Module 22 and 23: Section 11.1 through Section 11.4
Module 24: Section 11.4 through Section 11.13[1]

Table of Contents

Inductance and Magnetic Energy

11.1 Mutual Inductance

Suppose two coils are placed near each other, as shown in Figure 11.1.1
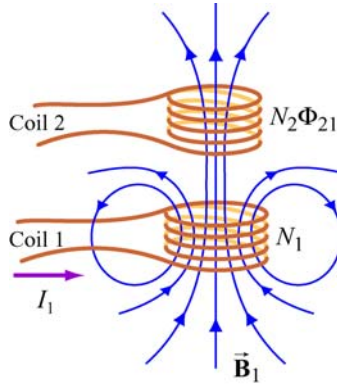
Figure 11.1.1 Changing current in coil 1 produces changing magnetic flux in coil 2.

The first coil has N_1 turns and carries a current I_1 which gives rise to a magnetic field $\vec{\mathbf{B}}_1$. Since the two coils are close to each other, some of the magnetic field lines through coil 1 will also pass through coil 2. Let Φ_{21} denote the magnetic flux through one turn of coil 2 due to I_1. Now, by varying I_1 with time, there will be an induced emf associated with the changing magnetic flux in the second coil:

$$\varepsilon_{21} = -N_2 \frac{d\Phi_{21}}{dt} = -\frac{d}{dt} \iint_{\text{coil 2}} \vec{\mathbf{B}}_1 \cdot d\vec{\mathbf{A}}_2 \tag{11.1.1}$$

The time rate of change of magnetic flux Φ_{21} in coil 2 is proportional to the time rate of change of the current in coil 1:

$$N_2 \frac{d\Phi_{21}}{dt} = M_{21} \frac{dI_1}{dt} \tag{11.1.2}$$

where the proportionality constant M_{21} is called the mutual inductance. It can also be written as

$$M_{21} = \frac{N_2 \Phi_{21}}{I_1} \tag{11.1.3}$$

The SI unit for inductance is the henry (H):

$$1\text{ henry} = 1\text{ H} = 1\text{ T} \cdot \text{m}^2/\text{A} \tag{11.1.4}$$

We shall see that the mutual inductance M_{21} depends only on the geometrical properties of the two coils such as the number of turns and the radii of the two coils.

In a similar manner, suppose instead there is a current I_2 in the second coil and it is varying with time (Figure 11.1.2). Then the induced emf in coil 1 becomes

$$\varepsilon_{12} = -N_1 \frac{d\Phi_{12}}{dt} = -\frac{d}{dt} \iint_{\text{coil 1}} \vec{\mathbf{B}}_2 \cdot d\vec{\mathbf{A}}_1 \tag{11.1.5}$$

and a current is induced in coil 1.
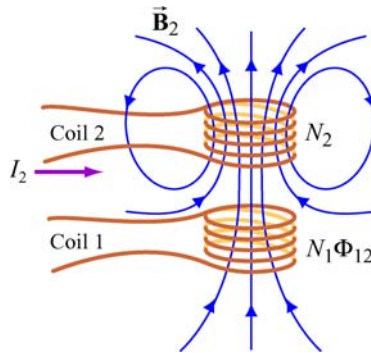
Figure 11.1.2 Changing current in coil 2 produces changing magnetic flux in coil 1.

This changing flux in coil 1 is proportional to the changing current in coil 2,

$$N_1 \frac{d\Phi_{12}}{dt} = M_{12} \frac{dI_2}{dt} \tag{11.1.6}$$

where the proportionality constant M_{12} is another mutual inductance and can be written as

$$M_{12} = \frac{N_1 \Phi_{12}}{I_2} \tag{11.1.7}$$

However, using the *reciprocity theorem* which combines Ampere's law and the Biot-Savart law, one may show that the constants are equal:

$$M_{12} = M_{21} \equiv M \tag{11.1.8}$$

Example 11.1 Mutual Inductance of Two Concentric Coplanar Loops

Consider two single-turn co-planar, concentric coils of radii R_1 and R_2, with $R_1 \gg R_2$, as shown in Figure 11.1.3. What is the mutual inductance between the two loops?
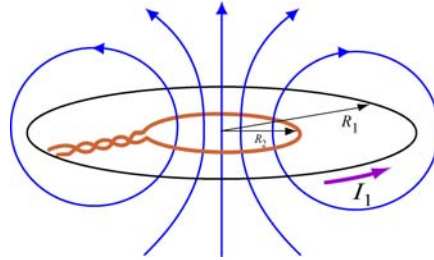
Figure 11.1.3 Two concentric current loop

Solution:

The mutual inductance can be computed as follows. Using Eq. (9.1.15) of Chapter 9, we see that the magnetic field at the center of the ring due to I_1 in the outer coil is given by

$$B_1 = \frac{\mu_0 I_1}{2R_1} \qquad (11.1.9)$$

Since $R_1 \gg R_2$, we approximate the magnetic field through the entire inner coil by B_1. Hence, the flux through the second (inner) coil is

$$\Phi_{21} = B_1 A_2 = \left(\frac{\mu_0 I_1}{2R_1}\right)\pi R_2^2 = \frac{\mu_0 \pi I_1 R_2^2}{2R_1} \qquad (11.1.10)$$

Thus, the mutual inductance is given by

$$M = \frac{\Phi_{21}}{I_1} = \frac{\mu_0 \pi R_2^2}{2R_1} \qquad (11.1.11)$$

The result shows that M depends only on the geometrical factors, R_1 and R_2, and is independent of the current I_1 in the coil.

11.2 Self-Inductance

Consider again a coil consisting of N turns and carrying current I in the counterclockwise direction, as shown in Figure 11.2.1. If the current is steady, then the magnetic flux through the loop will remain constant. However, suppose the current I changes with time,

then according to Faraday's law, an induced emf will arise to oppose the change. The induced current will flow clockwise if $dI/dt > 0$, and counterclockwise if $dI/dt < 0$. The property of the loop in which its own magnetic field opposes any change in current is called "self-inductance," and the emf generated is called the self-induced emf or back emf, which we denote as ε_L. All current-carrying loops exhibit this property. In particular, an inductor is a circuit element (symbol ⦿⦿⦿) which has a large self-inductance.
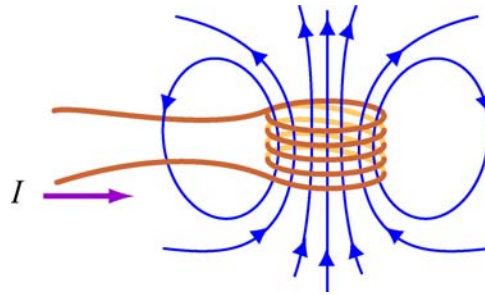
Figure 11.2.1 Magnetic flux through the current loop

Mathematically, the self-induced emf can be written as

$$\varepsilon_L = -N\frac{d\Phi_B}{dt} = -N\frac{d}{dt}\iint \vec{\mathbf{B}}\cdot d\vec{\mathbf{A}} \tag{11.2.1}$$

and is related to the self-inductance L by

$$\varepsilon_L = -L\frac{dI}{dt} \tag{11.2.2}$$

The two expressions can be combined to yield

$$L = \frac{N\Phi_B}{I} \tag{11.2.3}$$

Physically, the inductance L is a measure of an inductor's "resistance" to the change of current; the larger the value of L, the lower the rate of change of current.

Example 11.2 Self-Inductance of a Solenoid

Compute the self-inductance of a solenoid with N turns, length l, and radius R with a current I flowing through each turn, as shown in Figure 11.2.2.
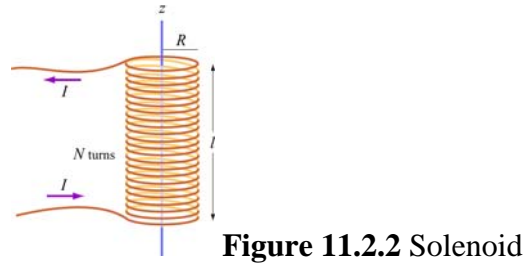
Figure 11.2.2 Solenoid

Solution:

Ignoring edge effects and applying Ampere's law, the magnetic field inside a solenoid is given by Eq. (9.4.3):

$$\vec{\mathbf{B}} = \frac{\mu_0 NI}{l}\hat{\mathbf{k}} = \mu_0 nI\ \hat{\mathbf{k}} \tag{11.2.4}$$

where $n = N/l$ is the number of turns per unit length. The magnetic flux through each turn is

$$\Phi_B = BA = \mu_0 nI \cdot (\pi R^2) = \mu_0 nI\pi R^2 \tag{11.2.5}$$

Thus, the self-inductance is

$$\boxed{L = \frac{N\Phi_B}{I} = \mu_0 n^2 \pi R^2 l} \tag{11.2.6}$$

We see that L depends only on the geometrical factors (n, R and l) and is independent of the current I.

