2(x,t) = E_{20}\cos(kx - \omega t + \phi)\hat{\mathbf{j}}$$

(a) Find the Poynting vector associated with the resultant electromagnetic wave.

(b) Find the intensity of the resultant electromagnetic wave

(c) Repeat the calculations above if the direction of propagation of the second electromagnetic wave is reversed so that

$$\vec{E}_1(x,t) = E_{10}\cos(kx - \omega t)\hat{\mathbf{j}}, \quad \vec{E}_2(x,t) = E_{20}\cos(kx + \omega t + \phi)\hat{\mathbf{j}}$$

13.14.5 Sinusoidal Electromagnetic Wave

The electric field of an electromagnetic wave is given by

$$\vec{E}(z,t) = E_0 \cos(kz - \omega t)(\hat{i} + \hat{j})$$

(a) What is the maximum amplitude of the electric field?

(b) Compute the corresponding magnetic field $\vec{B}$.

(c) Find the Ponyting vector $\vec{S}$.

(d) What is the radiation pressure if the wave is incident normally on a surface and is perfectly reflected?

13.14.6 Radiation Pressure of Electromagnetic Wave

A plane electromagnetic wave is described by

$$\vec{E} = E_0 \sin(kx - \omega t)\hat{j}, \qquad \vec{B} = B_0 \sin(kx - \omega t)\hat{k}$$

where $E_0 = cB_0$.

(a) Show that for any point in this wave, the density of the energy stored in the electric field equals the density of the energy stored in the magnetic field. What is the time-averaged *total* (electric plus magnetic) energy density in this wave, in terms of E_0? In terms of B_0?

(b) This wave falls on and is totally *absorbed* by an object. Assuming total absorption, show that the radiation pressure on the object is just given by the time-averaged total energy density in the wave. Note that the dimensions of energy density are the same as the dimensions of pressure.

(c) Sunlight strikes the Earth, just outside its atmosphere, with an average intensity of 1350 W/m². What is the time averaged total energy density of this sunlight? An object in orbit about the Earth totally absorbs sunlight. What radiation pressure does it feel?

13.14.7 Energy of Electromagnetic Waves

(a) If the electric field of an electromagnetic wave has an rms (root-mean-square) strength of 3.0×10^{-2} V/m, how much energy is transported across a 1.00-cm² area in one hour?

(b) The intensity of the solar radiation incident on the upper atmosphere of the Earth is approximately 1350 W/m². Using this information, estimate the energy contained in a 1.00-m³ volume near the Earth's surface due to radiation from the Sun.

13.14.8 Wave Equation

Consider a plane electromagnetic wave with the electric and magnetic fields given by

$$\vec{E}(x,t) = E_z(x,t)\hat{\mathbf{k}}, \quad \vec{B}(x,t) = B_y(x,t)\hat{\mathbf{j}}$$

Applying arguments similar to that presented in 13.4, show that the fields satisfy the following relationships:

$$\frac{\partial E_z}{\partial x} = \frac{\partial B_y}{\partial t}, \quad \frac{\partial B_y}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_z}{\partial t}$$

13.14.9 Electromagnetic Plane Wave

An electromagnetic plane wave is propagating in vacuum has a magnetic field given by

$$\vec{B} = B_0 f(ax+bt)\hat{\mathbf{j}} \qquad f(u) = \begin{cases} 1 & 0 < u < 1 \\ 0 & else \end{cases}$$

The wave encounters an infinite, dielectric sheet at $x = 0$ of such a thickness that half of the energy of the wave is reflected and the other half is transmitted and emerges on the far side of the sheet. The $+\hat{\mathbf{k}}$ direction is out of the paper.

(a) What condition between a and b must be met in order for this wave to satisfy Maxwell's equations?

(b) What is the magnitude and direction of the $\vec{E}$ field of the incoming wave?

(c) What is the magnitude and direction of the energy flux (power per unit area) carried by the incoming wave, in terms of B_0 and universal quantities?

(d) What is the pressure (force per unit area) that this wave exerts on the sheet while it is impinging on it?

13.14.10 Sinusoidal Electromagnetic Wave

An electromagnetic plane wave has an electric field given by

$$\vec{E} = (300\,\text{V/m})\cos\left(\frac{2\pi}{3}x - 2\pi \times 10^6 t\right)\hat{\mathbf{k}}$$

where x and t are in SI units and $\hat{\mathbf{k}}$ is the unit vector in the $+z$-direction. The wave is propagating through ferrite, a ferromagnetic insulator, which has a relative magnetic permeability $\kappa_m = 1000$ and dielectric constant $\kappa = 10$.

(a) What direction does this wave travel?

(b) What is the wavelength of the wave (in meters)?

(c) What is the frequency of the wave (in Hz)?

(d) What is the speed of the wave (in m/s)?

(e) Write an expression for the associated magnetic field required by Maxwell's equations. Indicate the vector direction of $\vec{\mathbf{B}}$ with a unit vector and a $+$ or $-$, and you should give a numerical value for the amplitude in units of tesla.

(g) The wave emerges from the medium through which it has been propagating and continues in vacuum. What is the new wavelength of the wave (in meters)?