Module 25:
Driven RLC Circuits

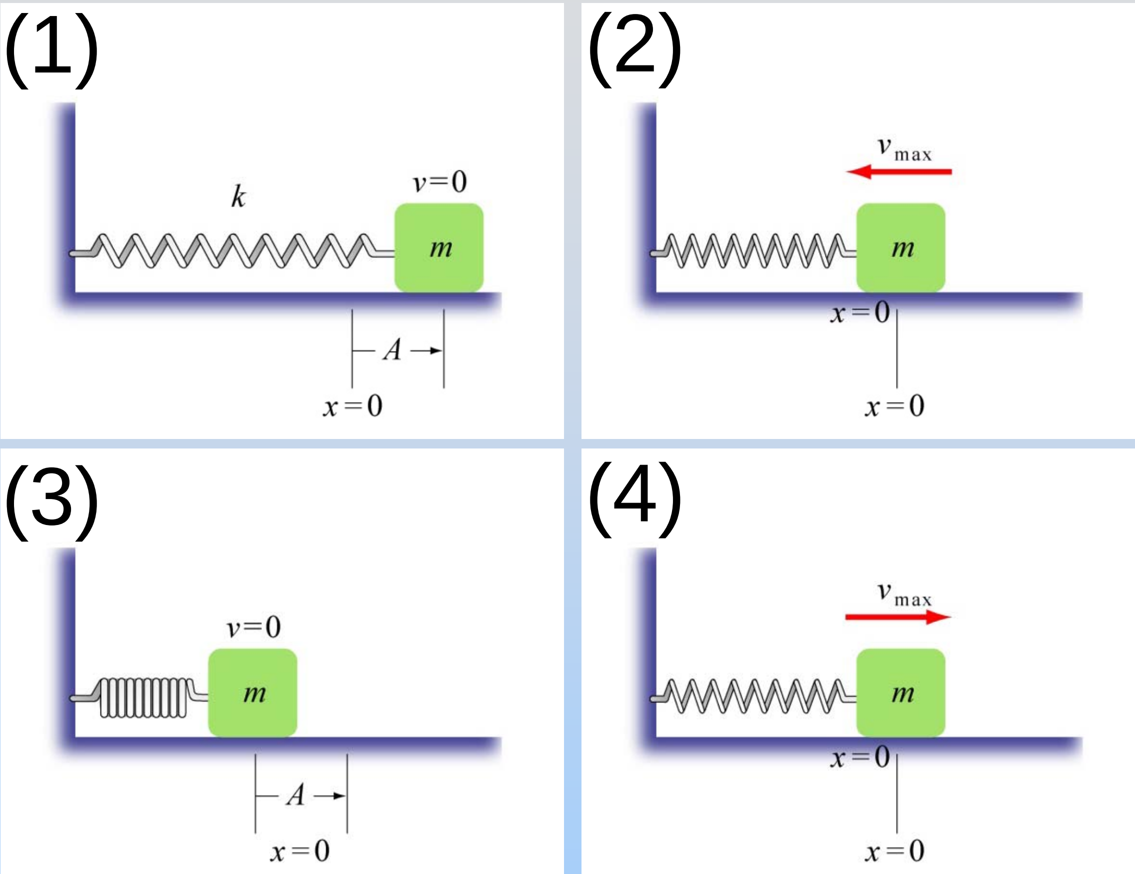
Module 25: Outline

Resonance & Driven LRC Circuits

Driven Oscillations: Resonance

Mass on a Spring: Simple Harmonic Motion A Second Look

Mass on a Spring



We solved this:

$$F = -kx = ma = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} + kx = 0$$

Simple Harmonic Motion

$$x(t) = x_0 \cos(\omega_0 t + \phi)$$

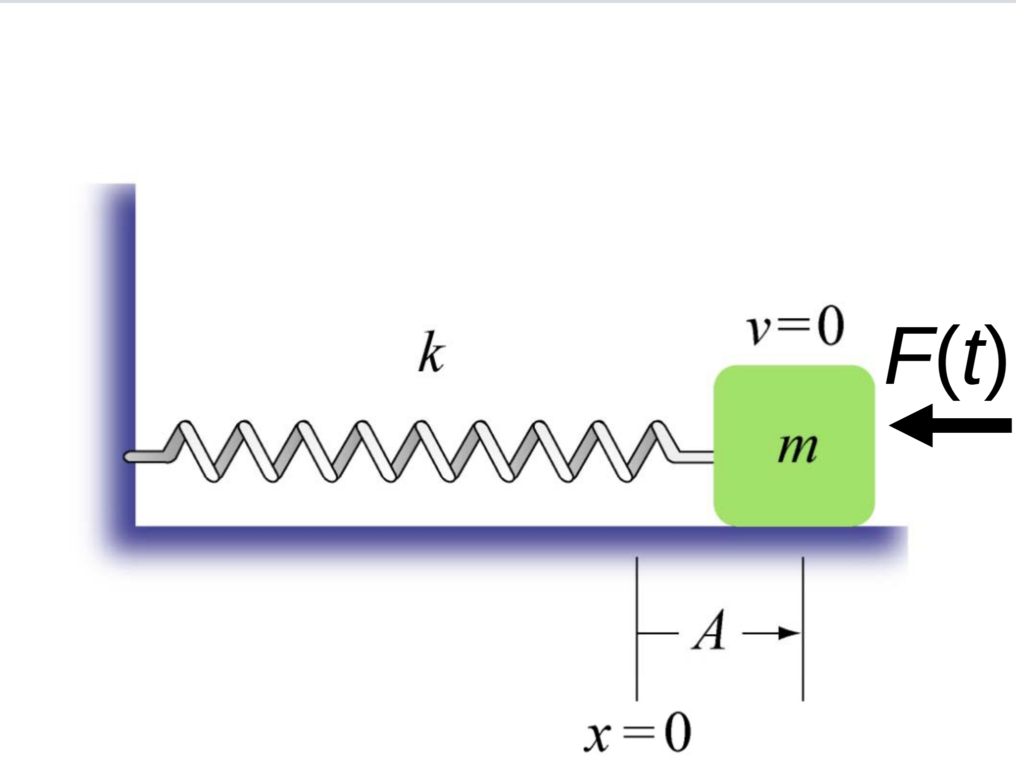
Moves at natural frequency

What if we now move the wall?

Push on the mass?

Demonstration:
Driven Mass on a Spring
Off Resonance

Driven Mass on a Spring



Now we get:

$$F = F(t) - kx = ma = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} + kx = F(t)$$

Assume harmonic force:

$$F(t) = F_0 \cos(\omega t)$$

Simple Harmonic Motion

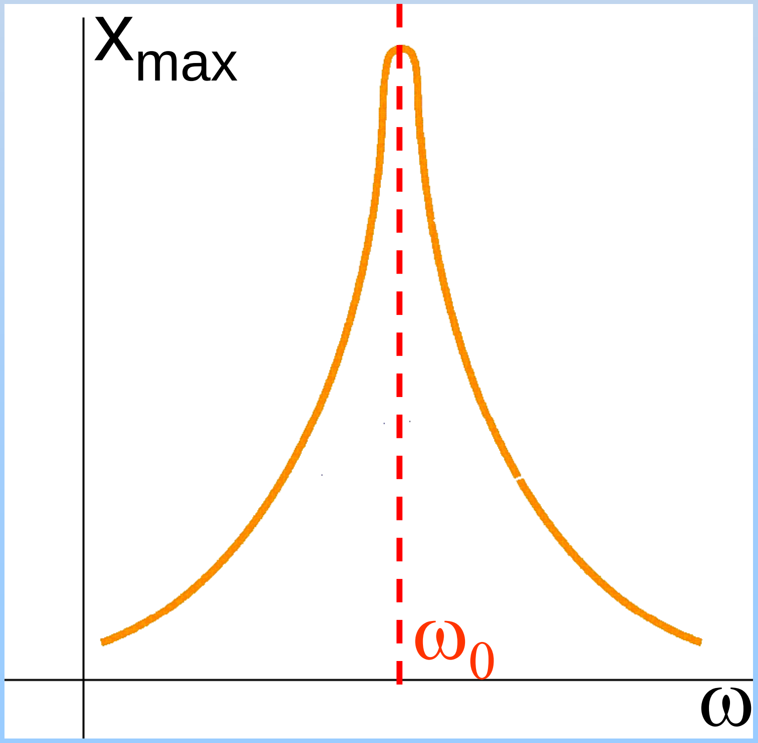
$$x(t) = x_{\max} \cos(\omega t + \phi)$$

Moves at driven frequency

Resonance

$$x(t) = x_{\max}\cos(\omega t + \phi)$$

Now the amplitude, x_{max}, depends on how close the drive frequency is to the natural frequency



Let's

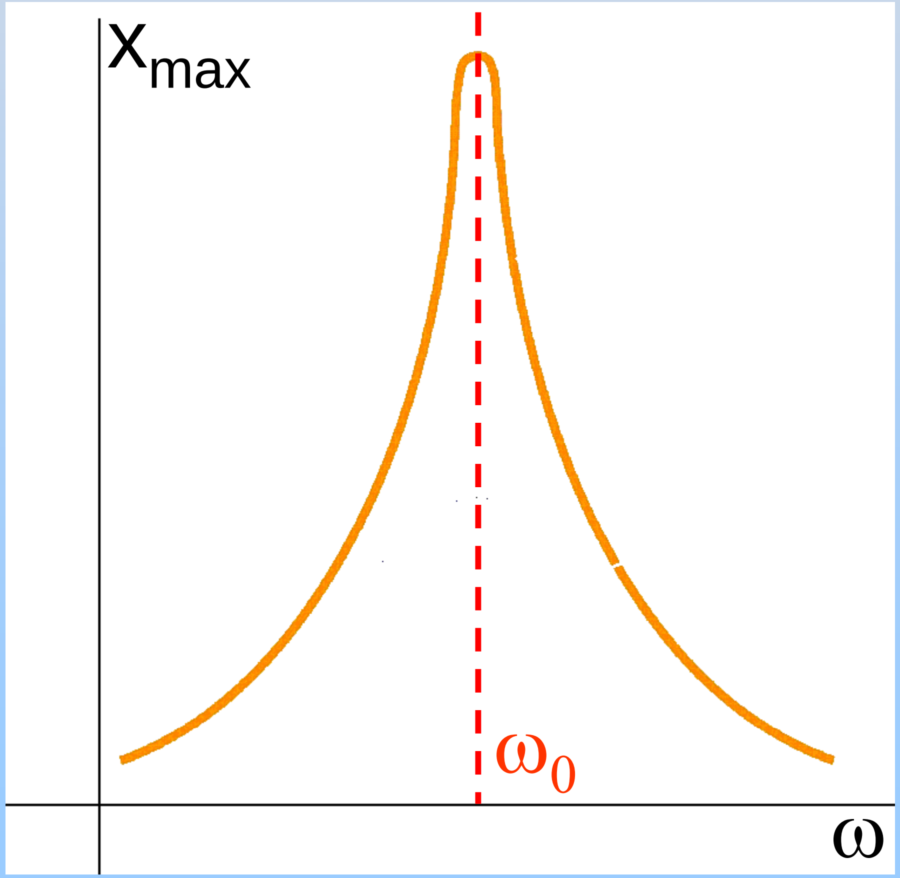
See…

Demonstration:
Driven Mass on a Spring

Resonance

$$x(t) = x_{\max} \cos(\omega t + \phi)$$

x_{max} depends on drive frequency



Many systems behave like this:

Swings

Some cars

Kendall T Station

…

Famous Resonance Examples

Electronic Analog: RLC Circuits

Analog: RLC Circuit

Recall:

Inductors are like masses (have inertia)

Capacitors are like springs (store/release energy)

Batteries supply external force (EMF)

Charge on capacitor is like position,

Current is like velocity – watch them resonate

Now we move to “frequency dependent batteries:”

AC Power Supplies/AC Function Generators

Demonstration:
RLC with Light Bulb

Concept Q.: RLC Circuit w/ Light bulb

As I slide the core into the inductor the light bulb changes brightness. Why?