Example 11.3 Self-Inductance of a Toroid

Calculate the self-inductance of a toroid which consists of N turns and has a rectangular cross section, with inner radius a, outer radius b and height h, as shown in Figure 11.2.3(a).
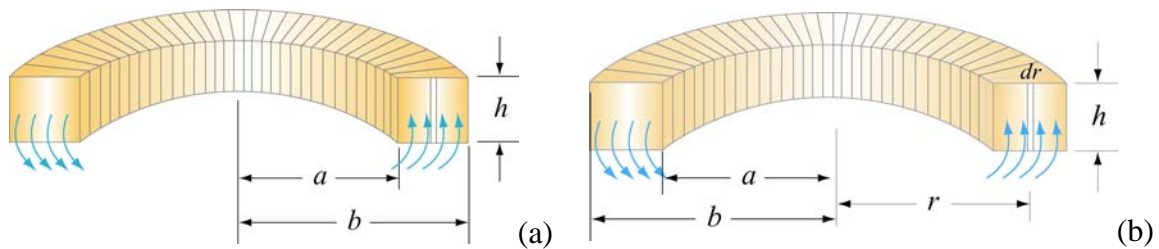
Figure 11.2.3 A toroid with N turns

Solution:

According to Ampere's law discussed in Section 9.3, the magnetic field is given by

$$\oint \vec{B} \cdot d\vec{s} = \oint B\, ds = B \oint ds = B(2\pi r) = \mu_0 NI \tag{11.2.7}$$

or

$$B = \frac{\mu_0 NI}{2\pi r} \tag{11.2.8}$$

The magnetic flux through one turn of the toroid may be obtained by integrating over the rectangular cross section, with $dA = h\, dr$ as the differential area element (Figure 11.2.3b):

$$\Phi_B = \iint \vec{B} \cdot d\vec{A} = \int_a^b \left(\frac{\mu_0 NI}{2\pi r} \right) h\, dr = \frac{\mu_0 NIh}{2\pi} \ln\left(\frac{b}{a} \right) \tag{11.2.9}$$

The total flux is $N\Phi_B$. Therefore, the self-inductance is

$$L = \frac{N\Phi_B}{I} = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a} \right) \tag{11.2.10}$$

Again, the self-inductance L depends only on the geometrical factors. Let's consider the situation where $a \ll b - a$. In this limit, the logarithmic term in the equation above may be expanded as

$$\ln\left(\frac{b}{a} \right) = \ln\left(1 + \frac{b-a}{a} \right) \approx \frac{b-a}{a} \tag{11.2.11}$$

and the self-inductance becomes

$$L \approx \frac{\mu_0 N^2 h}{2\pi} \cdot \frac{b-a}{a} = \frac{\mu_0 N^2 A}{2\pi a} = \frac{\mu_0 N^2 A}{l} \tag{11.2.12}$$

where $A = h(b-a)$ is the cross-sectional area, and $l = 2\pi a$. We see that the self-inductance of the toroid in this limit has the same form as that of a solenoid.

Example 11.4 Mutual Inductance of a Coil Wrapped Around a Solenoid

A long solenoid with length l and a cross-sectional area A consists of N_1 turns of wire. An insulated coil of N_2 turns is wrapped around it, as shown in Figure 11.2.4.

(a) Calculate the mutual inductance M, assuming that all the flux from the solenoid passes through the outer coil.

(b) Relate the mutual inductance M to the self-inductances L_1 and L_2 of the solenoid and he coil.
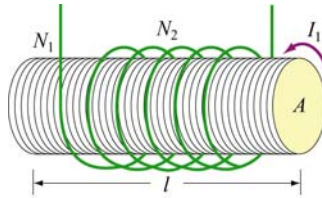
Figure 11.2.4 A coil wrapped around a solenoid

Solutions:

(a) The magnetic flux through each turn of the outer coil due to the solenoid is

$$\Phi_{21} = BA = \frac{\mu_0 N_1 I_1}{l} A \tag{11.2.13}$$

where $B = \mu_0 N_1 I_1 / l$ is the uniform magnetic field inside the solenoid. Thus, the mutual inductance is

$$\boxed{M = \frac{N_2 \Phi_{21}}{I_1} = \frac{\mu_0 N_1 N_2 A}{l}} \tag{11.2.14}$$

(b) From Example 11.2, we see that the self-inductance of the solenoid with N_1 turns is given by

$$L_1 = \frac{N_1 \Phi_{11}}{I_1} = \frac{\mu_0 N_1^2 A}{l} \tag{11.2.15}$$

where Φ_{11} is the magnetic flux through one turn of the solenoid due to the magnetic field produced by I_1. Similarly, we have $L_2 = \mu_0 N_2^2 A / l$ for the outer coil. In terms of L_1 and L_2, the mutual inductance can be written as

$$M = \sqrt{L_1 L_2} \tag{11.2.16}$$

More generally the mutual inductance is given by

$$M = k\sqrt{L_1 L_2}, \qquad 0 \le k \le 1 \tag{11.2.17}$$

where k is the "coupling coefficient." In our example, we have $k = 1$ which means that all of the magnetic flux produced by the solenoid passes through the outer coil, and vice versa, in this idealization.

11.3 Energy Stored in Magnetic Fields

Since an inductor in a circuit serves to oppose any change in the current through it, work must be done by an external source such as a battery in order to establish a current in the inductor. From the work-energy theorem, we conclude that energy can be stored in an inductor. The role played by an inductor in the magnetic case is analogous to that of a capacitor in the electric case.

The power, or rate at which an external emf ε_{ext} works to overcome the self-induced emf ε_L and pass current I in the inductor is

$$P_L = \frac{dW_{\text{ext}}}{dt} = I\varepsilon_{\text{ext}} \tag{11.3.1}$$

If only the external emf and the inductor are present, then $\varepsilon_{\text{ext}} = -\varepsilon_L$ which implies

$$P_L = \frac{dW_{\text{ext}}}{dt} = -I\varepsilon_L = +IL\frac{dI}{dt} \tag{11.3.2}$$

If the current is increasing with $dI/dt > 0$, then $P > 0$ which means that the external source is doing positive work to transfer energy to the inductor. Thus, the internal energy U_B of the inductor is increased. On the other hand, if the current is decreasing with $dI/dt < 0$, we then have $P < 0$. In this case, the external source takes energy away from the inductor, causing its internal energy to go down. The total work done by the external source to increase the current form zero to I is then

$$W_{\text{ext}} = \int dW_{\text{ext}} = \int_0^I LI'\,dI' = \frac{1}{2}LI^2 \tag{11.3.3}$$

This is equal to the magnetic energy stored in the inductor:

$$\boxed{U_B = \frac{1}{2}LI^2} \tag{11.3.4}$$

The above expression is analogous to the electric energy stored in a capacitor:

$$U_E = \frac{1}{2}\frac{Q^2}{C} \tag{11.3.5}$$

We comment that from the energy perspective there is an important distinction between an inductor and a resistor. Whenever a current I goes through a resistor, energy flows into the resistor and dissipates in the form of heat regardless of whether I is steady or time-dependent (recall that power dissipated in a resistor is $P_R = IV_R = I^2R$). On the other hand, energy flows into an ideal inductor only when the current is varying with $dI/dt > 0$. The energy is not dissipated but stored there; it is released later when the current decreases with $dI/dt < 0$. If the current that passes through the inductor is steady, then there is no change in energy since $P_L = LI(dI/dt) = 0$.

Example 11.5 Energy Stored in a Solenoid

A long solenoid with length l and a radius R consists of N turns of wire. A current I passes through the coil. Find the energy stored in the system.

Solution:

Using Eqs. (11.2.6) and (11.3.4), we readily obtain

$$U_B = \frac{1}{2}LI^2 = \frac{1}{2}\mu_0 n^2 I^2 \pi R^2 l \tag{11.3.6}$$

The result can be expressed in terms of the magnetic field strength $B = \mu_0 nI$:

$$U_B = \frac{1}{2\mu_0}(\mu_0 nI)^2(\pi R^2 l) = \frac{B^2}{2\mu_0}(\pi R^2 l) \tag{11.3.7}$$

Since $\pi R^2 l$ is the volume within the solenoid, and the magnetic field inside is uniform, the term

$$u_B = \frac{B^2}{2\mu_0} \tag{11.3.8}$$

may be identified as the magnetic energy density, or the energy per unit volume of the magnetic field. The above expression holds true even when the magnetic field is non-uniform. The result can be compared with the energy density associated with an electric field:

$$u_E = \frac{1}{2}\varepsilon_0 E^2 \tag{11.3.9}$$

11.3.1 Creating and Destroying Magnetic Energy Animation

Let's consider the process involved in creating magnetic energy. Figure 11.3.1 shows the process by which an external agent(s) creates magnetic energy. Suppose we have five rings that carry a number of free positive charges that are not moving. Since there is no current, there is no magnetic field. Now suppose a set of external agents come along (one for each charge) and simultaneously spin up the charges counterclockwise as seen from above, at the same time and at the same rate, in a manner that has been pre-arranged. Once the charges on the rings start to accelerate, there is a magnetic field in the space between the rings, mostly parallel to their common axis, which is stronger inside the rings than outside. This is the solenoid configuration.
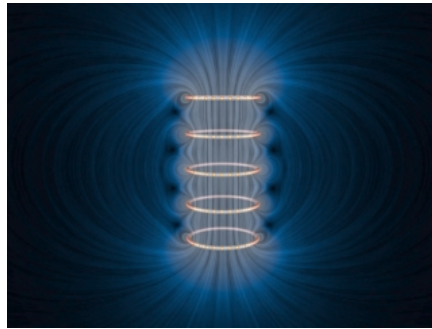

Figure 11.3.1 Creating (link) and destroying (link) magnetic field energy

As the magnetic flux through the rings grows, Faraday's law of induction tells us that there is an electric field induced by the time-changing magnetic field that is circulating clockwise as seen from above. The force on the charges due to this electric field is thus opposite the direction the external agents are trying to spin the rings up (counterclockwise), and thus the agents have to do additional work to spin up the charges because of their charge. This is the source of the energy that is appearing in the magnetic field between the rings — the work done by the agents against the "back emf."

