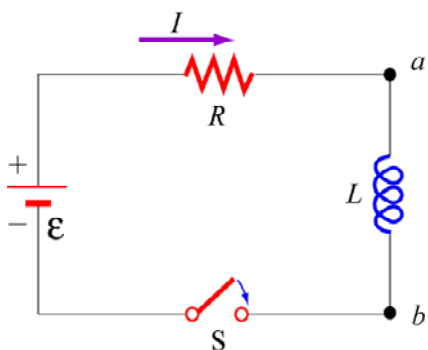


RL Circuits Challenge Problem Solutions

Problem 1:

RL Circuits



Consider the circuit at left, consisting of a battery (emf ε), an inductor L , resistor R and switch S .

For times $t < 0$ the switch is open and there is no current in the circuit. At $t = 0$ the switch is closed.

(a) Using Kirchhoff's loop rules (really Faraday's law now), write an equation relating the emf on the battery, the current in the circuit and the time derivative of the current in the circuit.

We know from thinking about it above that the results should look very similar to RC circuits. In other words:

$$I = A(X - \exp(-t/\tau))$$

- (b) Plug this expression into the differential equation you obtained in (a) in order to confirm that it indeed is a solution and to determine what the time constant τ and the constants A and X are. What would be a better label for A ? (HINT: You will also need to use the initial condition for current. What is $I(t=0)$?)
- (c) Now that you know the time dependence for the current I in the circuit you can also determine the voltage drop V_R across resistor and the EMF generated by the inductor. Do so, and confirm that your expressions match the plots in Fig. 2a or 2b.

Problem 1 Solutions:

(a) Walking in the direction of current, starting at the switch

$$\varepsilon - IR - L \frac{dI}{dt} = 0$$

(b)

$$0 = \varepsilon - A(X - e^{-t/\tau})R - L \frac{Ae^{-t/\tau}}{\tau} = (\varepsilon - ARX) + \left(AR - L \frac{A}{\tau} \right) e^{-t/\tau}$$

Both the constant and time dependent part must equal zero, giving us two equations. The third (because there are three unknowns) we can get from initial conditions:

$$I(t=0) = A(X - 1) = 0 \quad \Rightarrow X = 1$$

$$\varepsilon - ARX = 0 \quad \Rightarrow A = \frac{\varepsilon}{RX} = \frac{\varepsilon}{R}$$

$$\left(AR - L \frac{A}{\tau} \right) e^{-t/\tau} = 0 \quad \Rightarrow \tau = \frac{L}{R}$$

A better label for A would be I_f , the final current.

(c) We find:

$$I(t) = A(X - e^{-t/\tau}) = \frac{\varepsilon}{R}(1 - e^{-t/\tau}) \quad (\text{Fig. 2a})$$

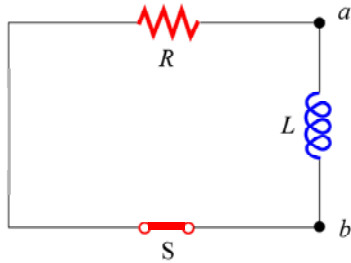
$$V_R(t) = IR = \varepsilon(1 - e^{-t/\tau}) \quad (\text{Fig. 2a})$$

$$\varepsilon_L(t) = -L \frac{dI}{dt} = -L \frac{\varepsilon}{R\tau} e^{-t/\tau} = -\varepsilon e^{-t/\tau} \quad (\text{Fig. 2b})$$

Looking at the EMF from the inductor you see that it starts the same as the battery (but in the opposite direction) which explains why no current initially flows. Then as time goes on it relaxes.

Problem 2:

'Discharging' an Inductor



After a long time T the current will reach an equilibrium value and inductor will be “fully charged.” At this point we turn off the battery ($\mathcal{E}=0$), allowing the inductor to ‘discharge,’ as pictured at left. Repeat each of the steps a-c in problem 1, noting that instead of $\exp(-t/\tau)$, our expression for current will now contain $\exp(-(t-T)/\tau)$.

