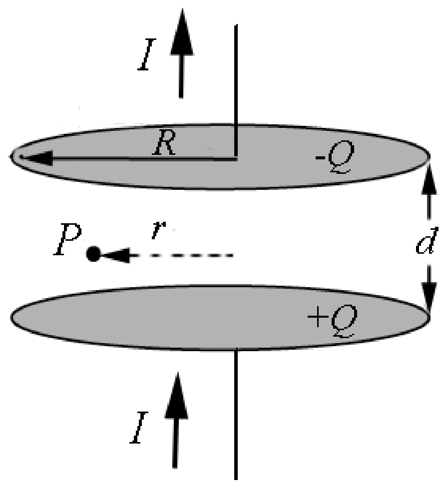
Poynting Vector and Energy Flow in a Capacitor
Challenge Problems

Problem 1:

A parallel-plate capacitor consists of two circular plates, each with radius R, separated by a distance d. A steady current I is flowing towards the lower plate and away from the upper plate, charging the plates.



a) What is the direction and magnitude of the electric field $\vec{\mathbf{E}}$ between the plates? You may neglect any fringing fields due to edge effects.

b) What is the total energy stored in the electric field of the capacitor?

c) What is the rate of change of the energy stored in the electric field?

d) What is the magnitude of the magnetic field $\vec{\mathbf{B}}$ at point P located between the plates at radius $r < R$ (see figure above). As seen from above, is the direction of the magnetic field *clockwise* or *counterclockwise*. Explain your answer.

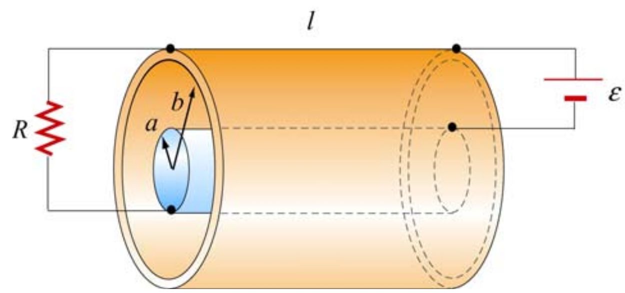
e) Make a sketch of the electric and magnetic field inside the capacitor.

f) What is the direction and magnitude of the Pointing vector $\vec{\mathbf{S}}$ at a distance $r = R$ from the center of the capacitor.

g) By integrating $\vec{\mathbf{S}}$ over an appropriate surface, find the power that flows into the capacitor.

h) How does your answer in part g) compare to your answer in part c)?

Problem 2:



A coaxial cable consists of two concentric long hollow cylinders of zero resistance; the inner has radius a, the outer has radius b, and the length of both is l, with $l >> b$, as shown in the figure. The cable transmits DC power from a battery to a load. The battery provides an electromotive force ε between the two conductors at one end of the cable, and the load is a resistance R connected between the two conductors at the other end of the cable. A current I flows down the inner conductor and back up the outer one. The battery charges the inner conductor to a charge $-Q$ and the outer conductor to a charge $+Q$.

(a) Find the direction and magnitude of the electric field $\vec{\mathbf{E}}$ everywhere.

(b) Find the direction and magnitude of the magnetic field $\vec{\mathbf{B}}$ everywhere.

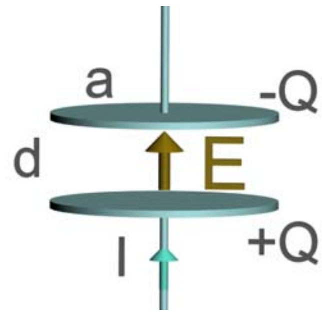
(c) Calculate the Poynting vector $\vec{\mathbf{S}}$ in the cable.

(d) By integrating $\vec{\mathbf{S}}$ over appropriate surface, find the power that flows into the coaxial cable.

(e) How does your result in (d) compare to the power dissipated in the resistor?

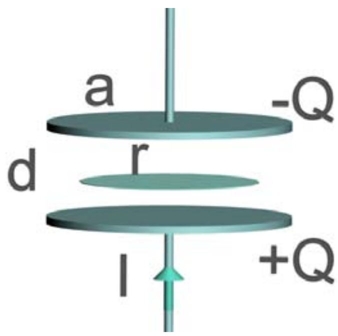
Problem 3:

A capacitor consists of two circular plates of radius a separated by a distance d (assume $d << a$). The center of each plate is connected to the terminals of a voltage source by a thin wire. A switch in the circuit is closed at time $t = 0$ and a current $I(t)$ flows in the circuit. The charge on the plate is related to the current according to $I(t) = \frac{dQ(t)}{dt}$. We begin by calculating the electric field between the plates. Throughout this problem you may ignore edge effects. We assume that the electric field is zero for $r > a$.



(a) Use Gauss' Law to find the electric field between the plates as a function of time t, in terms of $Q(t)$, a, ε_0, and π. The vertical direction is the $\hat{\mathbf{k}}$ direction.

(b) Now take an imaginary flat disk of radius $r < a$ inside the capacitor, as shown below.



Using your expression for $\vec{\mathbf{E}}$ above, calculate the electric flux through this flat disk of radius $r < a$ in the plane midway between the plates, in terms of r, $Q(t)$, a, and ε_0. Take the surface normal to the imaginary disk to be in the $\hat{\mathbf{k}}$ direction.

(c) Calculate the Maxwell displacement current,

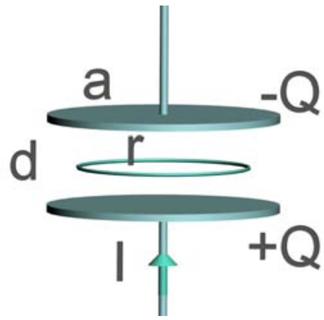
$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 \frac{d}{dt} \iint_{\text{disk}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

through the flat disc of radius $r < a$ in the plane midway between the plates, in terms of r, $I(t)$, and a. Remember, there is really not a "current" there, we just call it that to confuse you.

(d) What is the conduction current $\iint_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}}$ through the flat disk of radius $r < a$?

"Conduction" current just means the current due to the flow of real charge across the surface (e.g. electrons or ions).

(e) Since the capacitor plates have an axial symmetry and we know that the magnetic field due to a wire runs in azimuthal circles about the wire, we assume that the magnetic field between the plates is non-zero, and also runs in azimuthal circles.



Choose for an Amperian loop, a circle of radius $r < a$ in the plane midway between the plates. Calculate the line integral of the magnetic field around the circle, $\oint_{\text{circle}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$.

Express your answer in terms of $\left|\vec{\mathbf{B}}\right|$, π, and r. The line element $d\vec{\mathbf{s}}$ is right-handed with respect to $d\vec{\mathbf{A}}$, that is counterclockwise as seen from the top.

(f) Now use the results of your answers above, and apply the generalized Ampere' Law Equation to find the magnitude of the magnetic field at a distance $r < a$ from the axis. Your answer should be in terms of r, $I(t)$, μ_o, π, and a.