I am driving the circuit through resonance by…

1. continuously increasing the frequency of current oscillations in the circuit
2. continuously decreasing the frequency of current oscillations in the circuit
3. continuously increasing the natural frequency of oscillations in the circuit
4. continuously decreasing the natural frequency of oscillations in the circuit
5. I don't know

Start at Beginning:
AC Circuits

Problem: Discovery Mathlet Driven RLC Circuits

Alternating-Current Circuit

- direct current (dc) – current flows one way (battery)
- alternating current (ac) – current oscillates

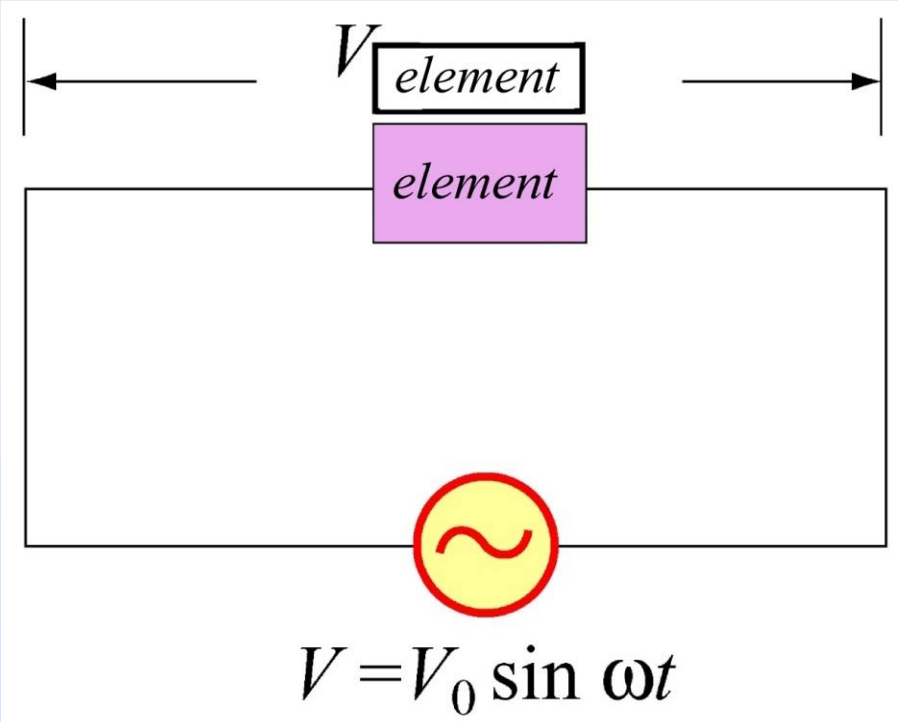
- sinusoidal voltage source

$$V(t) = V_0 \sin \omega t$$

$$a = 2\pi f : \text{ angular frequency}$$

$$V_0 : \text{ voltage amplitude}$$

AC Circuit: Single Element



$$V_{\boxed{element}} = V$$

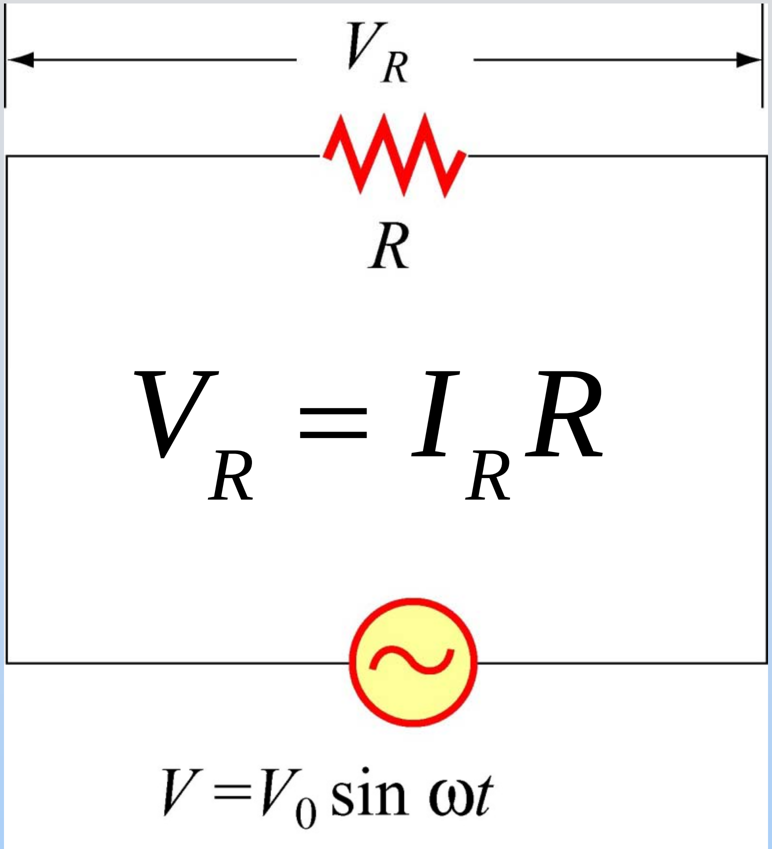
$$= V_0 \sin \omega t$$

$$I(t) = I_0 \sin(\omega t - \phi)$$

Questions:

1. What is I_0?
2. What is ϕ?

AC Circuit: Resistors



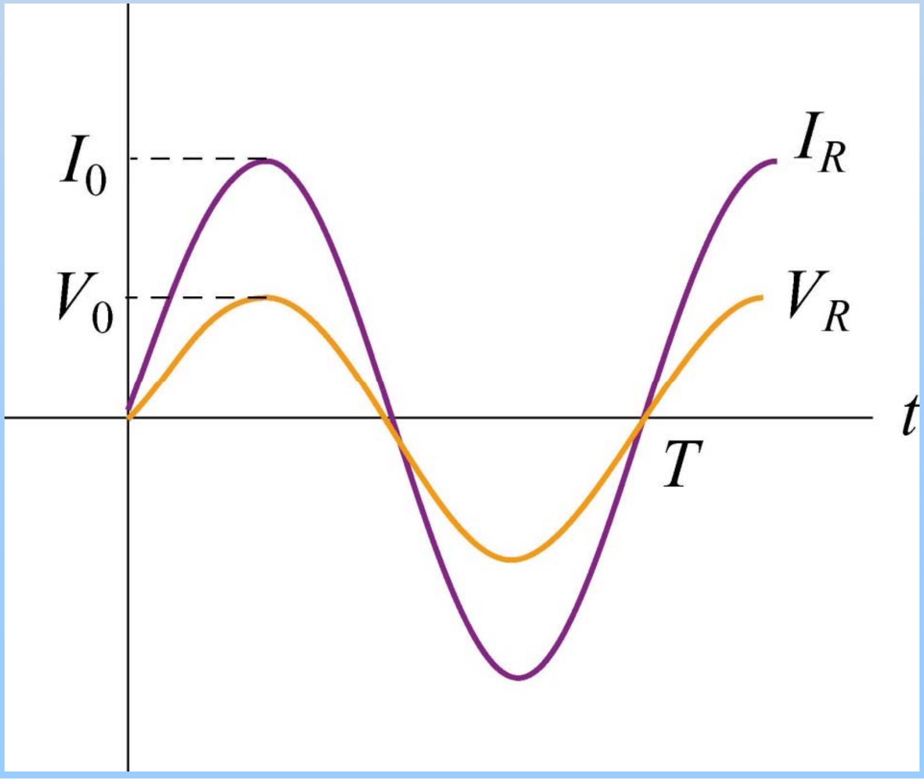
$$I_R = \frac{V_R}{R} = \frac{V_0}{R} \sin \omega t$$