- (a) Faraday’s law:
- (b) Confirm solution:
- (c) Determine V_R across resistor and the EMF generated by the inductor.

Problem 2 Solutions:

- (a) Walking in the direction of current, starting at the switch

$$-IR - L \frac{dI}{dt} = 0$$

- (b)

$$0 = -A \left(X - e^{-(t-T)/\tau} \right) R - L \frac{A e^{-(t-T)/\tau}}{\tau} = (-ARX) + \left(AR - L \frac{A}{\tau} \right) e^{-(t-T)/\tau}$$

Both the constant and time dependent part must equal zero, giving us two equations. The third (because there are three unknowns) we can get from initial conditions:

$$-ARX = 0 \quad \Rightarrow X = 0$$

$$\left(AR - L \frac{A}{\tau} \right) e^{-t/\tau} = 0 \quad \Rightarrow \tau = \frac{L}{R}$$

$$I(t=T) = A(X-1) = \frac{\mathcal{E}}{R} \quad \Rightarrow A = -\frac{\mathcal{E}}{R}$$

A better label for A would be I_0 , the initial current.

- (c) Everything is exponentially decaying with time:

$$I(t) = A \left(X - e^{-t/\tau} \right) = \frac{\mathcal{E}}{R} e^{-t/\tau} \quad (\text{Fig. 2b})$$

$$V_R(t) = IR = \mathcal{E} e^{-t/\tau} \quad (\text{Fig. 2b})$$

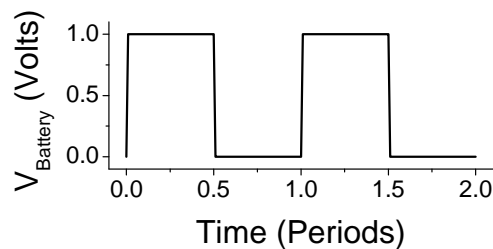
$$\mathcal{E}_L(t) = -L \frac{dI}{dt} = L \frac{\mathcal{E}}{R\tau} e^{-t/\tau} = \mathcal{E} e^{-t/\tau} \quad (\text{Fig. 2b})$$

Problem 3:

A Real Inductor

Consider a coil that does not behave as an ideal inductor, but rather as an ideal inductor in series with a resistor. For this reason you have no way to independently measure the voltage drop across the resistor or the EMF induced by the inductor, but instead must measure them together. None-the-less, you want to get information about both. In this problem you will figure out how.

- (a) Hook up the circuit of problem 1 (with the ideal inductor L of that problem now replaced by a coil that is a non-ideal inductor – an inductor L and resistor r in series). The battery will periodically turn on and off, displaying a voltage as shown here:

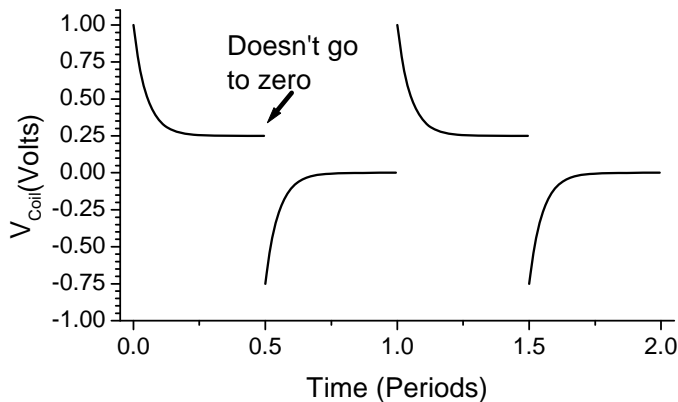
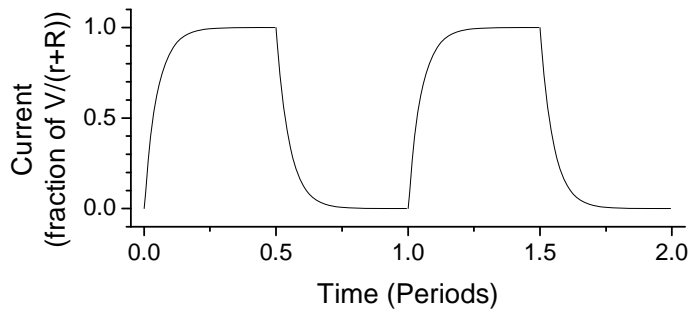


