Undriven RLC Circuits
Challenge Problem Solutions

Problem 1:

Show that

$$A\cos\omega t + B\sin\omega t = Q_{\mathrm{m}}\cos\left(\omega t + \phi\right),$$

where

$$Q_{\mathrm{m}} = \left(A^2 + B^2\right)^{1/2}, \text{ and } \phi = \tan^{-1}\left(-B/A\right).$$

Problem 1 Solution:

Use the identity

$$Q_{\mathrm{m}}\cos\left(\omega t + \phi\right) = Q_{\mathrm{m}}\cos(\omega t)\cos(\phi) - Q_{\mathrm{m}}\sin(\omega t)\sin(\phi)\ .$$

Thus

$$A\cos(\omega t) + B\sin(\omega t) = Q_{\mathrm{m}}\cos(\omega t)\cos(\phi) - Q_{\mathrm{m}}\sin(\omega t)\sin(\phi)\ .$$

Comparing coefficients we see that

$$A = Q_{\mathrm{m}}\cos\phi\ .$$

$$B = -Q_{\mathrm{m}}\sin\phi\ .$$

Therefore

$$\left(A^2 + B^2\right) = Q_{\mathrm{m}}{}^2\left(\cos^2\phi + \sin^2\phi\right).$$

So

$$Q_{\mathrm{m}} = \left(A^2 + B^2\right)^{1/2}.$$

Also

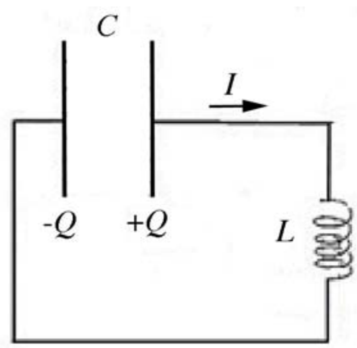
$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{-B/Q_{\mathrm{m}}}{A/Q_{\mathrm{m}}} = -\frac{B}{A}\ .$$

Hence

$$\phi = \tan^{-1}(-B/A)\ .$$

Problem 2:

At the moment depicted in the LC circuit the current is non-zero and the capacitor plates are charged (as shown in the figure below). The energy in the circuit is stored



a) only in the electric field and is decreasing.

b) only in the electric field and is constant.

c) only in the magnetic field and is decreasing.

d) only in the magnetic field and is constant.

e) in both the electric and magnetic field and is constant.

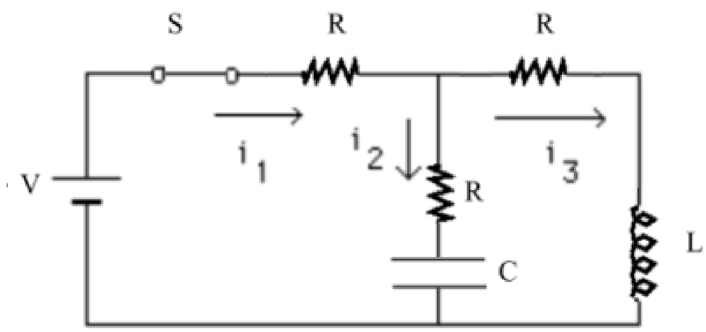
f) in both the electric and magnetic field and is decreasing.

Problem 2 Solution:

e. Since there is no resistance there is no dissipation of energy so energy is constant in time. At the moment depicted in the figure, the capacitor is charged so there is a non-zero electric field associated with the capacitor. There is a non-zero current in the circuit and so there is a non-zero magnetic field. Therefore the energy in the circuit is stored in both the electric and magnetic field and is constant.

Problem 3:

A circuit consists of a battery with emf V, an inductor with inductance L, a capacitor with capacitance C, and three resistors, each with resistance R, as shown in the sketch. The capacitor is initially uncharged and there is no current flowing anywhere in the circuit. The switch S has been open for a long time, and is then closed, as shown in the diagram. If we wait a long time after the switch is closed, the currents in the circuit are given by:



a) $i_1 = \dfrac{2V}{3R} \quad i_2 = \dfrac{V}{3R} \quad i_3 = \dfrac{V}{3R}$.

b) $i_1 = \dfrac{V}{2R} \quad i_2 = 0 \quad i_3 = \dfrac{V}{2R}$.

c) $i_1 = \dfrac{V}{3R} \quad i_2 = 0 \quad i_3 = \dfrac{V}{3R}$.

d) $i_1 = \dfrac{V}{2R} \quad i_2 = \dfrac{V}{2R} \quad i_3 = 0$.