Over the course of the "create" animation associated with Figure 11.3.1, the agents moving the charges to a higher speed against the induced electric field are continually doing work. The electromagnetic energy that they are creating at the place where they are doing work (the path along which the charges move) flows both inward and outward. The direction of the flow of this energy is shown by the animated texture patterns in Figure 11.3.1. This is the electromagnetic energy flow that increases the strength of the magnetic field in the space between the rings as each positive charge is accelerated to a higher and higher speed. When the external agents have gotten up the charges to a pre-determined speed, they stop the acceleration. The charges then move at a constant speed, with a constant field inside the solenoid, and zero "induced" electric field, in accordance with Faraday's law of induction.

We also have an animation of the "destroy" process linked to Figure 11.3.1. This process proceeds as follows. Our set of external agents now simultaneously start to spin down the moving charges (which are still moving counterclockwise as seen from above), at the

same time and at the same rate, in a manner that has been pre-arranged. Once the charges on the rings start to decelerate, the magnetic field in the space between the rings starts to decrease in magnitude. As the magnetic flux through the rings decreases, Faraday's law tells us that there is now an electric field induced by the time-changing magnetic field that is circulating counterclockwise as seen from above. The force on the charges due to this electric field is thus in the same direction as the motion of the charges. In this situation the agents have work done on them as they try to spin the charges down.

Over the course of the "destroy" animation associated with Figure 11.3.1, the strength of the magnetic field decreases, and this energy flows from the field back to the path along which the charges move, and is now being provided *to* the agents trying to spin down the moving charges. The energy provided to those agents as they destroy the magnetic field is exactly the amount of energy that they put into creating the magnetic field in the first place, neglecting radiative losses (such losses are small if we move the charges at speeds small compared to the speed of light). This is a totally reversible process if we neglect such losses. That is, the amount of energy the agents put into creating the magnetic field is exactly returned to the agents as the field is destroyed.

There is one final point to be made. Whenever electromagnetic energy is being created, an electric charge is moving (or being moved) against an electric field ($q\,\vec{\mathbf{v}}\cdot\vec{\mathbf{E}} < 0$). Whenever electromagnetic energy is being destroyed, an electric charge is moving (or being moved) along an electric field ($q\,\vec{\mathbf{v}}\cdot\vec{\mathbf{E}} > 0$). This is the same rule we saw above when we were creating and destroying electric energy above.

11.3.2 Magnets and Conducting Rings Animation

In the example of Faraday's law that we gave above, the sense of the electric field associated with a time-changing magnetic field was always such as to try to resist change. We consider another example of Faraday's law that illustrates this same tendency in a different way.
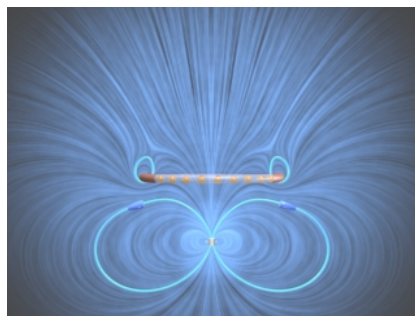

Figure 11.3.2 A perfectly conducting ring falls on the axis of a permanent magnet. The induced currents and the resulting magnetic field stresses are such as to slow the fall of the ring. If the ring is light enough (or the magnet strong enough), the ring will levitate above the magnet (link)

In Figure 11.3.2, we show a permanent magnet that is fixed at the origin with its dipole moment pointing upward. On the *z*-axis above the magnet, we have a co-axial, conducting, non-magnetic ring with radius *a*, inductance *L,* and resistance *R*. The center of the conducting ring is constrained to move along the vertical axis. The ring is released from rest and falls under gravity toward the stationary magnet. Eddy currents arise in the ring because of the changing magnetic flux and induced electric field as the ring falls toward the magnet, and the sense of these currents is to repel the ring when it is above the magnet.

This physical situation can be formulated mathematically in terms of three coupled ordinary differential equations for the position of the ring, its velocity, and the current in the ring. We consider in Figure 11.3.2 the particular situation where the resistance of the ring (which in our model can have any value) is identically zero, and the mass of the ring is small enough (or the field of the magnet is large enough) so that the ring levitates above the magnet. We let the ring begin at rest a distance 2*a* above the magnet. The ring begins to fall under gravity. When the ring reaches a distance of about *a* above the ring, its acceleration slows because of the increasing current in the ring. As the current increases, energy is stored in the magnetic field, and when the ring comes to rest, all of the initial gravitational potential of the ring is stored in the magnetic field. That magnetic energy is then returned to the ring as it "bounces" and returns to its original position a distance 2*a* above the magnet. Because there is no dissipation in the system for our particular choice of *R* in this example, this motion repeats indefinitely.

What are the important points to be learned from this animation? Initially, all the free energy in this situation is stored in the gravitational potential energy of the ring. As the ring begins to fall, that gravitational energy begins to appear as kinetic energy in the ring. It also begins to appear as energy stored in the magnetic field. The compressed field below the ring enables the transmission of an upward force to the moving ring as well as a downward force to the magnet. But that compression also stores energy in the magnetic field. It is plausible to argue based on the animation that the kinetic energy of the downwardly moving ring is decreasing as more and more energy is stored in the magnetostatic field, and conversely when the ring is rising.

Figure 11.3.3 shows a more realistic case in which the resistance of the ring is finite. Now energy is not conserved, and the ring eventually falls past the magnet. When it passes the magnet, the sense of the induced electric field and thus of the eddy currents reverses, and the ring is now attracted to the magnet above it, which again retards its fall.

There are many other examples of the falling ring and stationary magnet, or falling magnet and stationary ring, given in the animations at this link. All of them show that the effect of the electric field associated with a time-changing magnetic field is to try to keep things the same. In the limiting case of zero resistance, it can in fact achieve this goal, e.g. in Figure 11.3.2 the magnetic flux through the ring never changes over the course of the motion.
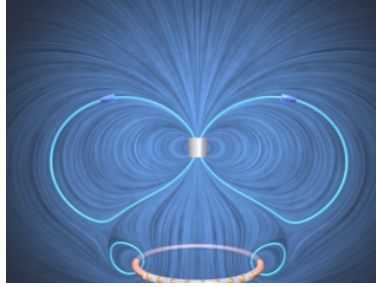
Figure 11.3.3 A ring with finite resistance falls on the axis of a magnet. We show the ring after it has fallen past the magnet, when it is attracted to the magnet above it (link)

11.4 *RL* Circuits

11.4.1 Self-Inductance and the Modified Kirchhoff's Loop Rule

The addition of time-changing magnetic fields to simple circuits means that the closed line integral of the electric field around a circuit is no longer zero. Instead, we have, for any open surface

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d}{dt} \iint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} \qquad (11.4.1)$$

Any circuit where the current changes with time will have time-changing magnetic fields, and therefore induced electric fields. How do we solve simple circuits taking such effects into account? We discuss here a consistent way to understand the consequences of introducing time-changing magnetic fields into circuit theory -- that is, inductance.

As soon as we introduce time-changing magnetic fields, the electric potential difference between two points in our circuit is no longer well-defined, because when the line integral of the electric field around a closed loop is nonzero, the potential difference between two points, say *a* and *b*, is no longer independent of the path taken to get from *a* to *b*. That is, the electric field is no longer a conservative field, and the electric potential is no longer an appropriate concept, since we can no longer write $\vec{\mathbf{E}}$ as the negative gradient of a scalar potential. However, we can still write down in a straightforward fashion the equation that determines the behavior of a circuit.
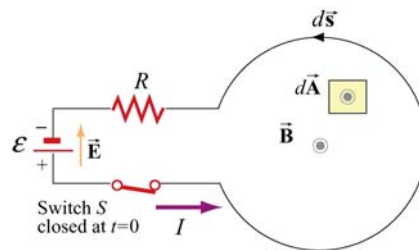

Figure 11.4.1 One-loop inductor circuit

To show how to do this, consider the circuit shown in Figure 11.4.1. We have a battery, a resistor, a switch *S* that is closed at *t* = 0, and a "one-loop inductor." It will become clear what the consequences of this "inductance" are as we proceed. For *t* > 0, current will flow in the direction shown (from the positive terminal of the battery to the negative, as usual). What is the equation that governs the behavior of our current $I(t)$ for *t* > 0?