$$= I_0 \sin\left(\omega t - 0\right)$$

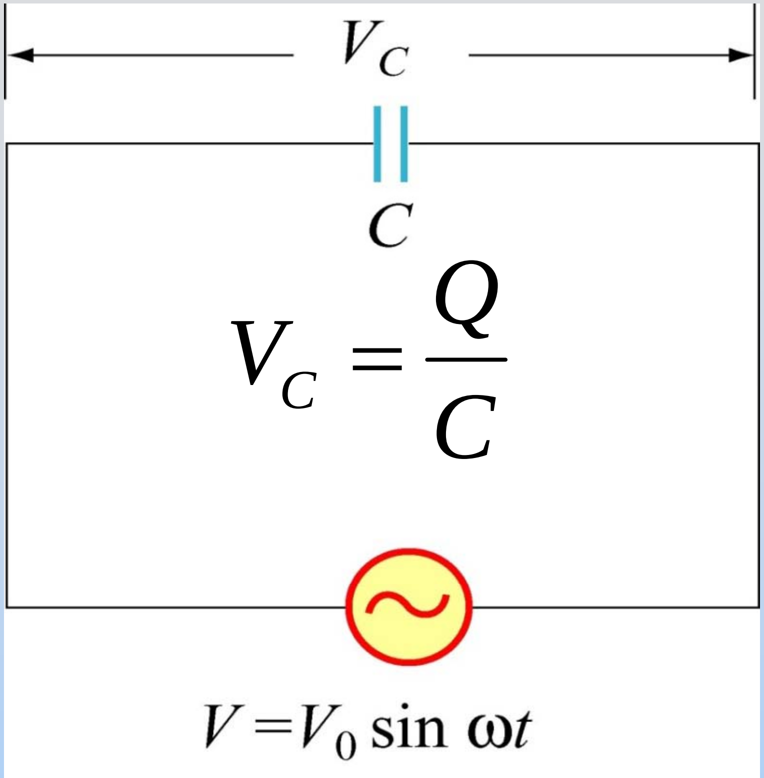
$$I_0 = \frac{V_0}{R}$$

$$\varphi = 0$$

I_R and V_R are **in phase**



AC Circuit: Capacitors



$$V_C = \frac{Q}{C}$$

$$I_C(t) = \frac{dQ}{dt}$$

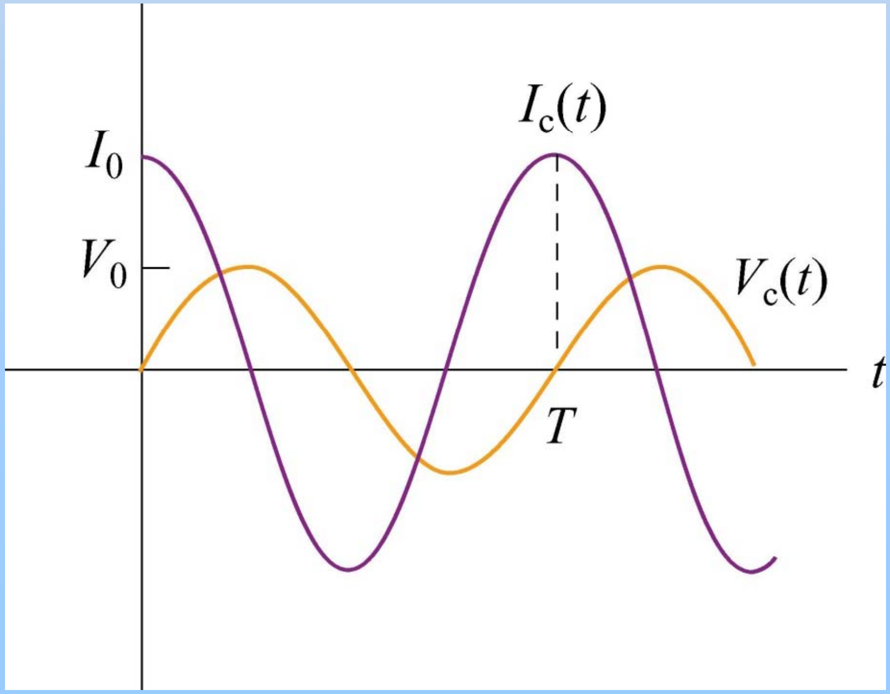
$$= \omega C V_0 \cos \omega t$$

$$= I_0 \sin(\omega t - \pi / 2)$$

$$I_0 = \omega C V_0$$

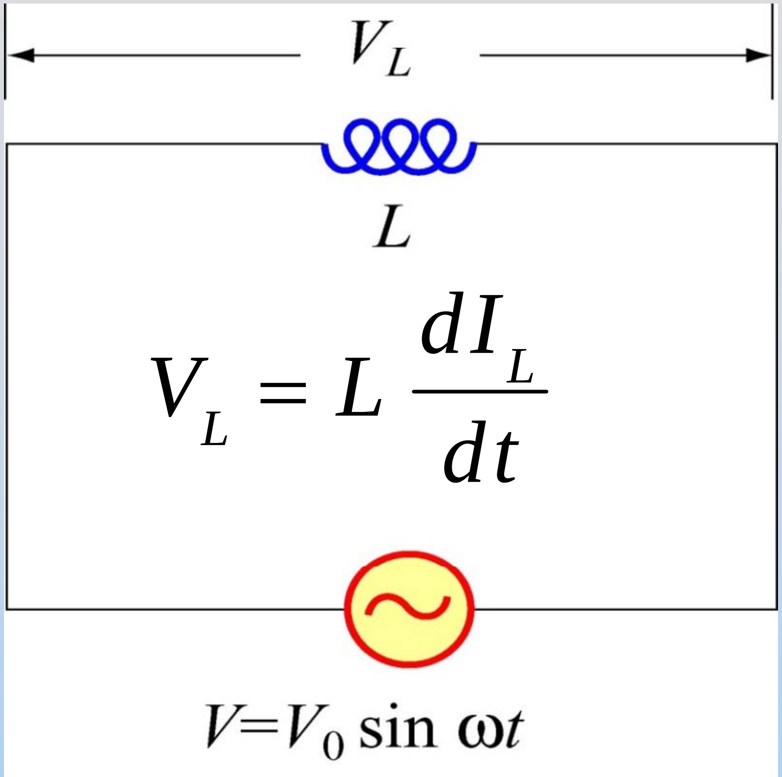
$$\phi = -\pi / 2$$

I_C **leads** V_C by $\pi/2$



$$Q(t) = CV_C = CV_0 \sin \omega t$$

AC Circuit: Inductors



$$V_L = L\frac{dI_L}{dt}$$

$$\frac{dI_L}{dt} = \frac{V_L}{L} = \frac{V_0}{L}\sin\omega t$$

$$I_L(t) = \frac{V_0}{L}\int \sin\omega t \; dt$$

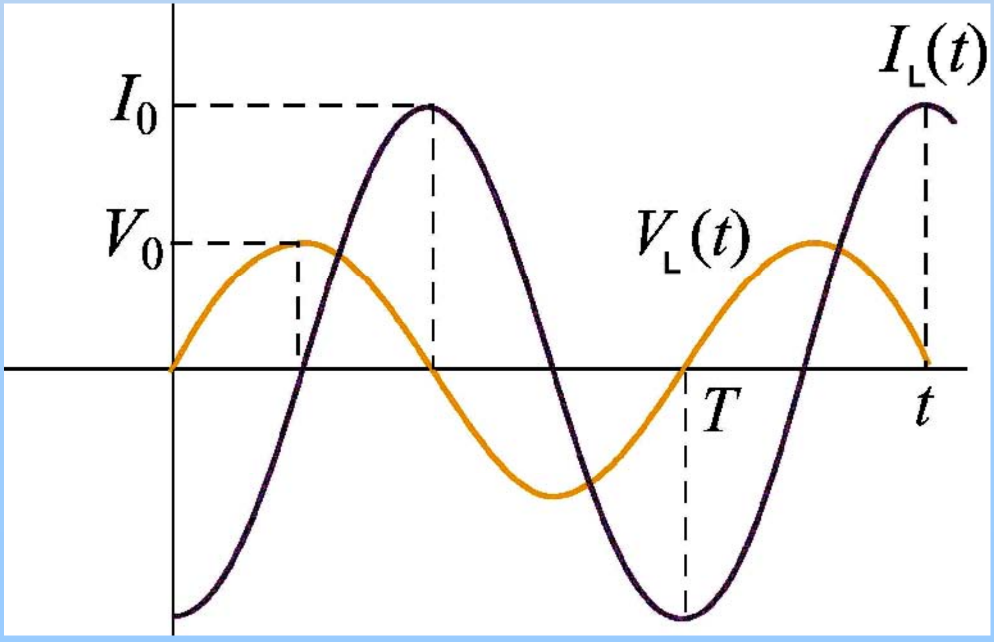
$$= -\frac{V_0}{\omega L}\cos\omega t$$

$$= I_0 \sin(\omega t - \pi/2)$$

$$I_0 = \frac{V_0}{\omega L}$$

$$\phi = \pi/2$$

***I_L* lags *V_L* by $\pi/2$**



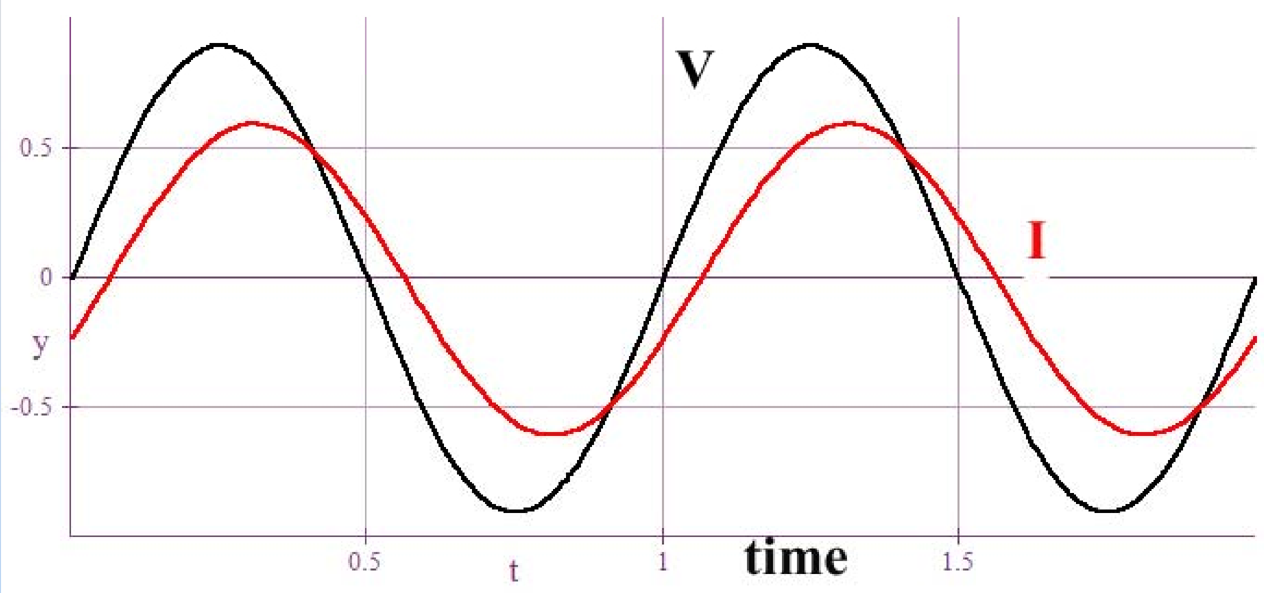
AC Circuits: Summary

Element	I_0	Current vs. Voltage	Resistance Reactance
Resistor	$\frac{V_{0R}}{R}$	In Phase	$R = R$
Capacitor	$\omega C V_{0C}$	Leads	$X_C = \frac{1}{\omega C}$
Inductor	$\frac{V_{0L}}{\omega L}$	Lags	$X_L = \omega L$

Although derived from single element circuits, these relationships hold generally!

Concept Question: Leading or Lagging?