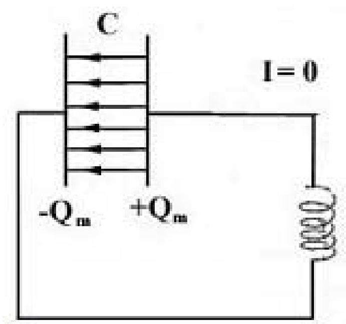
e) None of the above.

Problem 3 Solution :

b. If we wait a long time after the switch is closed, the capacitor is completely charged and no current flows in that branch, $i_2 = 0$. Also the current has reached steady state and is not changing in time so there is no effect from the self-inductance. Hence the inductor acts like a resistance-less wire. (Note that real inductors do have finite resistance as you saw in your lab.) Therefore the same current flow through resistors 1 and 3 and is given by $i_1 = i_3 = V/2R$.

Problem 4:

In an LC circuit, the electric and magnetic fields are shown in the figure. Which of the following is true? **Explain your answer** At the moment depicted in the figure, the energy in the circuit is stored in



1. the electric field and is decreasing
2. the electric field and is constant.
3. the magnetic field and is decreasing.
4. the magnetic field and is constant.
5. in both the electric and magnetic field and is constant.
6. in both the electric and magnetic field and is decreasing.

Problem 4 Solution:
2. The total energy in the LC circuit is constant since there is no resistance and hence no dissipation of electromagnetic energy into thermal energy. At the moment depicted in the figure the capacitor is completely charged, with maximum electric field strength between the plates, and electric potential energy stored in the electric field. Immediately after the instant depicted in the figure, the capacitor starts to discharge, the electric field strength and electric field energy decreases. At the moment depicted in the figure no current is flowing, so the magnetic field strength is zero and magnetic field energy is zero. Immediately after the instant depicted in the figure the current flows clockwise and the magnetic field strength and field energy increases. But the decrease in the electric field energy is equal to the increase in the magnetic field energy.

Problem 5:

In a freely oscillating *LC* circuit, (no driving voltage), suppose the maximum charge on the capacitor is $Q_{\max}$. Assume the circuit has zero resistance.

a) In terms of the maximum charge on the capacitor, what value of charge is present on the capacitor when the energy in the magnetic field is three times the energy in the electric field.

b) How much time has elapsed from when the capacitor is fully charged for this condition to arise?

c) If the resistance is non-zero, will the natural frequency of oscillation compared to the natural frequency of the ideal *LC* circuit (with zero resistance)

i) increase
ii) stay the same
iii) decrease

Problem 5 Solution:

(a) The total energy is constant hence

$$U = \frac{Q_{\max}{}^{2}}{2C} = U_{elec} + U_{mag} = \frac{Q^{2}}{2C} + U_{mag}$$

Suppose $U_{mag} = 3U_{elec}$. Then the electromagnetic energy in the system is

$$\frac{Q_{\max}{}^{2}}{2C} = U_{elec} + U_{mag} = 4U_{elec} = 4\frac{Q^{2}}{2C}$$

We can solve the above equation for the charge on the capacitor and find that

$$Q = \frac{Q_{\max}}{2}.$$

(b) The initial conditions for the charge and the current at time $t = 0$ are

$$Q(t=0) = Q_{\max}$$
$$I(t=0) = 0$$

Therefore the charge on the capacitor varies in time according to

$$Q(t) = Q_{\max} \cos \omega_0 t$$

where $\omega_0 = \sqrt{1/LC}$. When

$$Q(t) = Q_{\max} / 2 = Q_{\max} \cos \omega_0 t$$

we can solve for the time,

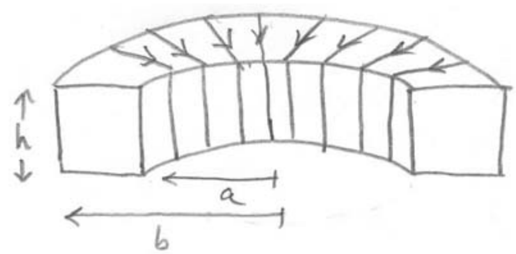
$$t = \frac{1}{\omega_0} \cos^{-1}(1/2) = \sqrt{LC}\,(\pi/3)$$