To investigate this, apply Faraday's law to the open surface bounded by our circuit, where we take $d\vec{\mathbf{A}}$ to be out of the page, and $d\vec{\mathbf{s}}$ right-handed with respect to that choice (counter-clockwise). First, what is the integral of the electric field around this circuit? There is an electric field in the battery, directed from the positive terminal to the negative terminal, and when we go through the battery in the direction of $d\vec{\mathbf{s}}$ that we have chosen, we are moving against that electric field, so that $\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} < 0$. Thus the contribution of the battery to our integral is $-\varepsilon$. Then, there is an electric field in the resistor, in the direction of the current, so when we move through the resistor in that direction, $\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$ is greater than zero, and that contribution to our integral is $+IR$. What about when we move through our one-loop inductor? There is **no** electric field in this loop if the resistance of the wire making up the loop is zero. Thus, going around the closed loop *clockwise* against the current, we have

$$\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\varepsilon + IR \qquad (11.4.2)$$

Now, what is the magnetic flux Φ_B through our open surface? First of all, we arrange the geometry so that the part of the circuit which includes the battery, the switch, and the resistor makes only a small contribution to Φ_B as compared to the (much larger in area) part of the open surface which includes our one-loop inductor. Second, we know that Φ_B is positive in that part of the surface, because current flowing counterclockwise will produce a magnetic field $\vec{\mathbf{B}}$ pointing out of the page, which is the same direction we have assumed for $d\vec{\mathbf{A}}$. Thus, the dot product $\vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} > 0$. Note that $\vec{\mathbf{B}}$ is the *self* magnetic field —that is, the magnetic field produced by the current flowing in the circuit, and not by any external currents.

From Section 11.1, we also see that the magnetic flux Φ_B is proportional to *I*, and may be written as $\Phi_B = LI$, where *L* is the self-inductance which depends on the geometry of the circuit. The time rate of change of Φ_B is just $L(dI/dt)$, so that we have from Faraday's law

$$\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\varepsilon + IR = -\frac{d\Phi_B}{dt} = -L\frac{dI}{dt} \qquad (11.4.3)$$

We can write the governing equation for $I(t)$ from above as

$$\Delta V = \varepsilon - IR - L\frac{dI}{dt} = 0 \qquad (11.4.4)$$

where the expression has been cast in a form that resembles Kirchhoff's loop rule, namely that the sum of the potential drops around a circuit is zero. To preserve the loop rule, we must specify the "potential drop" across an inductor.

$\varepsilon_L = V_b - V_a = -L(dI/dt) < 0$

(a)

$\varepsilon_L = V_b - V_a = -L(dI/dt) > 0$
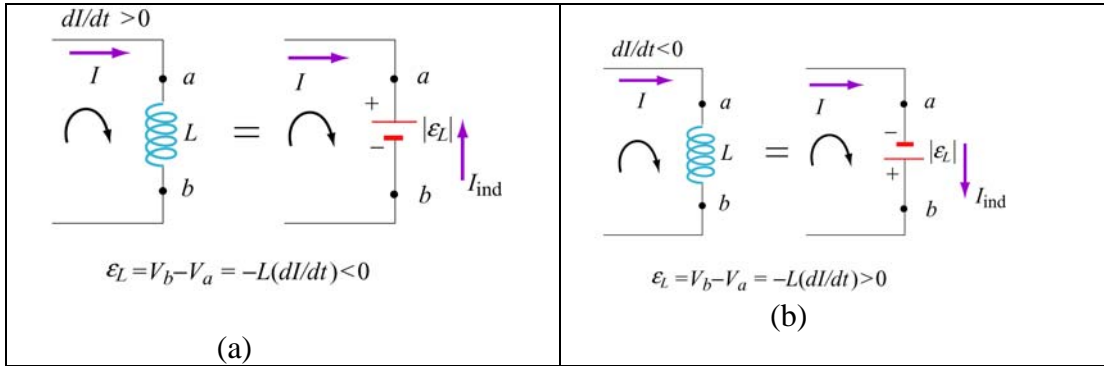
(b)

Figure 11.4.2 Modified Kirchhoff's rule for inductors (a) with increasing current, and (b) with decreasing current.

The modified rule for inductors may be obtained as follows: The polarity of the self-induced emf is such as to oppose the change in current, in accord with Lenz's law. If the rate of change of current is positive, as shown in Figure 11.4.2(a), the self-induced emf ε_L sets up an induced current I_{ind} moving in the opposite direction of the current I to oppose such an increase. The inductor could be replaced by an emf $|\varepsilon_L| = L|dI/dt| = +L(dI/dt)$ with the polarity shown in Figure 11.4.2(a). On the other hand, if $dI/dt < 0$, as shown in Figure 11.4.2(b), the induced current I_{ind} set up by the self-induced emf ε_L flows in the same direction as I to oppose such a decrease.

We see that whether the rate of change of current in increasing ($dI/dt > 0$) or decreasing ($dI/dt < 0$), in both cases, the change in potential when moving from a to b along the direction of the current I is $V_b - V_a = -L(dI/dt)$. Thus, we have

---

Kirchhoff's Loop Rule Modified for Inductors:

If an inductor is traversed in the direction of the current, the "potential change" is $-L(dI/dt)$. On the other hand, if the inductor is traversed in the direction opposite of the current, the "potential change" is $+L(dI/dt)$.

Use of this modified Kirchhoff's rule will give the correct equations for circuit problems that contain inductors. However, keep in mind that it is **misleading** at best, and at some level **wrong** in terms of the physics. Again, we emphasize that Kirchhoff's loop rule was originally based on the fact that the line integral of $\vec{\mathbf{E}}$ around a closed loop was zero.

With time-changing magnetic fields, this is no longer so, and thus the sum of the "potential drops" around the circuit, if we take that to mean the negative of the closed loop integral of $\vec{\mathbf{E}}$, is **no longer zero** –in fact it is $+L(dI/dt)$.

11.4.2 Rising Current

Consider the *RL* circuit shown in Figure 11.4.3. At $t=0$ the switch is closed. We find that the current does not rise immediately to its maximum value ε/R. This is due to the presence of the self-induced emf in the inductor.
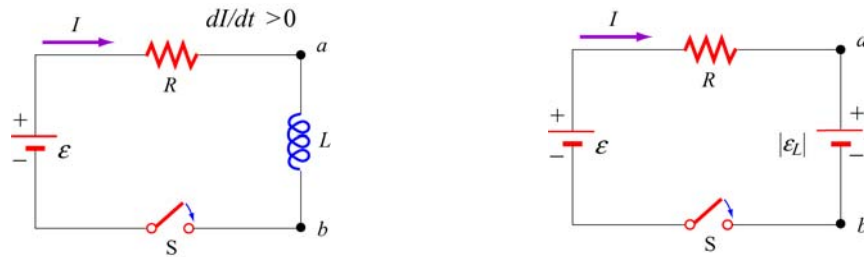

Figure 11.4.3 (a) *RL* Circuit with rising current. (b) Equivalent circuit using the modified Kirchhoff's loop rule.

Using the modified Kirchhoff's rule for increasing current, $dI/dt>0$, the *RL* circuit is described by the following differential equation:

$$\varepsilon - IR - |\varepsilon_L| = \varepsilon - IR - L\frac{dI}{dt} = 0 \qquad (11.4.5)$$

Note that there is an important distinction between an inductor and a resistor. The potential difference across a resistor depends on *I*, while the potential difference across an inductor depends on dI/dt. The self-induced emf does not oppose the current itself, but the change of current dI/dt.

The above equation can be rewritten as

$$\frac{dI}{I-\varepsilon/R} = -\frac{dt}{L/R} \qquad (11.4.6)$$

Integrating over both sides and imposing the condition $I(t=0)=0$, the solution to the differential equation is

$$I(t) = \frac{\varepsilon}{R}\left(1-e^{-t/\tau}\right) \qquad (11.4.7)$$

where

$$\tau = \frac{L}{R} \quad (11.4.8)$$

is the time constant of the RL circuit. The qualitative behavior of the current as a function of time is depicted in Figure 11.4.4.
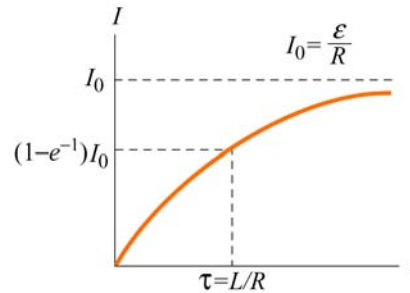

Figure 11.4.4 Current in the RL circuit as a function of time

Note that after a sufficiently long time, the current reaches its equilibrium value ε / R. The time constant τ is a measure of how fast the equilibrium state is attained; the larger the value of L, the longer it takes to build up the current. A comparison of the behavior of current in a circuit with or without an inductor is shown in Figure 11.4.5 below.

Similarly, the magnitude of the self-induced emf can be obtained as

$$|\varepsilon_L| = \left| -L \frac{dI}{dt} \right| = \varepsilon e^{-t/\tau} \quad (11.4.9)$$

which is at a maximum when $t = 0$ and vanishes as t approaches infinity. This implies that a sufficiently long time after the switch is closed, self-induction disappears and the inductor simply acts as a conducting wire connecting two parts of the circuit.
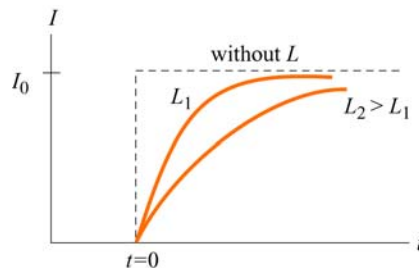

Figure 11.4.5 Behavior of current in a circuit with or without an inductor

To see that energy is conserved in the circuit, we multiply Eq. (11.4.7) by I and obtain

$$I\varepsilon = I^2 R + LI \frac{dI}{dt} \quad (11.4.10)$$

The left-hand side represents the rate at which the battery delivers energy to the circuit. On the other hand, the first term on the right-hand side is the power dissipated in the resistor in the form of heat, and the second term is the rate at which energy is stored in the inductor. While the energy dissipated through the resistor is irrecoverable, the magnetic energy stored in the inductor can be released later.