Sketch the current through the battery as well as what a voltmeter hooked across the coil would show versus time for the two periods shown above. Assume that the period of the battery turning off and on is comparable to but longer than several time constants of the circuit.

- (b) How can you tell from your plot of the voltmeter across the coil that the coil is not an ideal inductor? Indicate the relevant feature clearly on the plot. Can you determine the resistance of the coil, r , from this feature?
- (c) In the lab you will find it easier to make measurements if you do NOT use an additional resistor R , but instead simply hook the battery directly to the coil. (Why? Because the time constant is difficult to measure with extra resistance in the circuit). Plot the current through the battery and the reading on a voltmeter across the coil for this case. We will only bother to measure the current. Why?
- (d) For this case (only a battery & coil) how will you determine the resistance of the coil, r ? How will you determine its inductance L ?

Problem 3 Solutions:

- (a)



(b) The voltage measured across the coil doesn't go to zero because even when the inductor is "off" the coil resistance still has a voltage drop across it. You can determine r from this voltage $-r = V/I$ (in this case I made r $1/4$ of the total resistance, that is, $1/3$ of R).

(c) The current is the same as the current above (although the time constant will be longer because of the lower resistance). The voltage measured across the coil will be the same as the voltage measured across the battery because they are the only two things in the circuit, so there is no need to measure it.

(d) In this case we can determine the resistance from the final current ($r = V/I$) and the inductance from the time constant.

Problem 4:

The Coil

Consider a coil made of thin copper wire (**radius** ~ 0.25 mm) and has about 600 turns of average diameter 25 mm over a length of 25 mm. What approximately should the resistance and inductance of the coil be? The resistivity of copper at room temperature is around 20 nΩ·m. Note that your calculations can only be approximate because this is not at all an ideal solenoid (where length >> diameter).

Problem 4 Solution:

The resistance (NOTE: I screwed up and meant **radius** was 0.25 mm, not diameter)

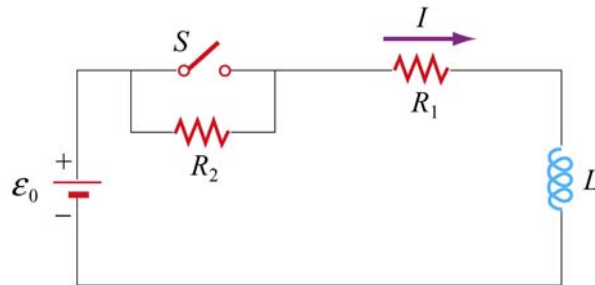
$$R = \frac{\rho L}{A} = \frac{\rho \cdot N \pi d}{\pi a^2} \approx \frac{(20 \text{ n}\Omega \text{ m}) \cdot (600)(25 \text{ mm})}{(0.25 \text{ mm})^2} \approx 4.8 \Omega$$

The inductance of a solenoid we calculated in class to be:

$$L = \mu_0 n^2 \pi R^2 l \approx (4\pi \times 10^{-7} \text{ T m A}^{-1}) \left(\frac{600}{25 \text{ mm}} \right)^2 \pi \left(\frac{25 \text{ mm}}{2} \right)^2 (25 \text{ mm}) \approx 9 \text{ mH}$$

Problem 5:

The LR circuit shown in the figure contains a resistor R_1 and an inductance L in series with a battery of emf ε_0 . The switch S is initially closed. At $t = 0$, the switch S is opened, so that an additional very large resistance R_2 (with $R_2 \gg R_1$) is now in series with the other elements.