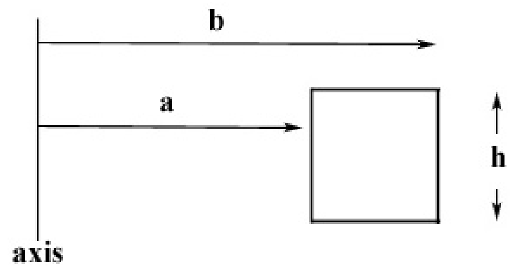
(c) The frequency of oscillation for an underdamped RLC circuit is less than the frequency of oscillation for an ideal LC according to

$$f_{RLC} = \frac{1}{2\pi}\sqrt{(\omega_{LC})^2 - \left(\frac{R}{2L}\right)^2} = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC}\right)^2 - \left(\frac{R}{2L}\right)^2} .$$

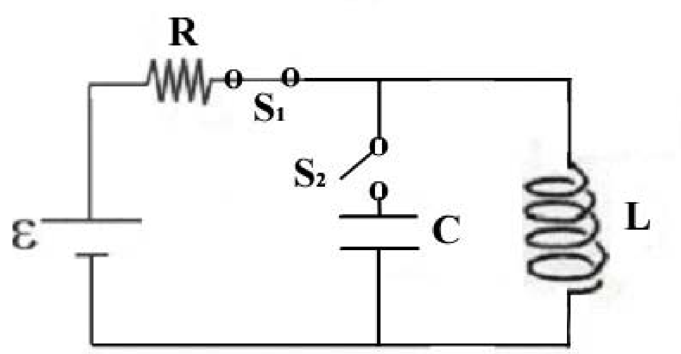
Problem 6:

A toroidoil coil has N turns, and an inner radius a, outer radius b, and height h. The coil has a rectangular cross section shown in the figures below.





The coil is connected via a switch, S_1, to an ideal voltage source with electromotive force $\mathcal{E}$. The circuit has total resistance R. Assume all the self-inductance L in the circuit is due to the coil. At time $t = 0$ S_1 is closed and S_2 remains open.



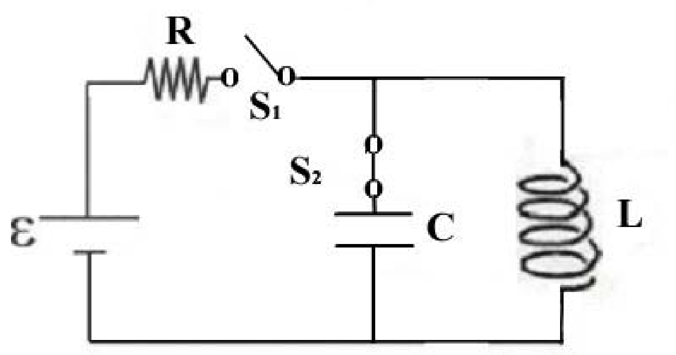
a) When a current I is flowing in the circuit, find an expression for the magnitude of the magnetic field inside the coil as a function of distance r from the axis of the coil.

b) What is the self-inductance L of the coil?

c) What is the current in the circuit a very long time $(t >> L/R)$ after S_1 is closed?

d) How much energy is stored in the magnetic field of the coil a very long time $(t >> L/R)$ after S_1 is closed?

For the next two parts, assume that a very long time $(t >> L/R)$ after the switch S_1 was closed, the voltage source is disconnected from the circuit by opening S_1, and by simultaneously closing S_2 the toroid is connected to a capacitor of capacitance C. Assume there is negligible resistance in this new circuit.