11.4.3 Decaying Current

Next we consider the *RL* circuit shown in Figure 11.4.6. Suppose the switch S_1 has been closed for a long time so that the current is at its equilibrium value ε / R. What happens to the current when at $t = 0$ switches S_1 is opened and S_2 closed?

Applying modified Kirchhoff's loop rule to the right loop for decreasing current, $dI/dt < 0$, yields

$$|\varepsilon_L| - IR = -L\frac{dI}{dt} - IR = 0 \tag{11.4.11}$$

which can be rewritten as

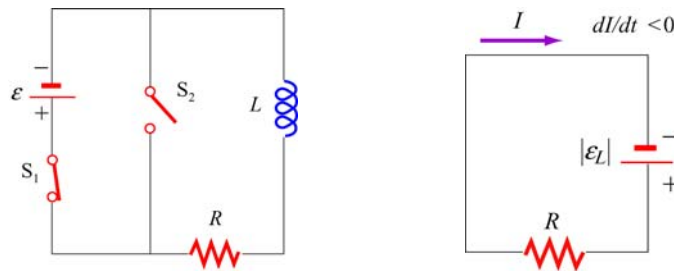$$\frac{dI}{I} = -\frac{dt}{L/R} \tag{11.4.12}$$

Figure 11.4.6 (a) *RL* circuit with decaying current, and (b) equivalent circuit.

The solution to the above differential equation is

$$I(t) = \frac{\varepsilon}{R}e^{-t/\tau} \tag{11.4.13}$$

where $\tau = L/R$ is the same time constant as in the case of rising current. A plot of the current as a function of time is shown in Figure 11.4.7.
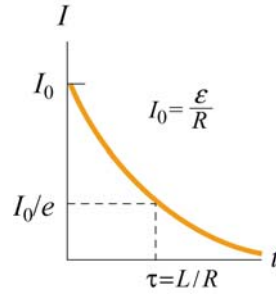
Figure 11.4.7 Decaying current in an *RL* circuit

11.5 *LC* Oscillations

Consider an *LC* circuit in which a capacitor is connected to an inductor, as shown in Figure 11.5.1.
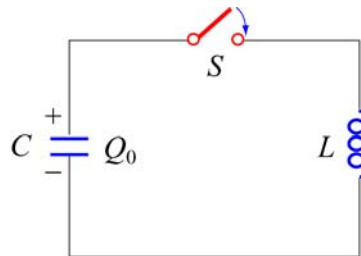

Figure 11.5.1 *LC* Circuit

Suppose the capacitor initially has charge Q_0. When the switch is closed, the capacitor begins to discharge and the electric energy is decreased. On the other hand, the current created from the discharging process generates magnetic energy which then gets stored in the inductor. In the absence of resistance, the total energy is transformed back and forth between the electric energy in the capacitor and the magnetic energy in the inductor. This phenomenon is called electromagnetic oscillation.

The total energy in the *LC* circuit at some instant after closing the switch is

$$U = U_C + U_L = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2 \qquad (11.5.1)$$

The fact that *U* remains constant implies that

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2\right) = \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = 0 \qquad (11.5.2)$$

or

$$\boxed{\frac{Q}{C} + L\frac{d^2Q}{dt^2} = 0}$$ (11.5.3)

where $I = -dQ/dt$ (and $dI/dt = -d^2Q/dt^2$). Notice the sign convention we have adopted here. The negative sign implies that the current I is equal to the rate of *decrease* of charge in the capacitor plate immediately after the switch has been closed. The same equation can be obtained by applying the modified Kirchhoff's loop rule clockwise:

$$\frac{Q}{C} - L\frac{dI}{dt} = 0$$ (11.5.4)

followed by our definition of current.

The general solution to Eq. (11.5.3) is

$$Q(t) = Q_0 \cos(\omega_0 t + \phi)$$ (11.5.5)

where Q_0 is the amplitude of the charge and ϕ is the phase. The angular frequency ω_0 is given by

$$\omega_0 = \frac{1}{\sqrt{LC}}$$ (11.5.6)

The corresponding current in the inductor is

$$I(t) = -\frac{dQ}{dt} = \omega_0 Q_0 \sin(\omega_0 t + \phi) = I_0 \sin(\omega_0 t + \phi)$$ (11.5.7)

where $I_0 = \omega_0 Q_0$. From the initial conditions $Q(t=0) = Q_0$ and $I(t=0) = 0$, the phase ϕ can be determined to be $\phi = 0$. Thus, the solutions for the charge and the current in our *LC* circuit are

$$Q(t) = Q_0 \cos \omega_0 t$$ (11.5.8)

and

$$I(t) = I_0 \sin \omega_0 t$$ (11.5.9)

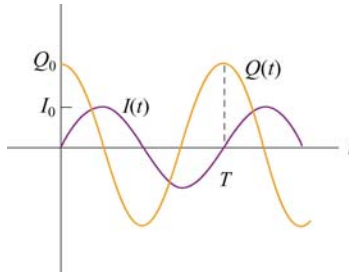The time dependence of $Q(t)$ and $I(t)$ are depicted in Figure 11.5.2.

Figure 11.5.2 Charge and current in the LC circuit as a function of time

Using Eqs. (11.5.8) and (11.5.9), we see that at any instant of time, the electric energy and the magnetic energies are given by

$$U_E = \frac{Q^2(t)}{2C} = \left(\frac{Q_0^2}{2C}\right)\cos^2 \omega_0 t \qquad (11.5.10)$$

and

$$U_B = \frac{1}{2}LI^2(t) = \frac{LI_0^2}{2}\sin^2 \omega t = \frac{L(-\omega_0 Q_0)^2}{2}\sin^2 \omega_0 t = \left(\frac{Q_0^2}{2C}\right)\sin^2 \omega_0 t \quad (11.5.11)$$

respectively. One can easily show that the total energy remains constant:

$$U = U_E + U_B = \left(\frac{Q_0^2}{2C}\right)\cos^2 \omega_0 t + \left(\frac{Q_0^2}{2C}\right)\sin^2 \omega_0 t = \frac{Q_0^2}{2C} \qquad (11.5.12)$$

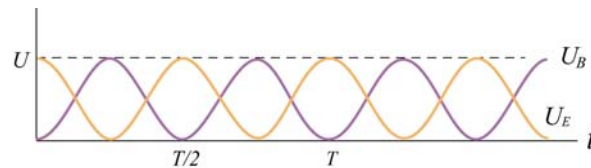The electric and magnetic energy oscillation is illustrated in Figure 11.5.3.

Figure 11.5.3 Electric and magnetic energy oscillations

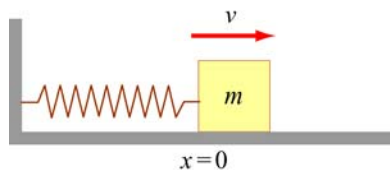The mechanical analog of the LC oscillations is the mass-spring system, shown in Figure 11.5.4.

Figure 11.5.4 Mass-spring oscillations

If the mass is moving with a speed v and the spring having a spring constant k is displaced from its equilibrium by x, then the total energy of this mechanical system is

$$U = K + U_{sp} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{11.5.13}$$

where K and U_{sp} are the kinetic energy of the mass and the potential energy of the spring, respectively. In the absence of friction, U is conserved and we obtain

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0 \tag{11.5.14}$$

Using $v = dx/dt$ and $dv/dt = d^2x/dt^2$, the above equation may be rewritten as

$$\boxed{m\frac{d^2x}{dt^2} + kx = 0} \tag{11.5.15}$$

The general solution for the displacement is

$$x(t) = x_0\cos(\omega_0 t + \phi) \tag{11.5.16}$$

where

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{11.5.17}$$

is the angular frequency and x_0 is the amplitude of the oscillations. Thus, at any instant in time, the energy of the system may be written as

$$
\begin{aligned}
U &= \frac{1}{2}mx_0^2\omega_0^2\sin^2(\omega_0 t + \phi) + \frac{1}{2}kx_0^2\cos^2(\omega_0 t + \phi) \\
&= \frac{1}{2}kx_0^2\left[\sin^2(\omega_0 t + \phi) + \cos^2(\omega_0 t + \phi)\right] = \frac{1}{2}kx_0^2
\end{aligned}
\tag{11.5.18}
$$

