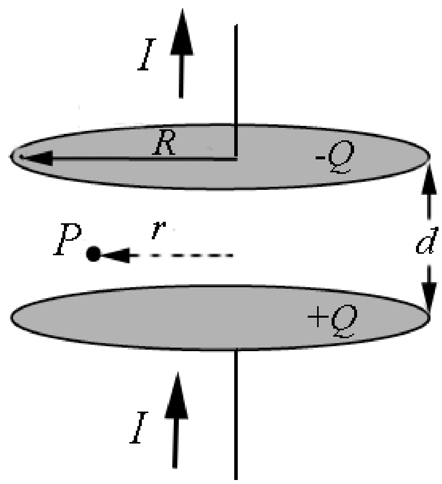
# Poynting Vector and Energy Flow in a Capacitor
## Challenge Problem Solutions

**Problem 1:**

A parallel-plate capacitor consists of two circular plates, each with radius $R$, separated by a distance $d$. A steady current $I$ is flowing towards the lower plate and away from the upper plate, charging the plates.




a) What is the direction and magnitude of the electric field $\vec{\mathbf{E}}$ between the plates? You may neglect any fringing fields due to edge effects.

b) What is the total energy stored in the electric field of the capacitor?

c) What is the rate of change of the energy stored in the electric field?

d) What is the magnitude of the magnetic field $\vec{\mathbf{B}}$ at point $P$ located between the plates at radius $r < R$ (see figure above). As seen from above, is the direction of the magnetic field *clockwise* or *counterclockwise*. Explain your answer.

e) Make a sketch of the electric and magnetic field inside the capacitor.

f) What is the direction and magnitude of the Pointing vector $\vec{\mathbf{S}}$ at a distance $r = R$ from the center of the capacitor.

g) By integrating $\vec{\mathbf{S}}$ over an appropriate surface, find the power that flows into the capacitor.

h) How does your answer in part g) compare to your answer in part c)?

**Problem 1 Solutions:**

**(a)** If we ignore fringing fields then we can calculate the electric field using Gauss's Law,

$$\oiint_{\substack{closed\\surface}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \frac{Q_{enc}}{\varepsilon_0} .$$

By superposition, the electric field is non-zero between the plates and zero everywhere else. Choose a Gaussian cylinder passing through the lower plate with its end faces parallel to the plates. Let $A_{cap}$ denote the area of the endface. The surface charge density is given by $\sigma = Q / \pi R^2$. Let $\hat{\mathbf{k}}$ denote the unit vector pointing from the lower plate to the upper plate. Then Gauss' Law becomes

$$\left|\vec{\mathbf{E}}\right| A_{cap} = \frac{\sigma A_{cap}}{\varepsilon_0}$$

which we can solve for the electric field

$$\vec{\mathbf{E}} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{k}} = \frac{Q}{\pi R^2 \varepsilon_0}\hat{\mathbf{k}} .$$

**(b)** The total energy stored in the electric field is given by

$$U_{elec} = \frac{1}{2}\varepsilon_0 \int_{volume} E^2 dV = \frac{1}{2}\varepsilon_0 E^2 \pi R^2 d .$$

Substitute the result for the electric field intot he energy equation yields

$$U_{elec} = \frac{1}{2}\varepsilon_0 \left(\frac{Q}{\pi R^2 \varepsilon_0}\right)^2 \pi R^2 d = \frac{1}{2}\frac{Q^2 d}{\pi R^2 \varepsilon_0} .$$

(c) The rate of change of the stored electric energy is found by taking the time derivative of the energy equation

$$\frac{d}{dt}U_{elec} = \frac{Qd}{\pi R^2 \varepsilon_0}\frac{dQ}{dt} .$$

The current flowing to the plate is equal to

$$I = \frac{dQ}{dt}.$$

Substitute the expression for the current into the expression for the rate of change of the stored electric energy yields

$$\frac{d}{dt}U_{elec} = \frac{QId}{\pi R^2 \varepsilon_0}.$$

**(d)** We shall calculate the magnetic field by using the generalized Ampere's Law,

$$\oint_{closed\ path} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 \iint_{open\,surface} \vec{\mathbf{J}} \cdot d\vec{\mathbf{a}} + \mu_0 \varepsilon_0 \frac{d}{dt} \iint_{open\,surface} \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}}$$