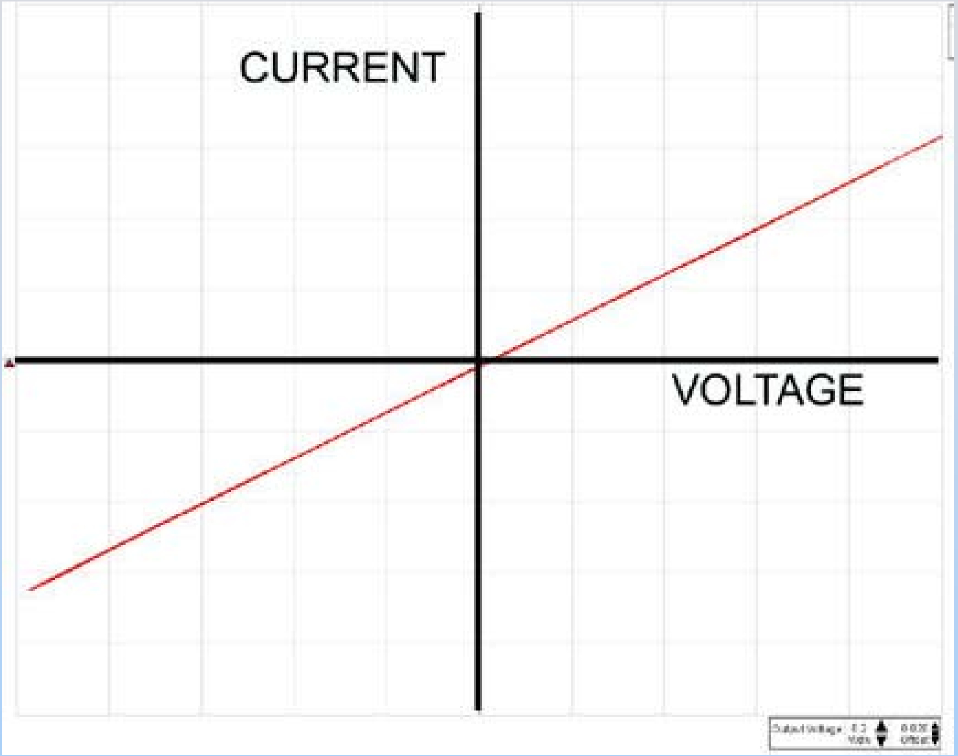
The plot shows the driving voltage V (black curve) and the current I (red curve) in a driven RLC circuit.   In this circuit,



1. The current leads the voltage
2. The current lags the voltage
3. Don’t have a clue.

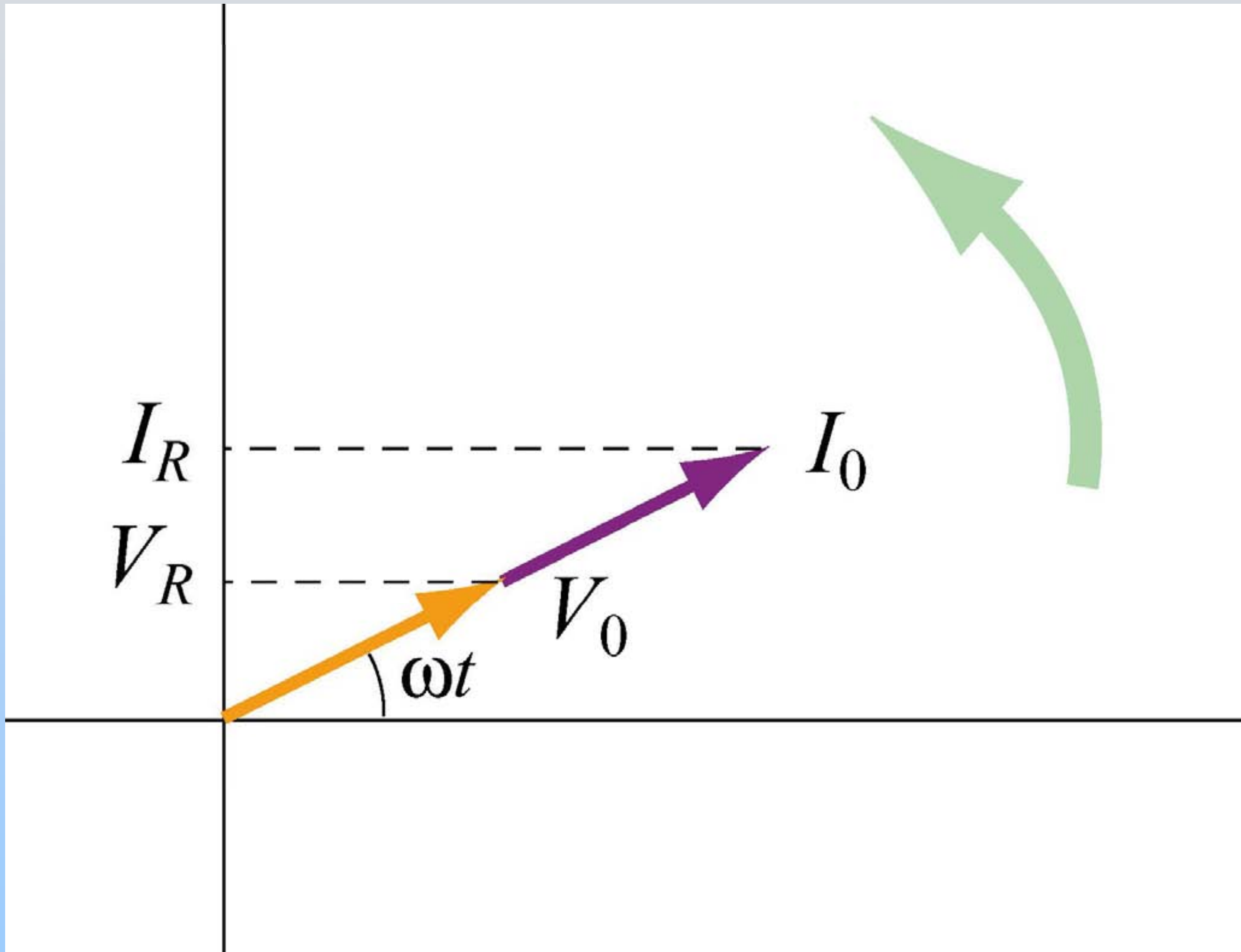
Concept Question: Leading or Lagging?

The graph shows current versus voltage in a driven RLC circuit at a given driving frequency. In this plot



1. The current leads the voltage by about 45º
2. The current lags the voltage by about 45º
3. The current and the voltage are in phase
4. Don’t have a clue