- (a) If the switch has been *closed* for a long time before $t = 0$, what is the steady current I_0 in the circuit?
- (b) While this current I_0 is flowing, at time $t = 0$, the switch S is opened. Write the differential equation for $I(t)$ that describes the behavior of the circuit at times $t \geq 0$. Solve this equation (by integration) for $I(t)$ under the approximation that $\varepsilon_0 = 0$. (Assume that the battery emf is negligible compared to the total emf around the circuit for times just after the switch is opened.) Express your answer in terms of the initial current I_0 , and R_1 , R_2 , and L .
- (c) Using your results from (b), find the value of the total emf around the circuit (which from Faraday's law is $-LdI/dt$) just after the switch is opened. Is your assumption in (b) that ε_0 could be ignored for times just after the switch is opened OK?
- (d) What is the magnitude of the potential drop across the resistor R_2 at times $t > 0$, just after the switch is opened? Express your answers in terms of ε_0 , R_1 , and R_2 . How does the potential drop across R_2 just after $t = 0$ compare to the battery emf ε_0 , if $R_2 = 100R_1$?

Problem 5 Solutions:

- (a) There is no induced emf before $t = 0$. Also, no current is flowing on R_2 . Therefore,

$$I_0 = \frac{\varepsilon_0}{R_1}$$

(b) The differential equation is

$$\varepsilon_0 - I(t)(R_1 + R_2) = L \frac{dI(t)}{dt}$$

Under the approximation that $\varepsilon_0 = 0$, the equation is

$$-I(t)(R_1 + R_2) = L \frac{dI(t)}{dt}$$

The solution with the initial condition $I(0) = I_0$ is given by

$$I(t) = I_0 \exp\left(-\frac{(R_1 + R_2)}{L} t\right)$$

(c)

$$\varepsilon = -L \left. \frac{dI(t)}{dt} \right|_{t=0} = I_0 (R_1 + R_2)$$

Since $I_0 = \frac{\varepsilon_0}{R_1}$,

$$\varepsilon = \frac{\varepsilon_0}{R_1} (R_1 + R_2) = \left(1 + \frac{R_2}{R_1}\right) \varepsilon_0 \gg \varepsilon_0 \quad (\because R_2 \gg R_1)$$

Thus, the assumption that ε_0 could be ignored for times just after the switch is open is OK.

(d) The potential drop across R_2 is given by

$$\Delta V_2 = \frac{R_2}{R_1 + R_2} \varepsilon = \left(\frac{R_2}{R_1 + R_2}\right) \left(1 + \frac{R_2}{R_1}\right) \varepsilon_0 = \frac{R_2}{R_1} \varepsilon_0$$

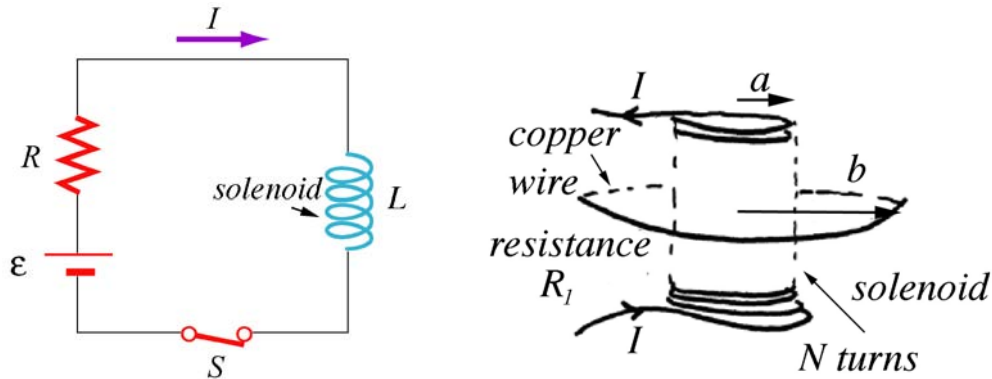
If $R_2 = 100R_1$,

$$\Delta V_2 = 100 \varepsilon_0$$

This is why you have to open a switch in a circuit with a lot of energy stored in the magnetic field very carefully, or you end up very dead!!