We choose a circle of radius $r < R$ passing through the point $P$ as the Amperian loop and the disk defined by the circle as the open surface with the circle as its boundary. We choose to circulate around the loop in the counterclockwise direction as seen from above. This means that flux in the positive $\hat{\mathbf{k}}$ -direction is positive.

The left hand side (LHS) of the generalized Ampere's Law becomes

$$LHS = \oint_{circle} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \left|\vec{\mathbf{B}}\right| 2\pi r .$$

The conduction current is zero passing through the disk, since no charges are moving between the plates. There is an electric flux passing through the disk. So the right hand side (RHS) of the generalized Ampere's Law becomes

$$RHS = \mu_0 \varepsilon_0 \frac{d}{dt} \iint_{disk} \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \mu_0 \varepsilon_0 \frac{d\left|\vec{\mathbf{E}}\right|}{dt} \pi r^2 .$$

Take the time derivative of the expression for the electric field and the expression for the current, and substitute it into the RHS of the generalized Ampere's Law:

$$RHS = \mu_0 \varepsilon_0 \frac{d\left|\vec{\mathbf{E}}\right|}{dt} \pi r^2 = \frac{\mu_0 I \pi r^2}{\pi R^2}$$

Equating the two sides of the generalized Ampere's Law yields

$$\left|\vec{\mathbf{B}}\right| 2\pi r = \frac{\mu_0 I \pi r^2}{\pi R^2}$$

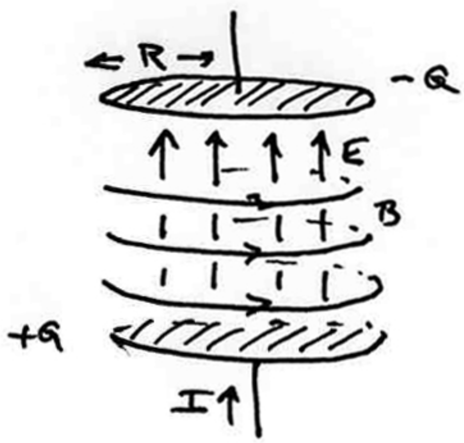
Finally the magnetic field between the plates is then

$$\left|\vec{\mathbf{B}}\right| = \frac{\mu_0 I}{2\pi R^2} r\ ;\ \ 0 < r < R\ .$$

The sign of the magnetic field is positive therefore the magnetic field points in the counterclockwise direction (consistent with our sign convention for the integration direction for the circle) as seen from above. Define the unit vector $\hat{\boldsymbol{\theta}}$ such that is it tangent to the circle pointing in the counterclockwise direction, then

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2\pi R^2} r\, \hat{\boldsymbol{\theta}}\ ;\ \ 0 < r < R\ .$$

**(e)**




**(f)** The Poynting vector at a distance $r = R$ is given by

$$\vec{\mathbf{S}}(r = R) = \frac{1}{\mu_0} \vec{\mathbf{E}} \times \vec{\mathbf{B}}\Big|_{r=R}\ .$$

Substituting the electric field and the magnetic field (setting $r = R$) into the above equation, and noting that $\hat{\mathbf{k}} \times \hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}}$, yields

$$\vec{\mathbf{S}}(r = R) = \frac{1}{\mu_0} \frac{Q}{\pi R^2 \varepsilon_0} \hat{\mathbf{k}} \times \frac{\mu_0 I}{2\pi R} \hat{\boldsymbol{\theta}} = \frac{Q}{\pi R^2 \varepsilon_0} \frac{I}{2\pi R} (-\hat{\mathbf{r}})\ .$$