Phasor Diagram: Resistor

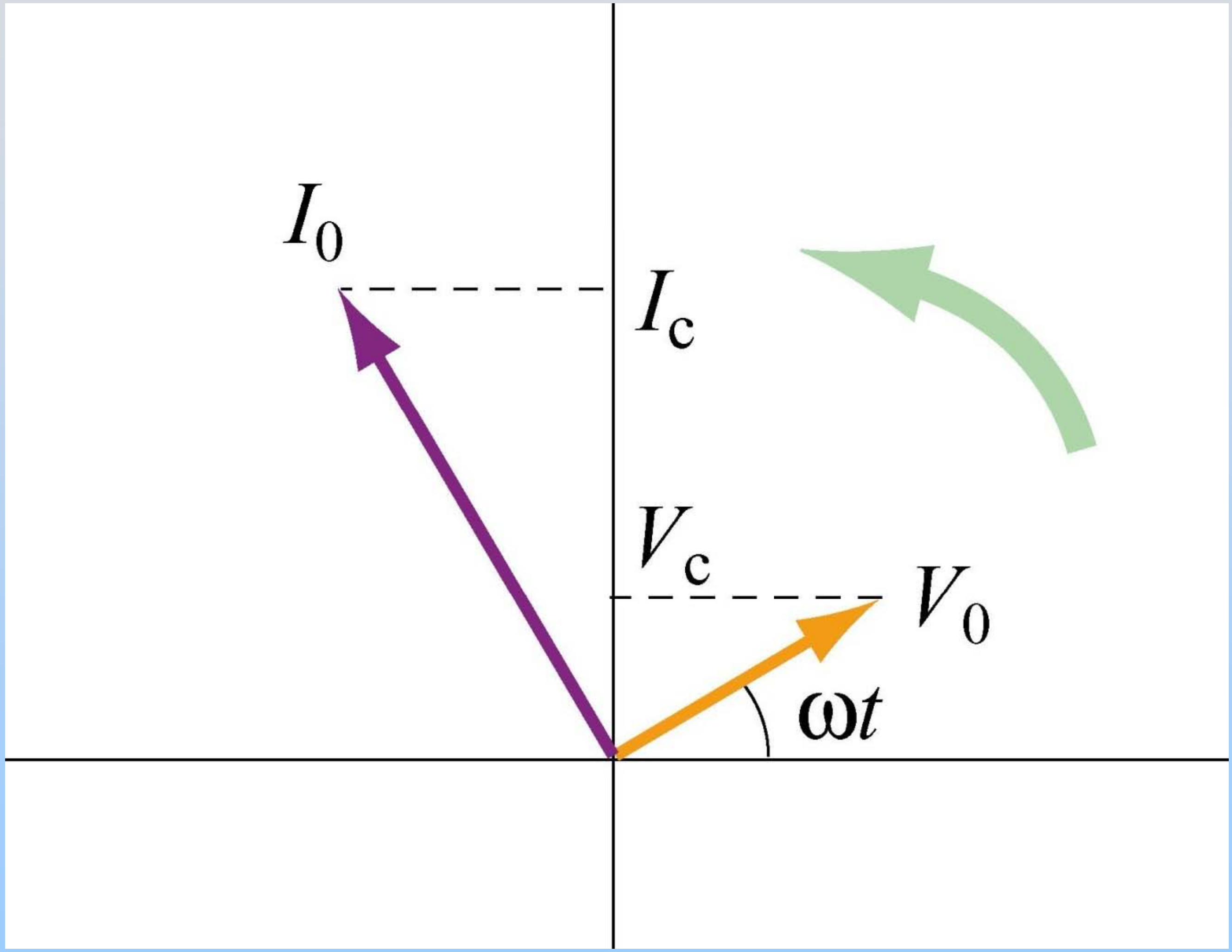


$$V_0 = I_0 R$$

$$\phi = 0$$

I_R and V_R are **in phase**

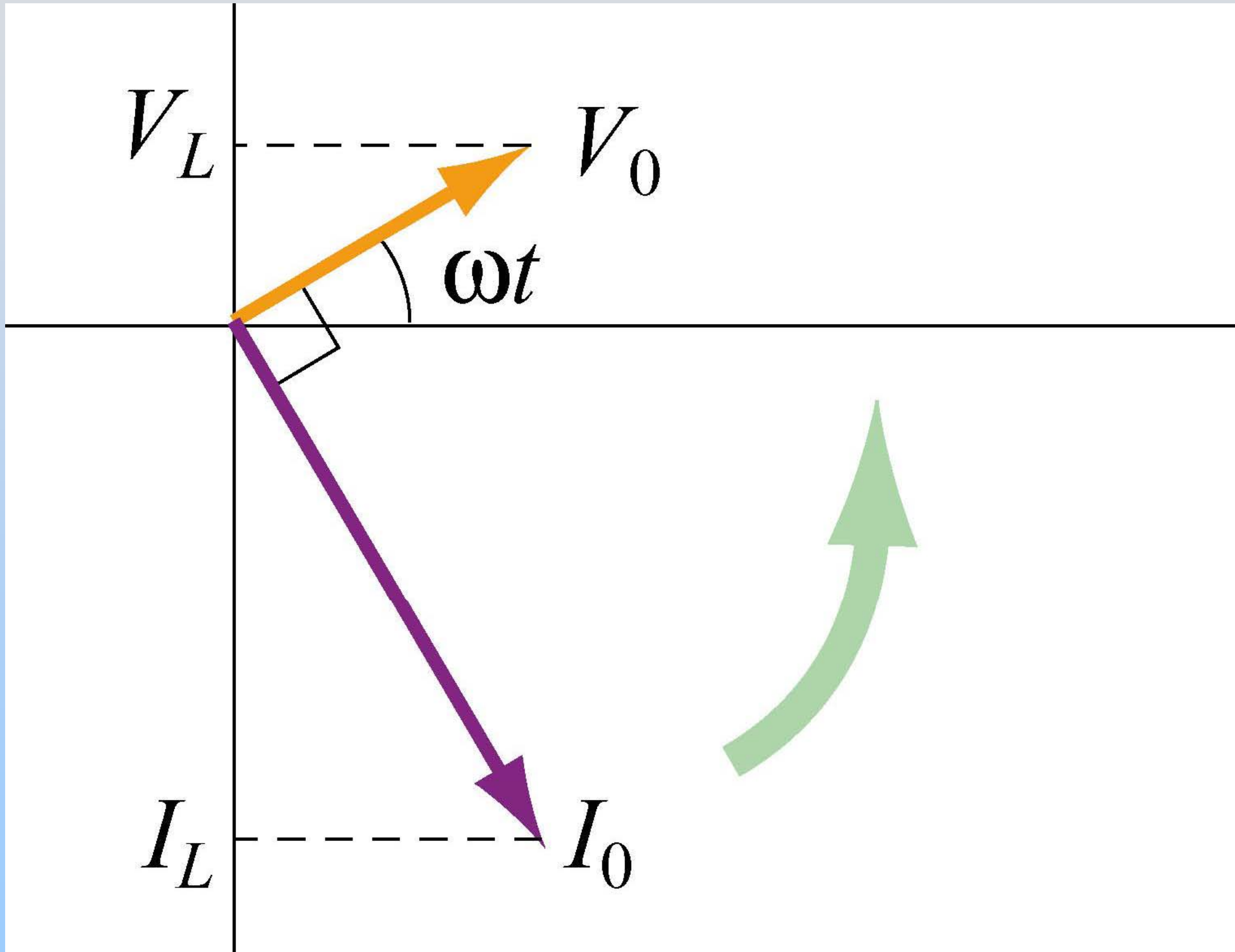
Phasor Diagram: Capacitor



$$V_0 = I_0 X_C$$
$$= I_0 \frac{1}{\omega C}$$
$$\phi = -\pi / 2$$

I_C **leads** V_C by $\pi/2$

Phasor Diagram: Inductor



$$V_0 = I_0 X_L$$
$$= I_0 \omega L$$
$$\phi = \pi / 2$$

I_L **lags** V_L by $\pi/2$

Put it all together: Driven RLC Circuits

Question of Phase

We had fixed phase of voltage:

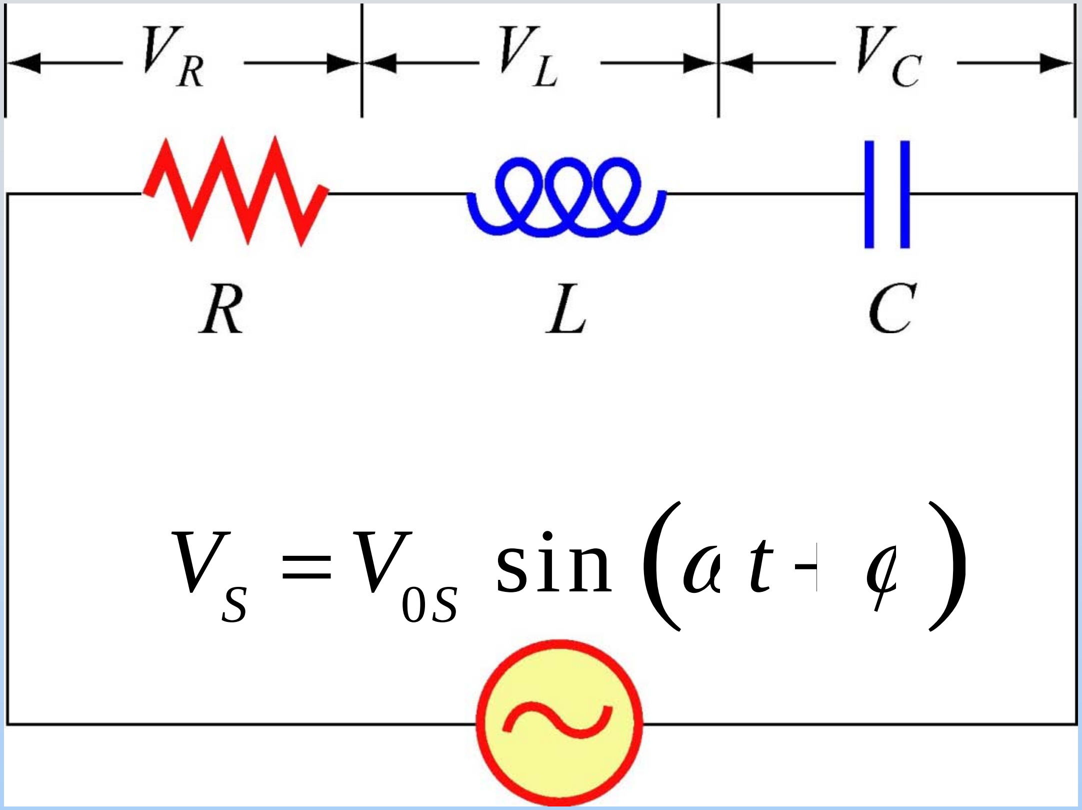
$$V_{\boxed{element}} = V_0 \sin \omega t \qquad I(t) = I_0 \sin(\omega t - \phi)$$

It's the same to write:

$$V_{\boxed{element}} = V_0 \sin(\omega t + \phi) \quad I(t) = I_0 \sin \omega t$$

(Just shifting zero of time)

Driven RLC Series Circuit



$$V_S = V_{0S}\sin\left(\omega t + \phi\right)$$

$$I(t) = I_0\sin(\omega t)$$

$$V_R = V_{R0}\sin\left(\omega t\right)$$

$$V_L = V_{L0}\sin\left(\omega t + \pi/2\right)$$

$$V_C = V_{C0}\sin\left(\omega t - \pi/2\right)$$

What is I_0 (and $V_{R0} = I_0 R, V_{L0} = I_0 X_L, V_{C0} = I_0 X_C$)?

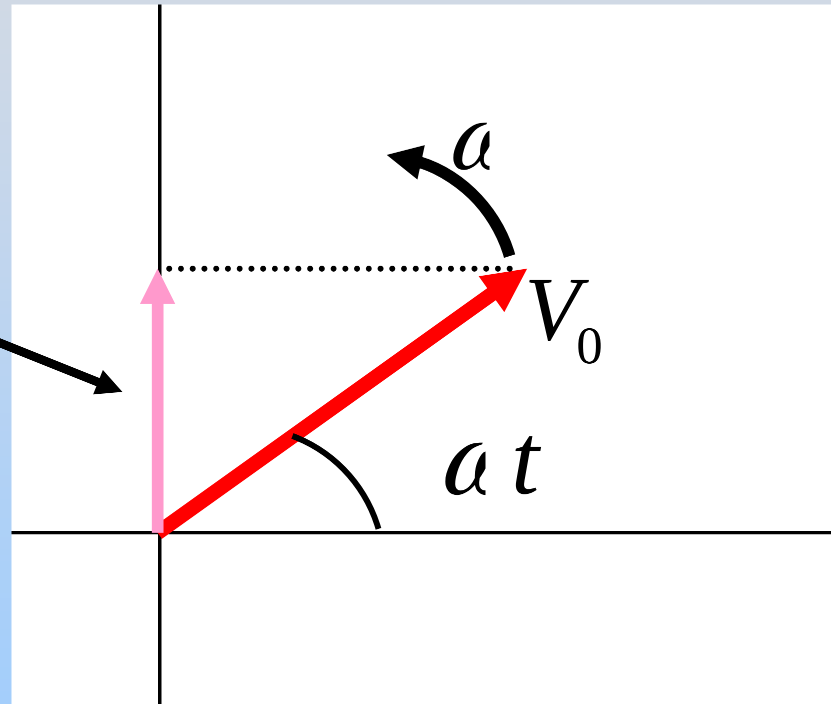
What is ϕ? Does the current lead or lag V_s?