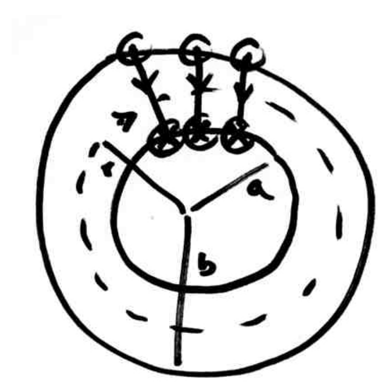


e) What is the maximum amount of charge that will appear on the capacitor?

f) How long will it take for the capacitor to first reach a maximal charge after S_2 has been closed?

Problem 6 Solution:

(a) The magnetic field is zero for $r < a$ and $r > b$. Choose a circle of radius r with $a < r < b$ for your Amperian loop (see figure below).



Then the left hand side of Ampere's Law, $\oint_{circle} \vec{\mathbf{B}} \cdot d\vec{\mathbf{r}} = \mu_0 I_{enc}$, becomes $\oint_{circle} \vec{\mathbf{B}} \cdot d\vec{\mathbf{r}} = B2\pi r$.

Since all N turns cuts through the Amperian circle, the right hand side of Ampere's law becomes $\mu_0 I_{enc} = \mu_0 NI$. So setting the two sides equal yields

$$B 2\pi r = \mu_0 NI \ .$$

Thus the magnitude of the magnetic field in the toroid is non-uniform (varies with distance r from the center) and is equal to is

$$B = \begin{cases} 0, & r < a \ and \ r > b \\ \dfrac{\mu_0 N\ I}{2\pi r}, & a < r < b \end{cases}$$

The field points in the clockwise direction when viewed from above

(b) We first need to find the magnitude of the magnetic flux through the toroid. We need to integrate the non-uniform magnetic field over the cross sectional area of one turn so we use for the area element $da = hdr$. Then the

$$\left| \int_{toroid} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} \right| = \left| N \int_{one\ turn} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} \right| = \left| N \int_{r=a}^{r=b} \frac{\mu_0 NI}{2\pi r} hdr \right| = \frac{\mu_0 N^2 h \ln(b/a)}{2\pi} I \ .$$

The magnetic flux through the toroid is proportional to the current,

$$N \int_{one turn} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = LI \ .$$

The constant of proportionality is called the self-inductance,

$$L = \left| N \int_{one turn} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} \Big/ I \right| .$$

The unit of self-inductance is the henry, [H], $[H] = [T \cdot m^2] / [A]$ and is given by

$$L_{toroid} = \frac{\mu_0 N^2 h \ln(b/a)}{2\pi}$$

(c) Solution: A very long time after the switch S_1 was closed, the current is steady so the inductor acts like a short and the current in the circuit is

$$I = \varepsilon / R$$

(d) The energy stored in the magnetic field is equal to

$$U_{mag} = \frac{1}{2} L I^2 = \frac{\mu_0 N^2 h \ln(b/a)}{4\pi R^2} \varepsilon^2 \, .$$

(e) When the switch S_2 is closed the current in the circuit is $I = \varepsilon / R$. The maximum amount of charge occurs when all the magnetic energy is converted to electrical energy

$$U_{elec} = \frac{Q_{\max}{}^2}{2C} = U_{mag,0} = \frac{1}{2} L I^2 = \frac{\mu_0 N^2 h \ln(b/a)}{4\pi R^2} \varepsilon^2 \, .$$

We can solve the above equation for the maximal charge on the capacitor

$$Q_{\max} = \sqrt{2CU_{mag,0}} = \sqrt{CLI_0{}^2} = \sqrt{\frac{2C\mu_0 N^2 h \ln(b/a)}{4\pi}} \, \varepsilon / R.$$

(f) It will take one quarter cycle, or

$$t = \frac{1}{4} T = \frac{1}{4} \frac{2\pi}{\omega_0} = \frac{\pi}{2} \sqrt{LC} = \frac{\pi}{2} \sqrt{\frac{C\mu_0 N^2 h \ln(b/a)}{2\pi}}$$

Problem 7:

For the underdamped RLC circuit, $R^2 < 4L/C$, let $\gamma = \left(1/LC - R^2/4L^2\right)^{1/2}$ and $\alpha = R/2L$.