So the Poynting vector points inward with magnitude

$$\left|\vec{\mathbf{S}}(r = R)\right| = \frac{Q}{\pi R^2 \varepsilon_0} \frac{I}{2\pi R}\ .$$

**(g)** The power flowing into the capacitor is the closed surface integral

$$P = \oiint_{closed\ surface} \vec{\mathbf{S}}(r=R)\cdot d\vec{\mathbf{a}}\,.$$

The Poynting vector points radially inward so the only contribution to this integral is from the cylindrical body of the capacitor. The unit normal associated with the area vector for a closed surface integral always points outward, so on the cylindrical body $d\vec{\mathbf{a}} = da\,\hat{\mathbf{r}}$. Use this definition for the area element and the power is then

$$P = \iint_{\substack{cylindrical\\ body}} \vec{\mathbf{S}}(r=R)\cdot d\vec{\mathbf{a}} = \iint_{\substack{cylindrical\\ body}} \frac{Q}{\pi R^2 \varepsilon_0}\frac{I}{2\pi R}(-\hat{\mathbf{r}})\cdot da\,\hat{\mathbf{r}}$$

The Poynting vector is constant and the area of the cylindrical body is $2\pi Rd$, so

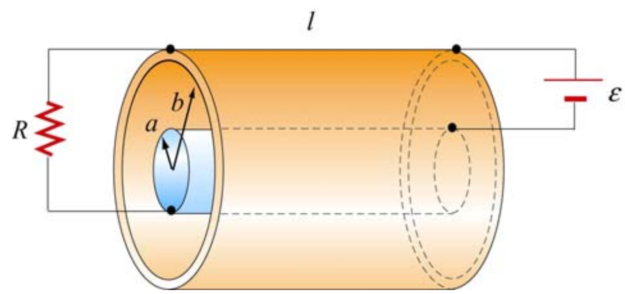
$$P = \iint_{\substack{cylindrical\\ body}} \frac{Q}{\pi R^2 \varepsilon_0}\frac{I}{2\pi R}(-\hat{\mathbf{r}})\cdot da\,\hat{\mathbf{r}} = -\frac{Q}{\pi R^2 \varepsilon_0}\frac{I}{2\pi R}2\pi Rd = -\frac{QId}{\pi R^2 \varepsilon_0}\,.$$

The minus sign correspond to power flowing into the region.

**(h)** The two expressions for power are equal so the power flowing in is equal to the change of energy stored in the electric fields.

**Problem 2:**

A coaxial cable consists of two concentric long hollow cylinders of zero resistance; the inner has radius $a$, the outer has radius $b$, and the length of both is $l$, with $l >> b$, as shown in the figure. The cable transmits DC power from a battery to a load. The battery provides an electromotive force $\varepsilon$ between the two conductors at one end of the cable, and the load is a resistance $R$ connected between the two conductors at the other end of the cable. A current $I$ flows down the inner conductor and back up the outer one. The battery charges the inner conductor to a charge $-Q$ and the outer conductor to a charge $+Q$.




(a) Find the direction and magnitude of the electric field $\vec{\mathbf{E}}$ everywhere.

(b) Find the direction and magnitude of the magnetic field $\vec{\mathbf{B}}$ everywhere.

(c) Calculate the Poynting vector $\vec{\mathbf{S}}$ in the cable.

(d) By integrating $\vec{\mathbf{S}}$ over appropriate surface, find the power that flows into the coaxial cable.

(e) How does your result in (d) compare to the power dissipated in the resistor?