In Figure 11.5.5 we illustrate the energy oscillations in the *LC* Circuit and the mass-spring system (harmonic oscillator).
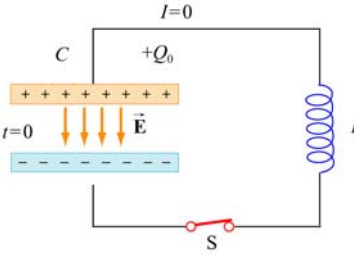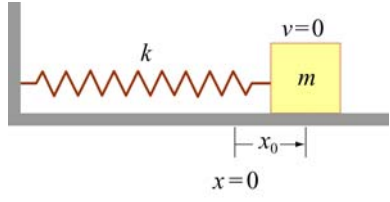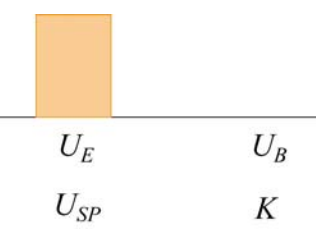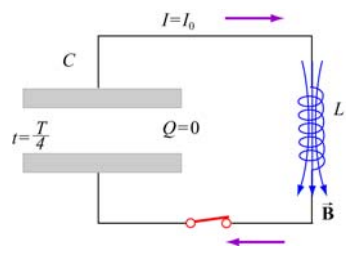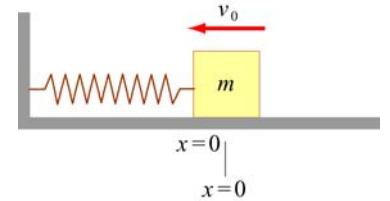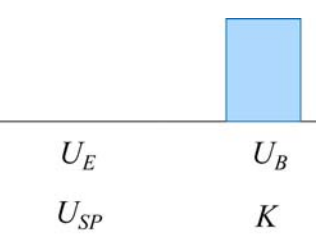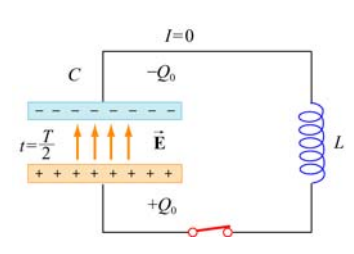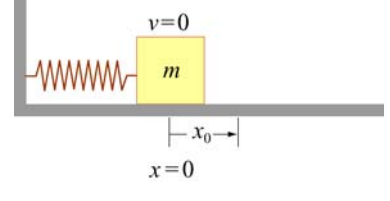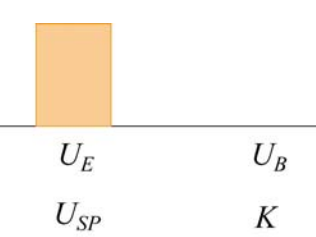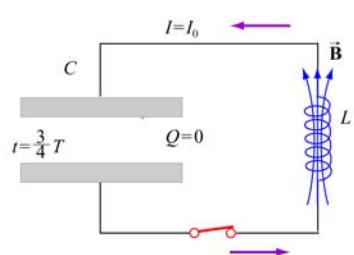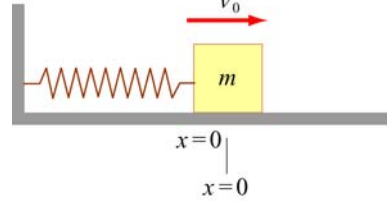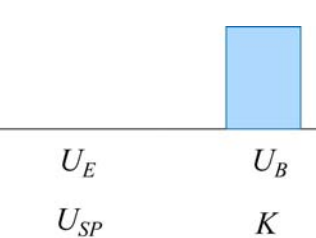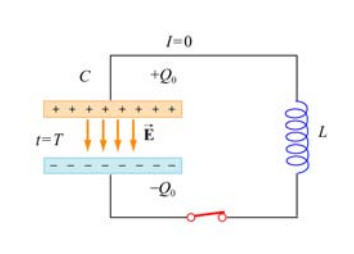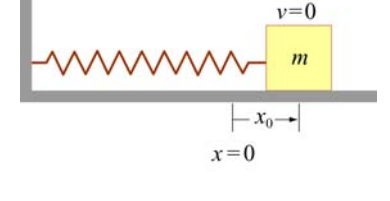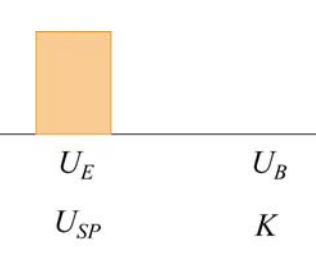
LC Circuit	Mass-spring System	Energy

Figure 11.5.5 Energy oscillations in the *LC* Circuit and the mass-spring system

11.6 The *RLC* Series Circuit

We now consider a series *RLC* circuit which contains a resistor, an inductor and a capacitor, as shown in Figure 11.6.1.
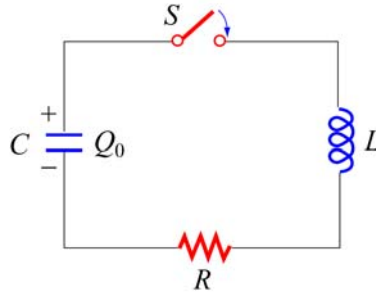
Figure 11.6.1 A series *RLC* circuit

The capacitor is initially charged to Q_0. After the switch is closed current will begin to flow. However, unlike the *LC* circuit energy will be dissipated through the resistor. The rate at which energy is dissipated is

$$\frac{dU}{dt} = -I^2 R \tag{11.6.1}$$

where the negative sign on the right-hand side implies that the total energy is decreasing. After substituting Eq. (11.5.2) for the left-hand side of the above equation, we obtain the following differential equation:

$$\frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = -I^2 R \tag{11.6.2}$$

Again, by our sign convention where current is equal to the rate of *decrease* of charge in the capacitor plates, $I = -dQ/dt$. Dividing both sides by I, the above equation can be rewritten as

$$\boxed{L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0} \tag{11.6.3}$$

For small R (the underdamped case, see Appendix 1), one can readily verify that a solution to the above equation is

$$Q(t) = Q_0 e^{-\gamma t} \cos(\omega' t + \phi) \tag{11.6.4}$$

where

$$\gamma = \frac{R}{2L} \tag{11.6.5}$$

is the damping factor and

$$\omega' = \sqrt{\omega_0^2 - \gamma^2} \tag{11.6.6}$$

is the angular frequency of the damped oscillations. The constants Q_0 and ϕ are real quantities to be determined from the initial conditions. In the limit where the resistance vanishes, $R = 0$, we recover the undamped, natural angular frequency $\omega_0 = 1/\sqrt{LC}$. There are three possible scenarios and the details are discussed in Appendix 1 (Section 11.8).

The mechanical analog of the series *RLC* circuit is the damped harmonic oscillator system. The equation of motion for this system is given by

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{11.6.7}$$

where the velocity-dependent term accounts for the non-conservative, dissipative force

$$F = -b\frac{dx}{dt} \tag{11.6.8}$$

with *b* being the damping coefficient. The correspondence between the *RLC* circuit and the mechanical system is summarized in Table 11.6.1. (Note that the sign of the current *I* depends on the physical situation under consideration.)

	RLC Circuit	Damped Harmonic Oscillator
Variable *s*	Q	x
Variable *ds/dt*	$\pm I$	v
Coefficient of *s*	$1/C$	k
Coefficient of *ds/dt*	R	b
Coefficient of d^2s/dt^2	L	m
Energy	$LI^2/2$	$mv^2/2$
	$Q^2/2C$	$kx^2/2$

Table 11.6.1 Correspondence between the *RLC* circuit and the mass-spring system

11.7 Summary

- Using Faraday's law of induction, the **mutual inductance** of two coils is given by

$$M_{12} = \frac{N_{12}\Phi_{12}}{I_1} = M_{21} = \frac{N_1\Phi_{21}}{I_2} = M$$

- The induced emf in coil 2 due to the change in current in coil 1 is given by

$$\varepsilon_2 = -M\frac{dI_1}{dt}$$

- The **self-inductance** of a coil with N turns is

$$L = \frac{N\Phi_B}{I}$$

  where Φ_B is the magnetic flux through one turn of the coil.

- The **self-induced emf** responding to a change in current inside a coil current is

$$\varepsilon_L = -L\frac{dI}{dt}$$

- The inductance of a solenoid with N turns, cross sectional area A and length l is

$$L = \frac{\mu_0 N^2 A}{l}$$

- If a battery supplying an emf ε is connected to an inductor and a resistor in series at time $t = 0$, then the current in this **RL circuit** as a function of time is

$$I(t) = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau}\right)$$

  where $\tau = L/R$ is the time constant of the circuit. If the battery is removed in the *RL* circuit, the current will decay as

$$I(t) = \left(\frac{\varepsilon}{R}\right)e^{-t/\tau}$$

- The **magnetic energy** stored in an inductor with current I passing through is

$$U_B = \frac{1}{2}LI^2$$

- The **magnetic energy density** at a point with magnetic field B is

$$u_B = \frac{B^2}{2\mu_0}$$

- The differential equation for an oscillating *LC* **circuit** is

$$\frac{d^2Q}{dt^2} + \omega_0^2 Q = 0$$

where $\omega_0 = \frac{1}{\sqrt{LC}}$ is the angular frequency of oscillation. The charge on the capacitor as a function of time is given by

$$Q(t) = Q_0 \cos(\omega_0 t + \phi)$$

and the current in the circuit is

$$I(t) = -\frac{dQ}{dt} = +\omega_0 Q_0 \sin(\omega_0 t + \phi)$$

- The total energy in an *LC* circuit is, using $I_0 = \omega_0 Q_0$,

$$U = U_E + U_B = \frac{Q_0^2}{2C}\cos^2 \omega_0 t + \frac{LI_0^2}{2}\sin^2 \omega_0 t = \frac{Q_0^2}{2C}$$

- The differential equation for an *RLC* **circuit** is

$$\frac{d^2Q}{dt^2} + 2\gamma\frac{dQ}{dt} + \omega_0^2 Q = 0$$

where $\omega_0 = \frac{1}{\sqrt{LC}}$ and $\gamma = R/2L$. In the underdamped case, the charge on the capacitor as a function of time is

$$Q(t) = Q_0 e^{-\gamma t} \cos(\omega' t + \phi)$$

where $\omega' = \sqrt{\omega_0^2 - \gamma^2}$.