(a) Show by direct substitution that the equation

$$0 = L\frac{d^2Q}{dt^2} + \frac{dQ}{dt}R + \frac{Q}{C}$$

has a solution of the form

$$Q(t) = Ae^{-\alpha t}\cos(\gamma t + \phi)$$

(b) Denote the current by

$$I(t) = \frac{dQ(t)}{dt} = Fe^{-\alpha t}\cos(\gamma t + \phi + \beta)$$

Find the constants F and β in terms of R, L and C as needed.

Problem 7 Solution:

(a) The first approach is by direct substitution. Calculate the first and second derivatives

$$\frac{dQ}{dt} = -\alpha Ae^{-\alpha t}\cos(\gamma t + \phi) - \gamma Ae^{-\alpha t}\sin(\gamma t + \phi)$$

$$\frac{d^2Q}{dt^2} = \alpha^2 Ae^{-\alpha t}\cos(\gamma t + \phi) + \alpha\gamma Ae^{-\alpha t}\sin(\gamma t + \phi) + \gamma\alpha Ae^{-\alpha t}\sin(\gamma t + \phi) - \gamma^2 Ae^{-\alpha t}\cos(\gamma t + \phi)$$

Then the differential equation becomes

$$0 = L\frac{d^2Q}{dt^2} + \frac{dQ}{dt}R + \frac{Q}{C}$$

becomes

$$\begin{aligned}
0 &= L\left(\alpha^2 Ae^{-\alpha t}\cos(\gamma t + \phi) + \alpha\gamma Ae^{-\alpha t}\sin(\gamma t + \phi) + \gamma\alpha Ae^{-\alpha t}\sin(\gamma t + \phi) - \gamma^2 Ae^{-\alpha t}\cos(\gamma t + \phi)\right) \\
&+ R\left(-\alpha Ae^{-\alpha t}\cos(\gamma t + \phi) - \gamma Ae^{-\alpha t}\sin(\gamma t + \phi)\right) \\
&+ \frac{1}{C}\left(Ae^{-\alpha t}\cos(\gamma t + \phi)\right)
\end{aligned}$$

This simplifies to

$$RHS = \left(L\left(\alpha^2 - \gamma^2\right) - \alpha R + \frac{1}{C} \right) Ae^{-\alpha t} \cos(\gamma t + \phi) + \left(\left(2L\alpha - R\right)\gamma \right) Ae^{-\alpha t} \sin(\gamma t + \phi)$$

Recall that

$$\alpha^2 - \gamma^2 = \left(R/2L\right)^2 - \left(1/LC - R^2/4L^2\right) = \frac{R^2}{2L} - \frac{1}{C}$$

Thus

$$L\left(\alpha^2 - \gamma^2\right) - \alpha R + \frac{1}{C} = \left(\frac{R^2}{2L} - \frac{1}{C} \right) - \frac{R^2}{2L} + \frac{1}{C} = 0$$

Also

$$\left(2L\alpha - R\right) = 2L\frac{R}{2L} - R = 0$$

So both coefficients on the RHS vanish hence

$$RHS = \left(L\left(\alpha^2 - \gamma^2\right) - \alpha R + \frac{1}{C} \right) Ae^{-\alpha t} \cos(\gamma t + \phi) + \left(\left(2L\alpha - R\right)\gamma \right) Ae^{-\alpha t} \sin(\gamma t + \phi) = 0$$