**Problem 2 Solutions:**

(a) Consider a Gaussian surface in the form of a cylinder with radius *r* and length *l*, coaxial with the cylinders. Inside the inner cylinder (*r*<*a*) and outside the outer cylinder (*r*>*b*) no charge is enclosed and hence the field is 0. In between the two cylinders (*a*<*r*<*b*) the charge enclosed by the Gaussian surface is –*Q*, the total flux through the Gaussian cylinder is

$$\Phi_E = \oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E(2\pi r l)$$

Thus, Gauss's law leads to $E(2\pi r l) = \dfrac{q_{\text{enc}}}{\varepsilon_0}$, or

$$\vec{\mathbf{E}} = \frac{q_{\text{enc}}}{2\pi r l}\hat{\mathbf{r}} = -\frac{Q}{2\pi\varepsilon_0 r l}\hat{\mathbf{r}} \text{ (inward) for } a < r < b \text{, 0 elsewhere}$$

(b) Just as with the E field, the enclosed current $I_{\text{enc}}$ in the Ampere's loop with radius *r* is zero inside the inner cylinder (*r*<*a*) and outside the outer cylinder (*r*>*b*) and hence the field there is 0. In between the two cylinders (*a*<*r*<*b*) the current enclosed is –*I*.

Applying Ampere's law, $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B(2\pi r) = \mu_0 I_{\text{enc}}$, we obtain

$\vec{\mathbf{B}} = -\dfrac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\varphi}}$ (clockwise viewing from the left side) for $a < r < b$, 0 elsewhere

(c) For $a < r < b$, the Poynting vector is

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}} = \frac{1}{\mu_0}\left(-\frac{Q}{2\pi\varepsilon_0 rl}\hat{\mathbf{r}}\right) \times \left(-\frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\varphi}}\right) = \left(\frac{QI}{4\pi^2\varepsilon_0 r^2 l}\right)\hat{\mathbf{k}} \quad \text{(from right to left)}$$

On the other hand, for $r < a$ and $r > b$, we have $\vec{\mathbf{S}} = 0$.

(d) With $d\vec{\mathbf{A}} = (2\pi r\, dr)\hat{\mathbf{k}}$, the power is

$$P = \oiint_S \vec{\mathbf{S}} \cdot d\vec{\mathbf{A}} = \frac{QI}{4\pi^2\varepsilon_0 l}\int_a^b \frac{1}{r^2}(2\pi r dr) = \frac{QI}{2\pi\varepsilon_0 l}\ln\left(\frac{b}{a}\right)$$

(e) Since

$$\varepsilon = \int \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = \int_a^b \frac{Q}{2\pi r l \varepsilon_0} dr = \frac{Q}{2\pi l \varepsilon_0}\ln\left(\frac{b}{a}\right) = IR$$

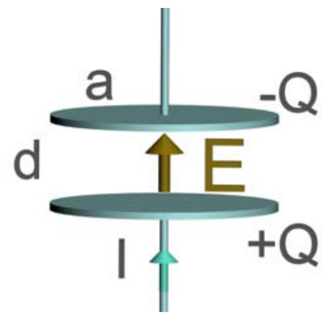
the charge $Q$ is related to the resistance $R$ by $Q = \dfrac{2\pi\varepsilon_0 lIR}{\ln(b/a)}$. The above expression for $P$ becomes

$$P = \left(\frac{2\pi\varepsilon_0 lIR}{\ln(b/a)}\right)\frac{I}{2\pi\varepsilon_0 l}\ln\left(\frac{b}{a}\right) = I^2 R$$

which is equal to the rate of energy dissipation in a resistor with resistance $R$.

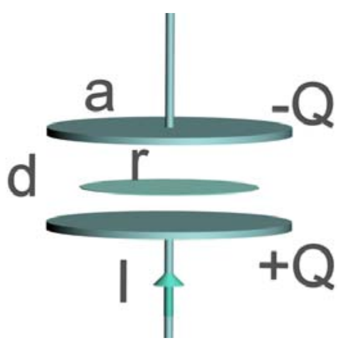
**Problem 3:**

A capacitor consists of two circular plates of radius $a$ separated by a distance $d$ (assume $d << a$). The center of each plate is connected to the terminals of a voltage source by a thin wire. A switch in the circuit is closed at time $t = 0$ and a current $I(t)$ flows in the circuit. The charge on the plate is related to the current according to $I(t) = \frac{dQ(t)}{dt}$. We begin by calculating the electric field between the plates. Throughout this problem you may ignore edge effects. We assume that the electric field is zero for $r > a$.