11.8 Appendix 1: General Solutions for the *RLC* Series Circuit

In Section 11.6, we have shown that the *LRC* circuit is characterized by the following differential equation

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C}=0 \tag{11.8.1}$$

whose solutions is given by

$$Q(t)=Q_0 e^{-\gamma t}\cos(\omega't+\phi) \tag{11.8.2}$$

where

$$\gamma=\frac{R}{2L} \tag{11.8.3}$$

is the damping factor and

$$\omega'=\sqrt{\omega_0^2-\gamma^2} \tag{11.8.4}$$

is the angular frequency of the damped oscillations. There are three possible scenarios, depending on the relative values of γ and ω_0.

Case 1: Underdamping

When $\omega_0 > \gamma$, or equivalently, ω' is real and positive, the system is said to be underdamped. This is the case when the resistance is small. Charge oscillates (the cosine function) with an exponentially decaying amplitude $Q_0 e^{-\gamma t}$. However, the frequency of this damped oscillation is less than the undamped oscillation, $\omega'<\omega_0$. The qualitative behavior of the charge on the capacitor as a function of time is shown in Figure 11.8.1.
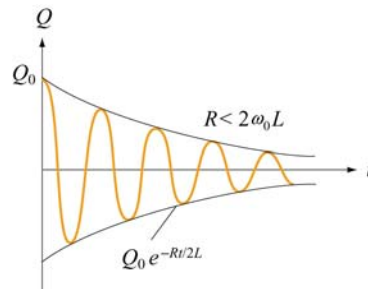
Figure 11.8.1 Underdamped oscillations

As an example, suppose the initial condition is $Q(t=0)=Q_0$. The phase is then $\phi=0$, and

$$Q(t)=Q_0 e^{-\gamma t}\cos\omega't \tag{11.8.5}$$

The corresponding current is

$$I(t)=-\frac{dQ}{dt}=Q_0\omega'e^{-\gamma t}\left[\sin\omega't+(\gamma/\omega')\cos\omega't\right] \tag{11.8.6}$$

For small R, the above expression may be approximated as

$$I(t)\approx\frac{Q_0}{\sqrt{LC}}e^{-\gamma t}\sin(\omega't+\delta) \tag{11.8.7}$$

where

$$\delta=\tan^{-1}\left(\frac{\gamma}{\omega'}\right) \tag{11.8.8}$$

The derivation is left to the readers as an exercise.

Case 2: Overdamping

In the overdamped case, $\omega_0<\gamma$, implying that ω' is imaginary. There is no oscillation in this case. By writing $\omega'=i\beta$, where $\beta=\sqrt{\gamma^2-\omega_0^2}$, one may show that the most general solution can be written as

$$Q(t)=Q_1 e^{-(\gamma+\beta)t}+Q_2 e^{-(\gamma-\beta)t} \tag{11.8.9}$$

where the constants Q_1 and Q_2 can be determined from the initial conditions.
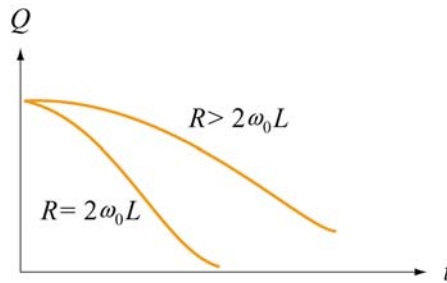
Figure 11.8.2 Overdamping and critical damping

Case 3: Critical damping

When the system is critically damped, $\omega_0 = \gamma$, $\omega' = 0$. Again there is no oscillation. The general solution is

$$Q(t) = (Q_1 + Q_2 t)e^{-\gamma t} \tag{11.8.10}$$

where Q_1 and Q_2 are constants which can be determined from the initial conditions. In this case one may show that the energy of the system decays most rapidly with time. The qualitative behavior of $Q(t)$ in overdamping and critical damping is depicted in Figure 11.8.2.

11.8.1 Quality Factor

When the resistance is small, the system is underdamped, and the charge oscillates with decaying amplitude $Q_0 e^{-\gamma t}$. The "quality" of this underdamped oscillation is measured by the so-called "quality factor," Q (not to be confused with charge.) The larger the value of Q, the less the damping and the higher the quality. Mathematically, Q is defined as

$$Q = \omega' \left(\frac{\text{energy stored}}{\text{average power dissipated}} \right) = \omega' \frac{U}{|dU/dt|} \tag{11.8.11}$$

Using Eq. (11.8.2), the electric energy stored in the capacitor is

$$U_E = \frac{Q(t)^2}{2C} = \frac{Q_0^{\ 2}}{2C} e^{-2\gamma t} \cos^2(\omega' t + \phi) \tag{11.8.12}$$

To obtain the magnetic energy, we approximate the current as

$$
\begin{aligned}
I(t) = -\frac{dQ}{dt} &= Q_0 \omega' e^{-\gamma t} \left[\sin(\omega' t + \phi) + \left(\frac{\gamma}{\omega'} \right) \cos(\omega' t + \phi) \right] \\
&\approx Q_0 \omega' e^{-\gamma t} \sin(\omega' t + \phi) \\
&\approx \frac{Q_0}{\sqrt{LC}} e^{-\gamma t} \sin(\omega' t + \phi)
\end{aligned}
\tag{11.8.13}
$$

assuming that $\omega' \square \ \gamma$ and $\omega'^2 \approx \omega_0^2 = 1/LC$. Thus, the magnetic energy stored in the inductor is given by

$$U_B = \frac{1}{2}LI^2 \approx \frac{LQ_0^{\ 2}}{2} \omega'^2 e^{-2\gamma t} \sin^2(\omega' t + \phi) \approx \frac{Q_0^{\ 2}}{2C} e^{-2\gamma t} \sin^2(\omega' t + \phi) \tag{11.8.14}$$

Adding up the two terms, the total energy of the system is

$$U = U_E + U_B \approx \frac{Q_0^{\ 2}}{2C}e^{-2\gamma t}\cos^2(\omega' t + \phi) + \frac{Q_0^{\ 2}}{2C}e^{-2\gamma t}\sin^2(\omega' t + \phi) = \left(\frac{Q_0^{\ 2}}{2C}\right)e^{-2\gamma t} \qquad (11.8.15)$$

Differentiating the expression with respect to t then yields the rate of change of energy:

$$\frac{dU}{dt} = -2\gamma\left(\frac{Q_0^{\ 2}}{2C}e^{-2\gamma t}\right) = -2\gamma U \qquad (11.8.16)$$

Thus, the quality factor becomes

$$\boxed{Q = \omega'\frac{U}{|dU/dt|} = \frac{\omega'}{2\gamma} = \frac{\omega' L}{R}} \qquad (11.8.17)$$

As expected, the smaller the value of R, the greater the value of Q, and therefore the higher the quality of oscillation.

11.9 Appendix 2: Stresses Transmitted by Magnetic Fields

  "…It appears therefore that the stress in the axis of a line of magnetic force is a tension, like that of a rope…"

J. C. Maxwell [1861].

In Chapter 9, we showed that the magnetic field due to an infinite sheet in the xy-plane carrying a surface current $\vec{\mathbf{K}} = K\hat{\mathbf{i}}$ is given by

$$\vec{\mathbf{B}} = \begin{cases} -\dfrac{\mu_0 K}{2}\hat{\mathbf{j}}, & z > 0 \\[2mm] \dfrac{\mu_0 K}{2}\hat{\mathbf{j}}, & z < 0 \end{cases} \qquad (11.9.1)$$

Now consider two sheets separated by a distance d carrying surface currents in the opposite directions, as shown in Figure 11.9.1.
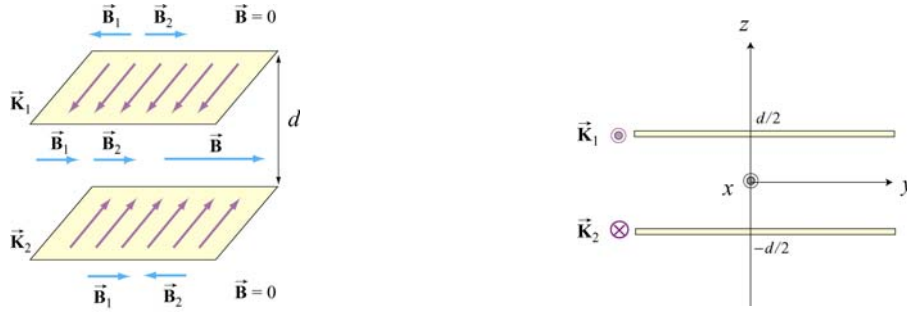
Figure 11.9.1 Magnetic field due to two sheets carrying surface current in the opposite directions

Using the superposition principle, we may show that the magnetic field is non-vanishing only in the region between the two sheets, and is given by

$$\vec{\mathbf{B}} = \mu_0 K \hat{\mathbf{j}}, \quad -d/2 < z < d/2 \tag{11.9.2}$$