(b) From part (a)

$$I = \frac{dQ}{dt} = -\alpha Ae^{-\alpha t} \cos(\gamma t + \phi) - \gamma Ae^{-\alpha t} \sin(\gamma t + \phi) = e^{-\alpha t} \left(C\cos(\gamma t + \phi) + D\sin(\gamma t + \phi) \right)$$

where

$$C = -\alpha A \text{ and } D = -\gamma A$$

Using the results from problem 1

$$I = e^{-\alpha t} \left(C\cos(\gamma t + \phi) + D\sin(\gamma t + \phi) \right) = Fe^{-\alpha t} \cos(\gamma t + \phi + \beta)$$

where

$$F = \left(C^2 + D^2\right)^{1/2} = \left(\left(-\alpha A\right)^2 + \left(-\gamma A\right)^2 \right)^{1/2} = A\left(\alpha^2 + \gamma^2\right)^{1/2}$$
$$= A\left(\left(R/2L\right)^2 + \left(1/LC - R^2/4L^2\right) \right)^{1/2} = \frac{A}{\sqrt{LC}}$$

and

$$\beta = \tan^{-1}(-D/C) = \tan^{-1}(-\gamma/\alpha) = \tan^{-1}\left(-\frac{\left(1/LC - R^2/4L^2\right)^{1/2}}{R/2L}\right)$$

Note: A second approach to part (a) explicitly solves the equation by first conjecturing that the solution is of the form

$$Q = Ae^{zt}$$

where z is a number (possibly complex). Then

$$\frac{dQ}{dt} = zAe^{zt}, \qquad \frac{d^2Q}{dt^2} = z^2 Ae^{zt}$$

so the circuit equation becomes

$$0 = \left(Lz^2 + zR + + \frac{1}{C}\right)Ae^{zt}$$

The condition for the solution is that the characteristic polynomial

$$Lz^2 + zR + + \frac{1}{C} = 0$$

This equation has solutions of the form

$$z = \frac{-R \pm \left(R^2 - 4L/C\right)^{1/2}}{2L}$$

When $R^2 < 4L/C$, we have two solutions for z, however the solutions are complex. Let $\gamma = \left(1/LC - R^2/4L^2\right)^{1/2}$ and $\alpha = R/2L$. Recall that the imaginary number $i = \sqrt{-1}$. Then $z_1 = -\alpha + i\gamma t$ and $z_2 = -\alpha - i\gamma t$. So the charge becomes

$$Q = A_1 e^{-\alpha + i\gamma t} + A_2 e^{-\alpha - i\gamma t} = \left(A_1 e^{i\gamma t} + A_2 e^{-i\gamma t}\right)e^{-\alpha t},$$

where A_1 and A_2 are constants.

We shall transform this expression into a more familiar equation involving sine and cosine functions with using the Euler formula,

$$e^{\pm i\gamma t} = \cos\gamma t \pm i \sin\gamma t \,.$$

We can rewrite our solution as

$$Q = \left(A_1\left(\cos\gamma t + i\sin\gamma t\right) + A_2\left(\cos\gamma t - i\sin\gamma t\right)\right)e^{-\alpha t}$$

A little rearrangement yields

$$Q = \left(\left(A_1 + A_2\right)\cos\gamma t + i\left(A_1 - A_2\right)\sin\gamma t\right)e^{-\alpha t}$$

Define two new constants

$$C = A_1 + A_2 \text{ and } D = i\left(A_1 - A_2\right).$$

Then our solution looks like

$$Q = \left(C\cos\gamma t + D\sin\gamma t\right)e^{-\alpha t}\,.$$

Now use the identity from problem 1: $C\cos\omega t + D\sin\omega t = A\cos\left(\omega t + \phi\right)$ where $A = \left(C^2 + D^2\right)^{1./2}$ and $\phi = \tan^{-1}(-D/C)$. Thus

$$Q(t) = Ae^{-\alpha t}\cos(\gamma t + \phi)$$

Problem 8:
***LC* Circuit**

An inductor having inductance L and a capacitor having capacitance C are connected in series. The current in the circuit increase linearly in time as described by $I = Kt$. The capacitor initially has no charge. Determine

(a) the voltage across the inductor as a function of time,

(b) the voltage across the capacitor as a function of time, and

(c) the time when the energy stored in the capacitor first exceeds that in the inductor.