(a) Use Gauss' Law to find the electric field between the plates as a function of time $t$, in terms of $Q(t)$, $a$, $\varepsilon_0$, and $\pi$. The vertical direction is the $\hat{\mathbf{k}}$ direction.

(b) Now take an imaginary flat disk of radius $r < a$ inside the capacitor, as shown below.




Using your expression for $\vec{\mathbf{E}}$ above, calculate the electric flux through this flat disk of radius $r < a$ in the plane midway between the plates, in terms of $r$, $Q(t)$, $a$, and $\varepsilon_0$. Take the surface normal to the imaginary disk to be in the $\hat{\mathbf{k}}$ direction.

(c) Calculate the Maxwell displacement current,

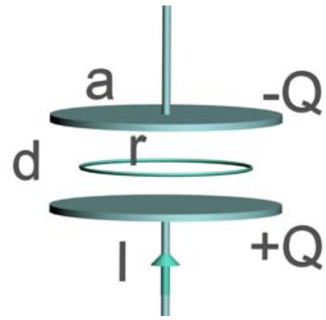
$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 \frac{d}{dt} \iint_{\text{disk}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

through the flat disc of radius $r < a$ in the plane midway between the plates, in terms of $r$, $I(t)$, and $a$. Remember, there is really not a "current" there, we just call it that to confuse you.

(d) What is the conduction current $\iint_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}}$ through the flat disk of radius $r < a$?

"Conduction" current just means the current due to the flow of real charge across the surface (e.g. electrons or ions).

(e) Since the capacitor plates have an axial symmetry and we know that the magnetic field due to a wire runs in azimuthal circles about the wire, we assume that the magnetic field between the plates is non-zero, and also runs in azimuthal circles.




Choose for an Amperian loop, a circle of radius $r < a$ in the plane midway between the plates. Calculate the line integral of the magnetic field around the circle, $\oint_{\text{circle}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$.

Express your answer in terms of $|\vec{\mathbf{B}}|$, $\pi$, and $r$. The line element $d\vec{\mathbf{s}}$ is right-handed with respect to $d\vec{\mathbf{A}}$, that is counterclockwise as seen from the top.

(f) Now use the results of your answers above, and apply the generalized Ampere' Law Equation to find the magnitude of the magnetic field at a distance $r < a$ from the axis. Your answer should be in terms of $r$, $I(t)$, $\mu_o$, $\pi$, and $a$.

**Problem 3 Solutions:**

(a) The electric field between the plates is

(b) The electric flux through the disk of radius $r$ is

$$\Phi_E = \iint_{\text{disk}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E\left(\pi r^2\right) = \frac{Q\pi r^2}{\varepsilon_0 \pi a^2} = \frac{Q}{\varepsilon_0} \frac{r^2}{a^2}$$

$$\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = \frac{Q(t)}{\varepsilon_0} \quad \Rightarrow \quad \vec{\mathbf{E}} = \frac{Q(t)}{\pi a^2 \varepsilon_0} \hat{\mathbf{k}} = \frac{\sigma}{\varepsilon_0} \hat{\mathbf{k}}$$

(c) Using the above equation, the displacement current is

$$I_d = \varepsilon_0 \frac{d}{dt} \iint_{\text{disk}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \varepsilon_0 \frac{d}{dt}\left( \frac{Q(t)}{\varepsilon_0} \frac{r^2}{a^2} \right) = \frac{dQ(t)}{dt} \frac{r^2}{a^2} = I(t) \frac{r^2}{a^2}$$

(d) The conduction current through the flat disk is zero.

(e) The line integral of the magnetic field around the circle is $\oint_{\text{circle}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B(2\pi r)$.

(f) The magnetic field at a distance $r < a$ is

$$B(2\pi r) = \mu_0 I_d = \mu_0 \left( I \frac{r^2}{a^2} \right) \quad \Rightarrow \quad B = \frac{\mu_0 I\, r}{2\pi a^2}$$

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