Problem 6:

Consider the circuit shown in the figure, consisting of a battery (emf \mathcal{E}), a resistor with resistance R , a long solenoid of radius a , height H that has N turns and a switch S . Coaxial with the solenoid at the center of the solenoid is a circular copper ring of wire of radius b with $b > a$ and resistance R_1 . At $t = 0$ the switch S is closed.



- (a) What is the rate that the current is changing the instant the switch is closed at $t = 0$? Express your answer in terms of R , \mathcal{E} , and L , the self-inductance of the solenoid, as needed.
- (b) What is the self-inductance L of the solenoid? You may assume that the solenoid is very long and so can ignore edge effects. Express your answer in terms of μ_0 , a , b , H , N , R_1 , R , and \mathcal{E} as needed. **Answers without any work shown will receive no credit.**
- (c) What is the induced current in the copper ring at the instant the switch is closed at $t = 0$? Express your answer in terms of μ_0 , a , b , H , N , R_1 , R , and \mathcal{E} as needed.

Problem 6 Solutions:

- (a) At $t = 0$, the current in the circuit is zero so the emf is related to the changing current by

$$\mathcal{E} = L \frac{dI}{dt}(t = 0).$$

Thus

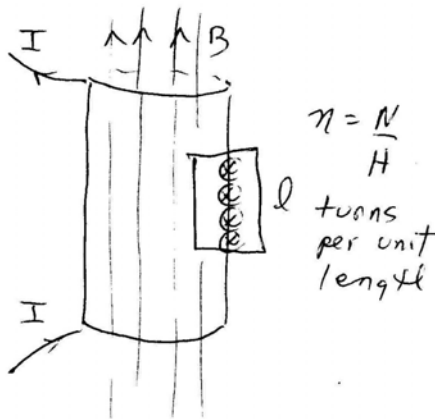
$$\frac{dI}{dt}(t=0) = \frac{\mathcal{E}}{L}$$

Alternatively, the loop equation is given by $\mathcal{E} - IR - L \frac{dI}{dt} = 0$. Thus at $t = 0$, the current in the circuit is zero and so $\mathcal{E} = L \frac{dI}{dt}(t=0)$. The current in the circuit is given by $I(t) = \frac{\mathcal{E}}{R}(1 - e^{-tR/L})$.

So

$$\frac{dI}{dt}(t=0) = \frac{\mathcal{E} R}{R L} e^{-tR/L}(t=0) = \frac{\mathcal{E}}{L}.$$

(b) The direction of the magnetic field upwards (see figure).



Choose an Amperian loop shown in the figure below, then Ampere's Law becomes $B\ell = \mu_0 n I \ell$. Therefore the magnitude of the magnetic field in the solenoid is

$$B = \mu_0 n I = \frac{\mu_0 N I}{H}.$$

The self inductance through the solenoid is

$$L = \frac{N \Phi_{loop}}{I} = \frac{N B \pi a^2}{I} = \frac{\mu_0 N^2 \pi a^2}{H}.$$

(c) The induced current is noting that the relevant area where the magnetic field is non-zero is πa^2

$$I_{ind} = \frac{1}{R_1} \frac{d\Phi}{dt} = \frac{1}{R_1} \frac{dB}{dt} \pi a^2 = \frac{1}{R_1} \frac{\mu_0 N}{H} \pi a^2 \frac{dI}{dt} (t=0)$$
$$\frac{1}{R_1} \frac{\mu_0 N}{H} \pi a^2 \frac{\mathcal{E}}{L} = \frac{1}{R_1} \frac{\mu_0 N}{H} \pi a^2 \frac{H\mathcal{E}}{\mu_0 N^2 \pi a^2} = \frac{\mathcal{E}}{NR_1}$$

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