Problem 8 Solutions:
(a)The voltage across the inductor is

$$\varepsilon_L = -L\frac{dI}{dt} = -L\frac{d}{dt}\left(Kt\right) = -LK$$

(b)Using $I = \dfrac{dQ}{dt}$, the charge on the capacitor as a function of time may be obtained as

$$Q(t) = \int_0^t I dt' = \int_0^t K t' dt' = \frac{1}{2}Kt^2$$

Thus, the voltage drop across the capacitor as a function of time is

$$\Delta V_C = -\frac{Q}{C} = -\frac{Kt^2}{2C}$$

(c)The energies stored in the capacitor and the inductor are

$$U_C = \frac{1}{2}C\left(\Delta V_C\right)^2 = \frac{1}{2}C\left(-\frac{Kt^2}{2C}\right)^2 = \frac{K^2t^4}{8C}$$
$$U_L = \frac{1}{2}LI^2 = \frac{1}{2}L(Kt)^2 = \frac{1}{2}LK^2t^2$$

The two energies are equal when

$$\frac{K^2t'^4}{8C} = \frac{1}{2}LK^2t'^2 \quad \Rightarrow \quad t' = 2\sqrt{LC}$$

Therefore, $U_C > U_L$ when $t > t'$.

Problem 9:
***LC* Circuit**

(a) Initially, the capacitor in a series *LC* circuit is charged. A switch is closed, allowing the capacitor to discharge, and after time *T* the energy stored in the capacitor is one-fourth its initial value. Determine *L* if *C* and *T* are known.

(b) A capacitor in a series *LC* circuit has an initial charge Q_0 and is being discharged. The inductor is a solenoid with N turns. Find, in terms of *L* and *C*, the flux through each of the *N* turns in the coil at time *t*, when the charge on the capacitor is *Q*(*t*).

(c) An *LC* circuit consists of a 20.0-mH inductor and a $0.500\text{-}\mu\text{F}$ capacitor. If the maximum instantaneous current is 0.100 A, what is the greatest potential difference across the capacitor?

Problem 9 Solutions:

(a) The energy stored in the capacitor is given by

$$U_C(t) = \frac{Q(t)^2}{2C} = \frac{\left(Q_0 \cos\omega_0 t\right)^2}{2C} = \frac{Q_0^2}{2C}\cos^2\omega_0 t$$

Thus,

$$\frac{U_C(T)}{U_C(0)} = \frac{\cos^2\omega_0 T}{\cos^2(0)} = \frac{\cos^2\omega_0 T}{1} = \frac{1}{4} \quad \Rightarrow \cos\omega_0 T = \frac{1}{2}$$

which implies that $\omega_0 T = \frac{\pi}{3}\,\text{rad} = 60°$. Therefore, with $\omega_0 = \frac{1}{\sqrt{LC}}$, we obtain

$$T = \frac{\pi}{3\omega_0} = \frac{\pi}{3}\sqrt{LC} \quad \Rightarrow \quad L = \frac{1}{C}\left(\frac{3T}{\pi}\right)^2$$

(b)

We can do this two ways, either is acceptable. First,we can make the explicit assumption that $Q(t) = Q_0 \cos \omega_0 t$ and the total flux through the inductor is $LI = L\dfrac{dQ}{dt} = -L\omega_0 Q_0 \sin \omega_0 t$

Therefore the flux through one turn of the inductor at time t is $\Phi_{\text{one turn}} = -\dfrac{L\omega_0 Q_0}{N} \sin \omega_0 t$

or in terms of L and C, $\Phi_{\text{one turn}} = -\sqrt{\dfrac{L}{C}}\dfrac{Q_0}{N} \sin \omega_0 t$. Or second, we can simply leave $Q(t)$ as an unspecified function of time and write (using the same arguments as above) that $\Phi_{\text{one turn}} = \dfrac{L}{N}\dfrac{dQ}{dt}$.

(c) The greatest potential difference across the capacitor when $U_{C\max} = U_{L\max}$, or

$$\frac{1}{2} C V_{C\max}^2 = \frac{1}{2} L\, I_{\max}^2 \quad \Rightarrow \quad V_{C\max} = \sqrt{\frac{L}{C}} I_{\max} = \sqrt{\frac{(20.0\,\text{mH})}{(0.500\,\mu\text{F})}}\left(0.100\text{A}\right) = 20\text{ V}$$

8.02SC Physics II: Electricity and Magnetism
Fall 2010