Must Solve: $V_S = V_R + V_L + V_C$

Recall: Phasor Diagram

Nice way of tracking magnitude & phase:

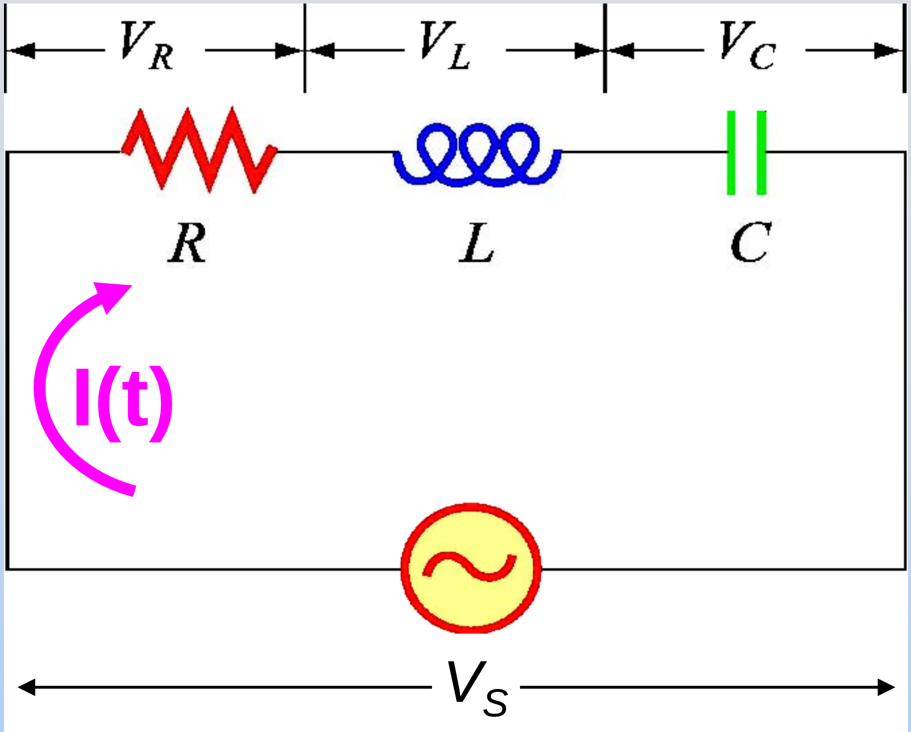
$$V(t) = V_0 \sin(\omega t)$$

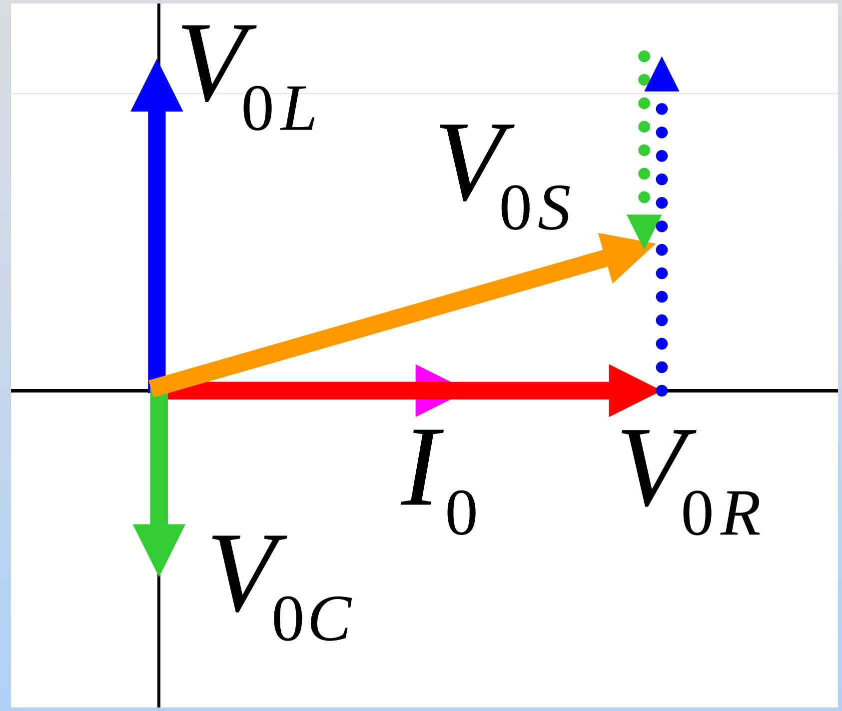


Notes: (1) As the phasor (red vector) rotates, the projection (pink vector) oscillates

(2) Do both for the current and the voltage

Driven RLC Series Circuit

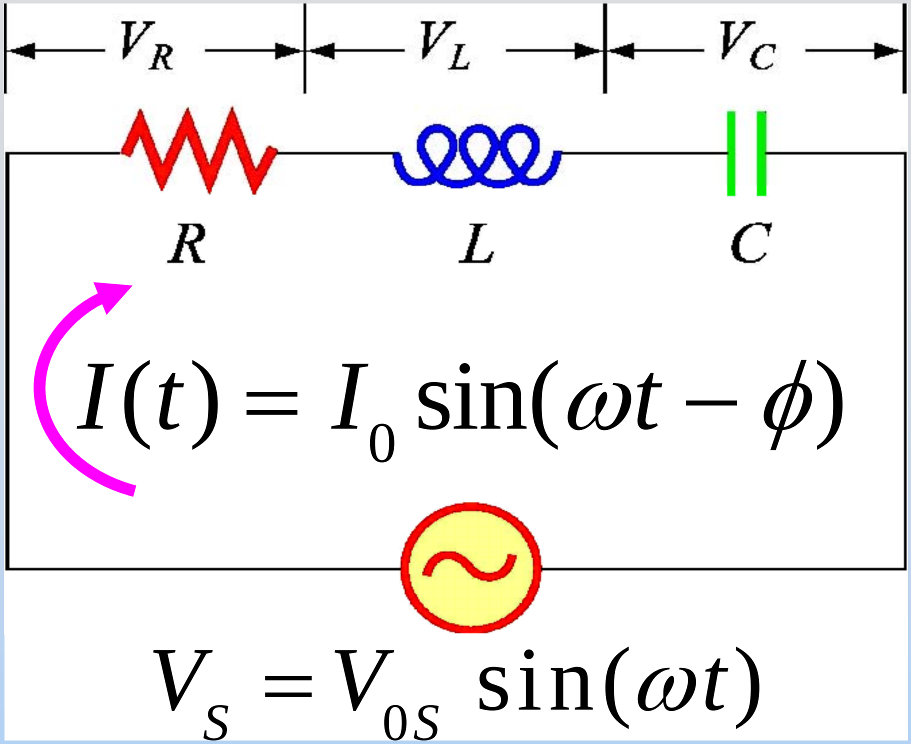


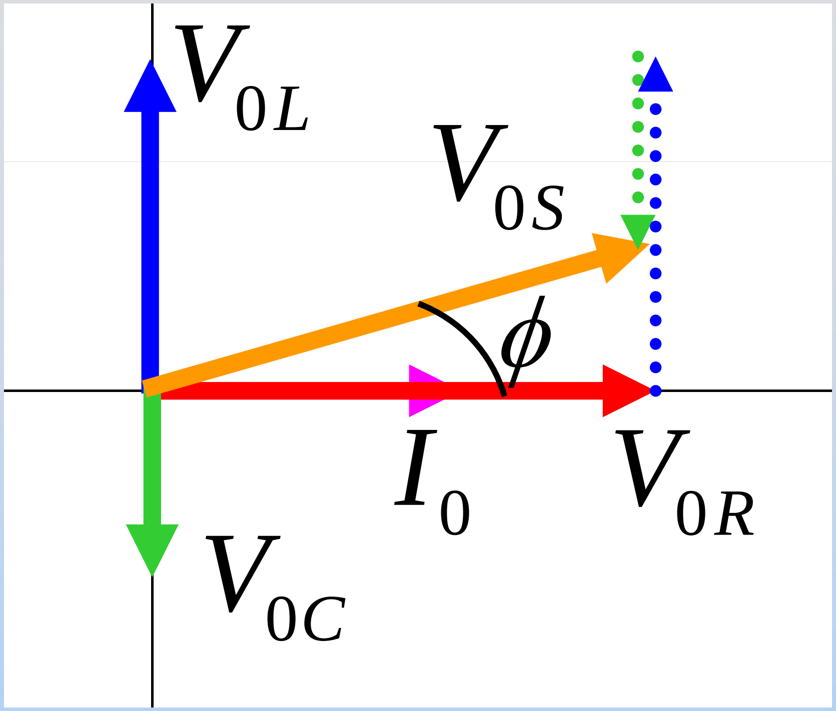


$$\text{Now Solve:} \quad V_S = V_R + V_L + V_C$$

Now we just need to read the phasor diagram!

Driven RLC Series Circuit





$$V_{0S} = \sqrt{V_{R0}^2 + (V_{L0} - V_{C0})^2} = I_0\sqrt{R^2 + (X_L - X_C)^2} \equiv I_0 Z$$

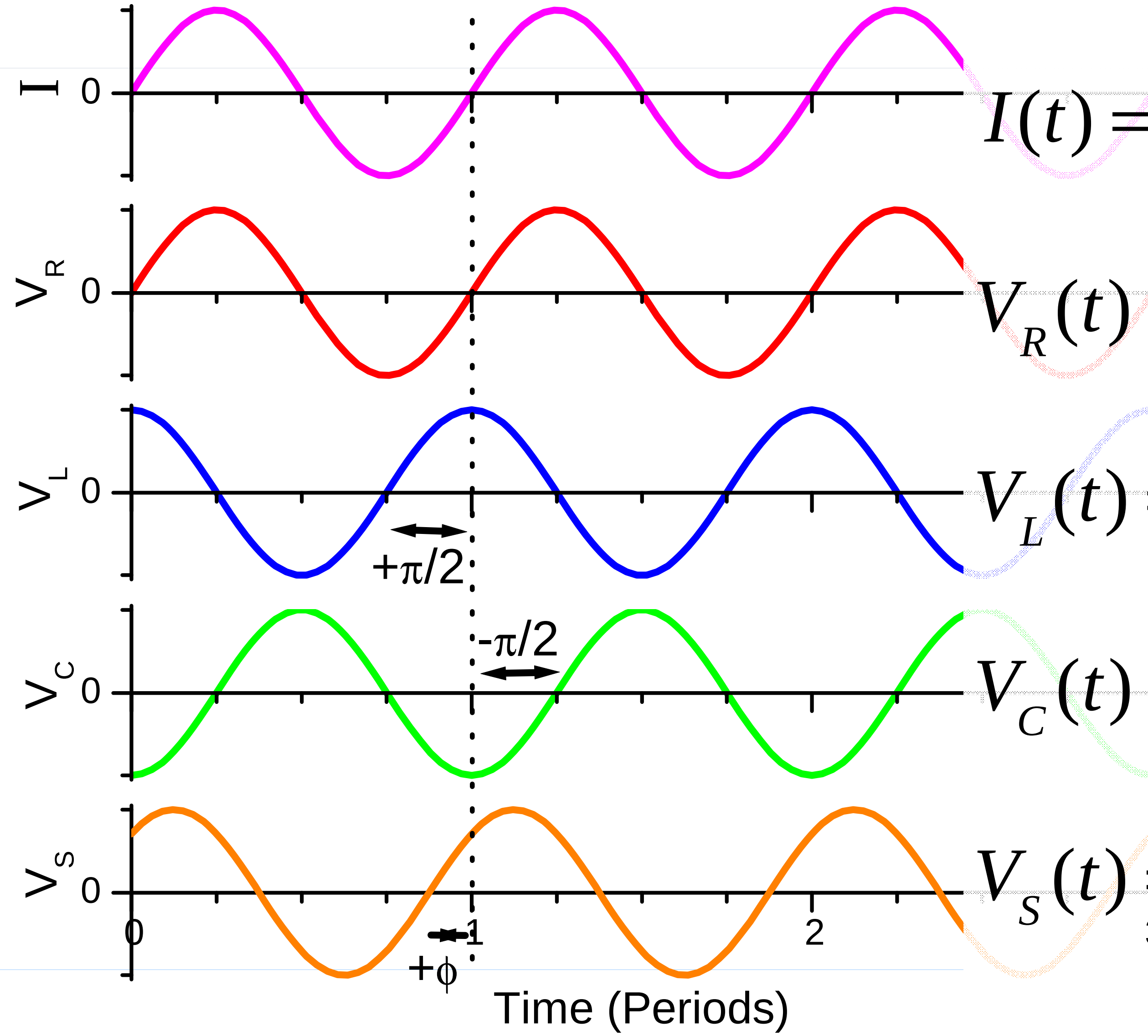
$$I_0 = \frac{V_{0S}}{Z}$$

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

Impedance

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

Plot I, V's vs. Time



$$I(t) = I_0 \sin(a\, t)$$

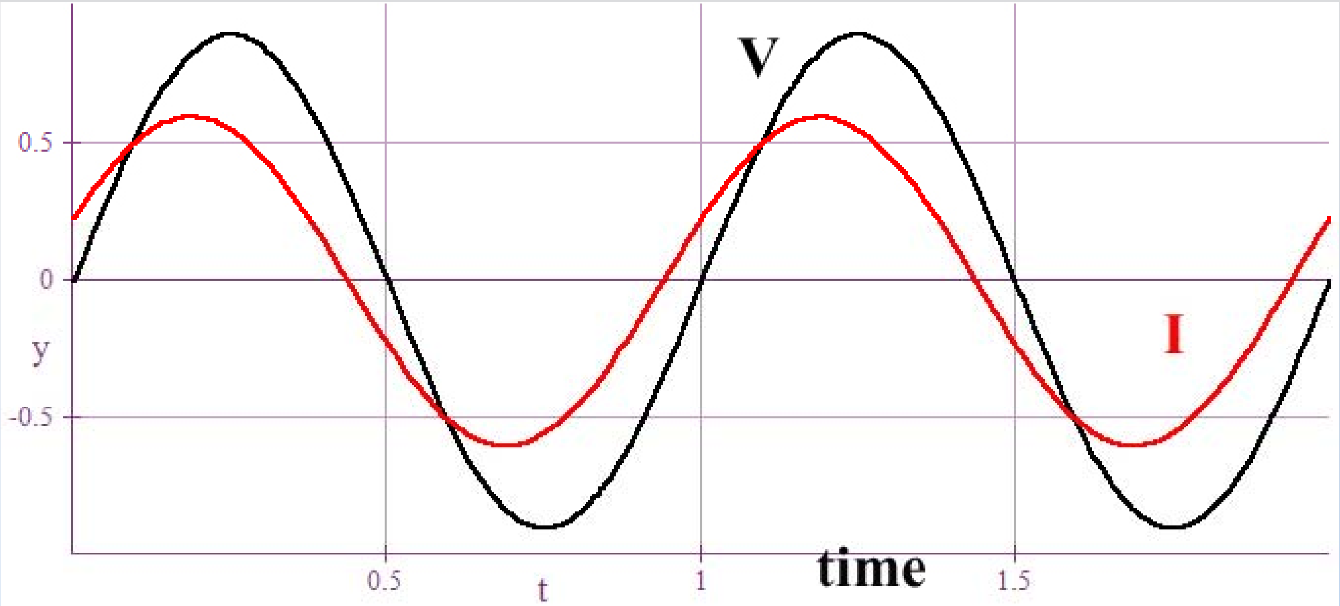
$$V_R(t) = I_0 R \sin(\omega t)$$

$$V_L(t) = I_0 X_L \sin(\omega t + \pi / 2)$$

$$V_C(t) = I_0 X_C \sin(\omega t - \pi / 2)$$

$$V_S(t) = V_{S0} \sin(\omega t + \phi)$$

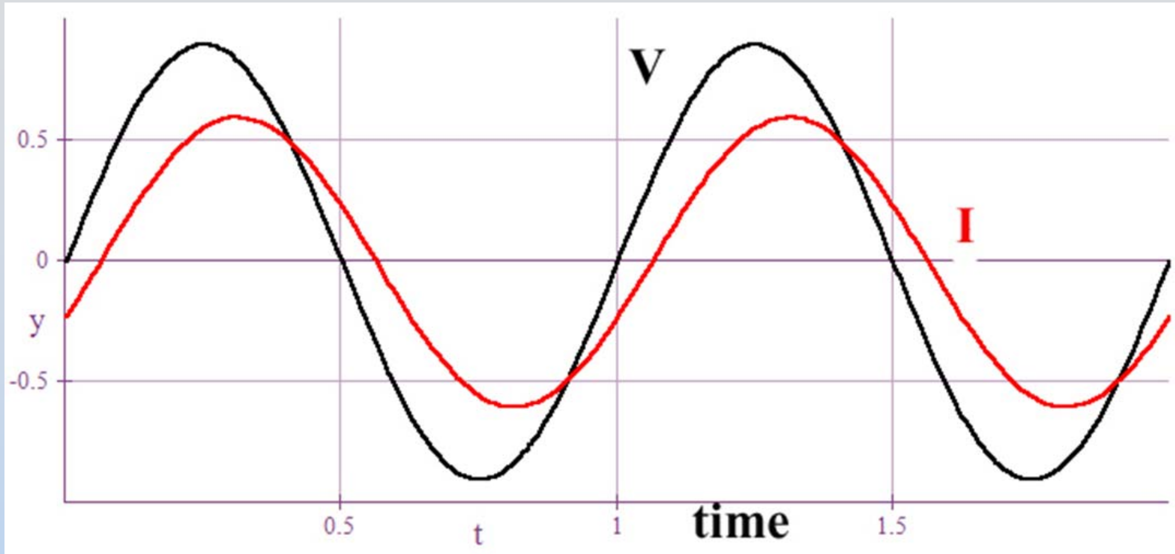
Concept Q.: Who Dominates?



The graph shows current & voltage vs. time in a driven RLC circuit at a particular driving frequency. At this frequency, the circuit is dominated by its

1. Inductance
2. Capacitance
3. I don't know

Concept Q.: What Frequency?



The graph shows current & voltage vs. time in a driven RLC circuit at a particular driving frequency. Is this frequency above or below the resonance frequency of the circuit?

1. Above the resonance frequency
2. Below the resonance frequency
3. I don’t know

RLC Circuits: Resonances

Resonance

$$I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}\,; \quad X_L = \omega L, \quad X_C = \frac{1}{\omega C}$$

At very low frequencies, C dominates (X_C>>X_L):
it fills up and keeps the current low
At very high frequencies, L dominates (X_L>>X_C):
the current tries to change but it won't let it
At intermediate frequencies we have **resonance**

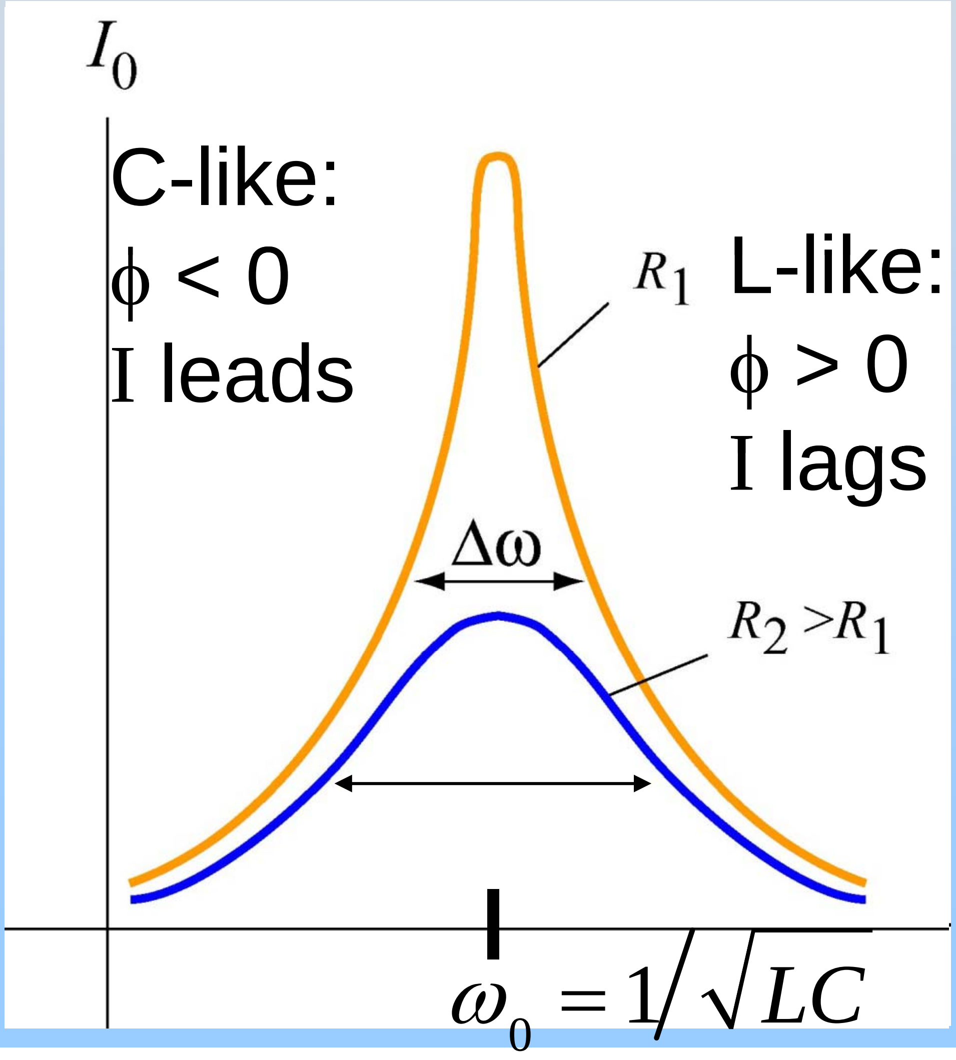
I_0 reaches maximum when $X_L = X_C$

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

Resonance

$$I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}; \quad X_L = \omega L, \quad X_C = \frac{1}{\omega C}$$

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$



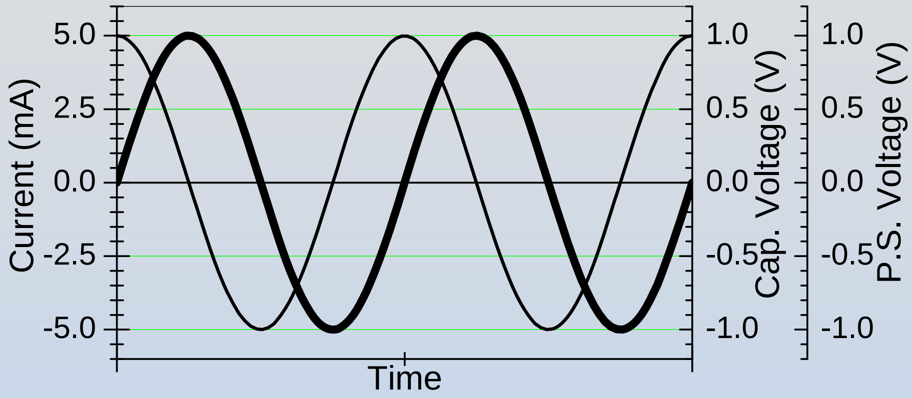
Demonstration:
RLC with Light Bulb

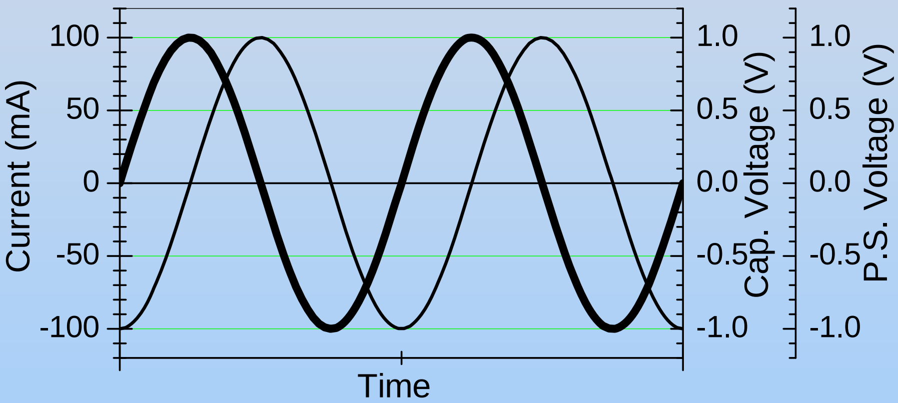
Concept Question: RLC Circuit With Light Bulb

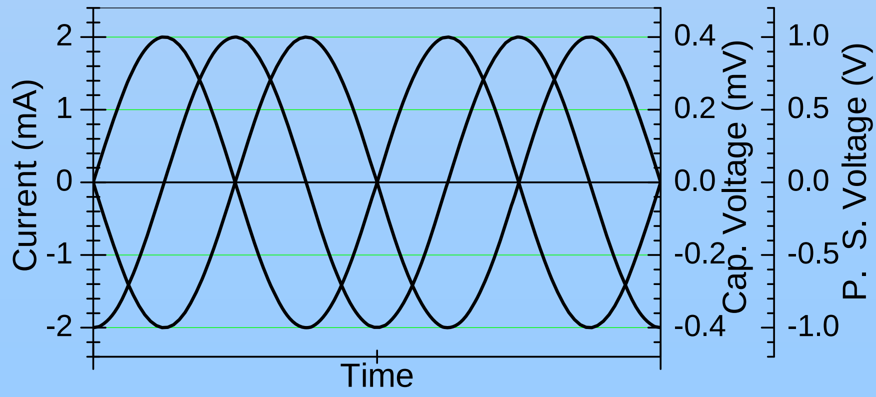
Imagine another light bulb connected in parallel to this LRC circuit. With the core pulled out that light bulb would be flashing:

1. before the LRC light bulb (leading)
2. after the LRC light bulb (lagging)
3. in time with the LRC light bulb
4. not at all
5. I don't know

Problem: RLC Circuit







- Consider plots of V_C, V_S and I made at 3 frequencies:
- a very low angular frequency (100 s^{-1}), a very high one (10^5 s^{-1}) and the resonance frequency, which is somewhere in between

- Each plot allows you to find one of R,L,C. In that order, which plot do you use, which frequency is it and what are the values of R, L & C?

Experiment 9:
Driven RLC Circuit

What to Learn from Lab

1) Properties of resonance? How can you tell when you are on resonance?

2) From plot I & V vs. t OR I vs. V:

   Which is leading (I or V)?

   L-like or C-like?

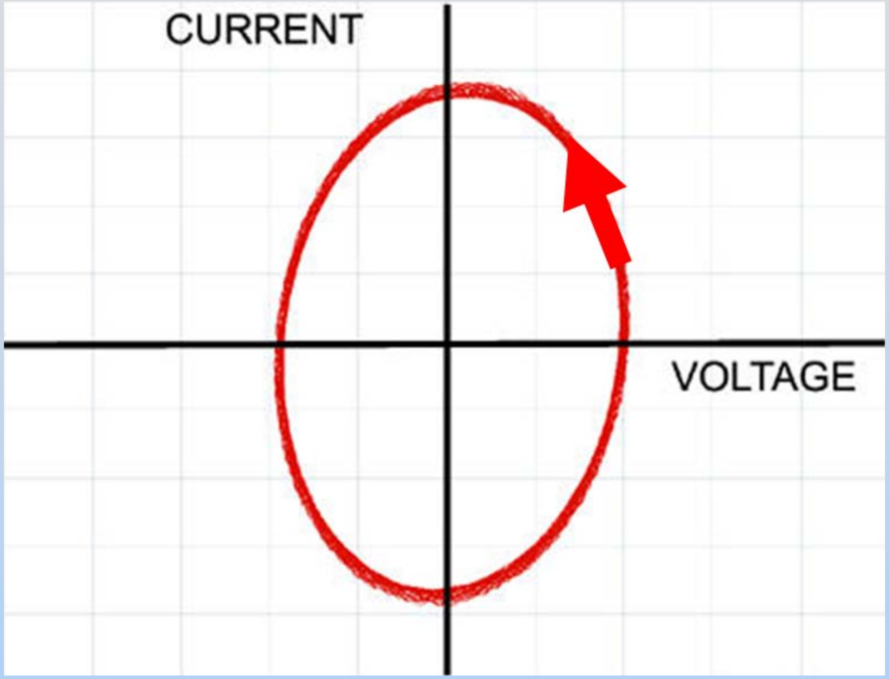
   Above or below resonance?

Concept Question Questions: Resonance

RLC Circuits:
leading/lagging

Concept Question: Leading or Lagging?

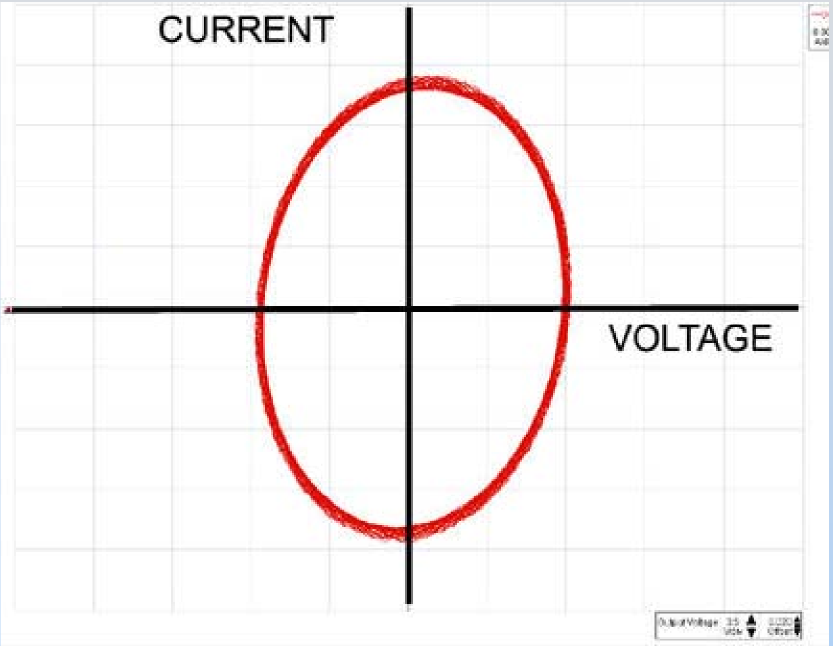
The graph shows current versus voltage in a driven RLC circuit at a given driving frequency. In this plot



1. Current lags voltage by ~90º
2. Current leads voltage by ~90º
3. Current and voltage are almost in phase
4. Not enough info (but they aren't in phase!)
5. I don't know.

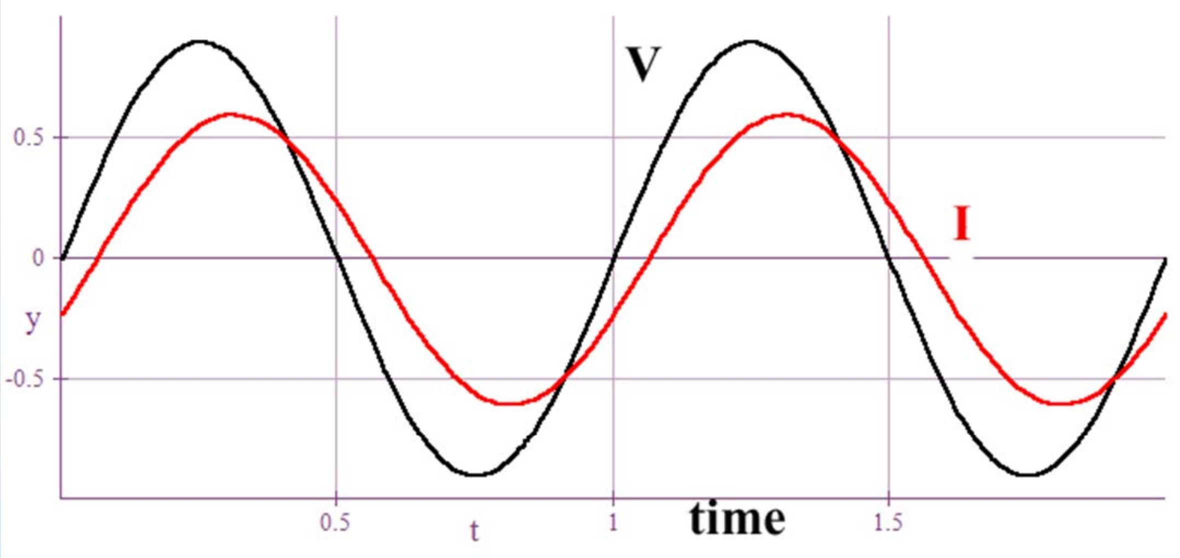
Concept Question: Leading or Lagging

The graph shows the current versus the voltage in a driven RLC circuit at a given driving frequency. In this plot



1. Current lags voltage by ~90º
2. Current leads voltage by ~90º
3. Current and voltage are almost in phase
4. We don't have enough information (but they aren't in phase!)
5. I don't know

Concept Question: What'd You Do?



The graph shows current & voltage vs. time in a driven RLC circuit. We had been in resonance a second ago but then either put in or took out the core from the inductor. Which was it?

1. Put in the core
2. Took out the core
3. I don't know

8.02SC Physics II: Electricity and Magnetism
Fall 2010