Using Eq. (11.3.8), the magnetic energy stored in this system is

$$U_B = \frac{B^2}{2\mu_0}(Ad) = \frac{(\mu_0 K)^2}{2\mu_0}(Ad) = \frac{\mu_0}{2}K^2(Ad) \tag{11.9.3}$$

where A is the area of the plate. The corresponding magnetic energy density is

$$u_B = \frac{U_B}{Ad} = \frac{\mu_0}{2}K^2 \tag{11.9.4}$$

Now consider a small current-carrying element $Id\vec{\mathbf{s}}_1 = (K\Delta y)\Delta x\hat{\mathbf{i}}$ on the upper plate (Recall that K has dimensions of current/length). The force experienced by this element due to the magnetic field of the lower sheet is

$$d\vec{\mathbf{F}}_{21} = Id\vec{\mathbf{s}}_1 \times \vec{\mathbf{B}}_2 = (K\Delta y\,\Delta x\hat{\mathbf{i}}) \times \left(\frac{\mu_0}{2}K\,\hat{\mathbf{j}}\right) = \frac{\mu_0}{2}K^2(\Delta x\Delta y)\hat{\mathbf{k}} \tag{11.9.5}$$

The force points in the $+\hat{\mathbf{k}}$ direction and therefore is repulsive. This is expected since the currents flow in opposite directions. Since $d\vec{\mathbf{F}}_{21}$ is proportional to the area of the current element, we introduce force per unit area, $\vec{\mathbf{f}}_{21}$, and write

$$\vec{\mathbf{f}}_{21} = \vec{\mathbf{K}}_1 \times \vec{\mathbf{B}}_2 = \frac{\mu_0}{2}K^2\hat{\mathbf{k}} = u_B\hat{\mathbf{k}} \tag{11.9.6}$$

using Eq. (11.9.4). The magnitude of the force per unit area, f_{21}, is exactly equal to the magnetic energy density u_B. Physically, f_{21} may be interpreted as the magnetic pressure

$$f_{21} = P = u_B = \frac{B^2}{2\mu_0} \tag{11.9.7}$$

The repulsive force experienced by the sheets is shown in Figure 11.9.2
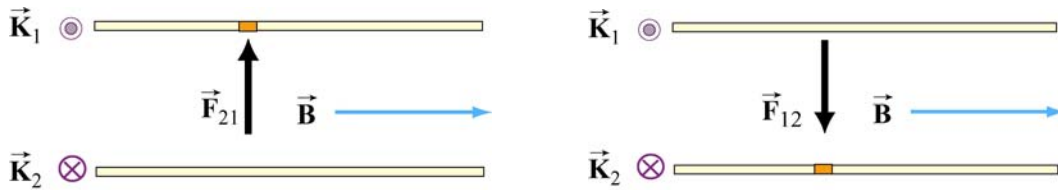

Figure 11.9.2 Magnetic pressure exerted on (a) the upper plate, and (b) the lower plate

Let's now consider a more general case of stress (pressure or tension) transmitted by fields. In Figure 11.9.3, we show an imaginary closed surface (an imaginary box) placed in a magnetic field. If we look at the face on the left side of this imaginary box, the field on that face is perpendicular to the outward normal to that face. Using the result illustrated in Figure 11.9.2, the field on that face transmits a pressure perpendicular to itself. In this case, this is a *push* to the *right*. Similarly, if we look at the face on the right side of this imaginary box, the field on that face is perpendicular to the outward normal to that face, the field on that face transmits a pressure perpendicular to itself. In this case, this is a *push* to the *left*.
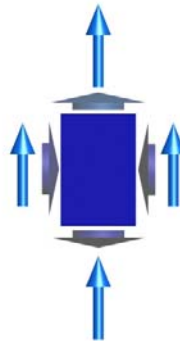
Figure 11.9.3 An imaginary box in a magnetic field (blue vectors). The short vectors indicate the directions of stresses transmitted by the field, either pressures (on the left or right faces of the box) or tensions (on the top and bottom faces of the box).

If we want to know the total electromagnetic force transmitted to the interior of this imaginary box in the left-right direction, we add these two transmitted stresses. If the electric or magnetic field is homogeneous, this total electromagnetic force transmitted to

the interior of the box in the left-right direction is a push to the left and an equal but opposite push to the right, and the transmitted force adds up to zero.

In contrast, if the right side of this imaginary box is sitting inside a long vertical solenoid, for which the magnetic field is vertical and constant, and the left side is sitting outside of that solenoid, where the magnetic field is zero, then there is a net push to the left, and we say that the magnetic field exerts a outward *pressure* on the walls of the solenoid. We can deduce this by simply looking at the magnetic field topology. At sufficiently high magnetic field, such forces will cause the walls of a solenoid to explode outward.

Similarly, if we look at the top face of the imaginary box in Figure 11.9.3, the field on that face is parallel to the outward normal to that face, and one may show that the field on that face transmits a tension along itself across that face. In this case, this is an *upward pull*, just as if we had attached a string under tension to that face, pulling upward. (The actual determination of the direction of the force requires an advance treatment using the Maxwell stress tensor.) On the other hand, if we look at the bottom face of this imaginary box, the field on that face is anti-parallel to the outward normal to that face, and Faraday would again have said that the field on that face transmits a tension along itself. In this case, this is a *downward pull*, just as if we had attached a string to that face, pulling downward. Note that this is a *pull* parallel to the outward surface normal, whether the field is into the surface or out of the surface, since the pressures or tensions are proportional to the squares of the field magnitudes.

If we want to know the total electromagnetic force transmitted to the interior of this imaginary box in the up-down direction, we add these two transmitted stresses. If the magnetic field is homogeneous, this total electromagnetic force transmitted to the interior of the box in the up-down direction is a pull upward plus an equal and opposite pull downward, and adds to zero.

The magnitude of these pressures and tensions on the various faces of the imaginary surface in Figure 11.9.3 is given by $B^2 / 2\mu_0$, as shown in Eq. (11.9.7). Our discussion may be summarized as follows:

Pressures and Tensions Transmitted by Magnetic Fields

Electromagnetic fields are mediators of the interactions between material objects. The fields transmit stresses through space. A magnetic field transmits a tension along itself and a pressure perpendicular to itself. The magnitude of the tension or pressure transmitted by a magnetic field is given by

$$P = u_B = \frac{1}{2\mu_o} B^2$$

11.9.1 A Charged Particle in a Time-Varying Magnetic Field Animation

As an example of the stresses transmitted by magnetic fields, consider a moving positive point charge at the origin in a rapidly changing time-dependent external field. This external field is uniform in space but varies in time according to the equation

$$\vec{B} = -B_0 \sin^4\left(\frac{2\pi t}{T}\right)\hat{\mathbf{k}} \qquad (11.9.8)$$

We assume that the variation of this field is so rapid that the charge moves only a negligible distance in one period T. Figure 11.9.4 shows two frames of an animation of the total magnetic field configuration for this situation. Figure 11.9.4(a) is at $t = 0$, when the vertical magnetic field is zero, and we see only the magnetic field of the moving charge (the charge is moving out of the page, so the field circulates clockwise). Frame 11.9.4(b) is at a quarter period later, when the vertically downward magnetic field is at a maximum. To the left of the charge, where the field of the charge is in the same direction as the external magnetic field (downward), the magnetic field is enhanced. To the right of the charge, where the field of the charge is opposite that of the external magnetic field, the magnetic field is reduced (and is zero at one point to the right of the charge).

(a)   (b)
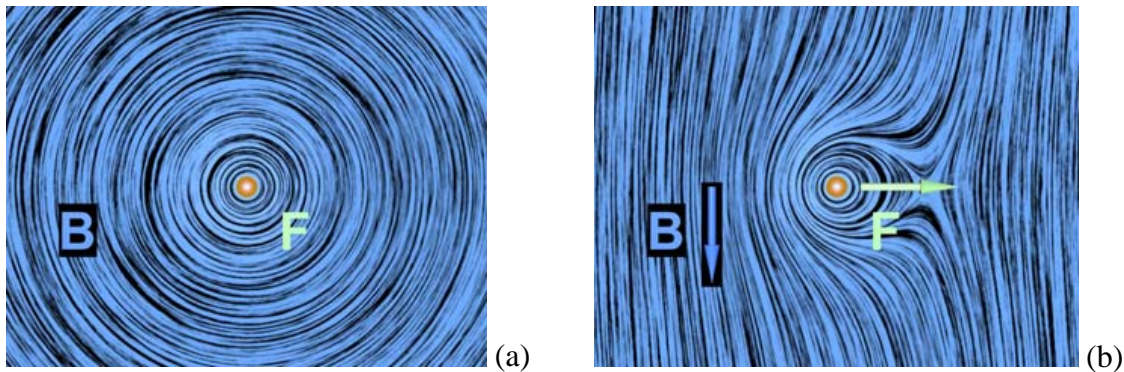
Figure 11.9.4 Two frames of an animation of the magnetic field around a positive charge moving out of the page in a time-changing magnetic field that points downward. The blue vector is the magnetic field and the white vector is the force on the point charge [(link)](link)

We interpret the field configuration in Figure 11.9.4(b) as indicating a net force to the right on the moving charge. This occurs because the pressure of the magnetic field is much higher on the left as compared to the right. Note that if the charge had been moving into the page instead of out of the page, the force would have been to the left, because the magnetic pressure would have been higher on the right. The animation of Figure 11.9.4 shows dramatically the inflow of energy into the neighborhood of the charge as the external magnetic field grows, with a resulting build-up of stress that transmits a sideways force to the moving positive charge.

We can estimate the magnitude of the force on the moving charge in Figure 11.9.4(b) as follows. At the time shown in Figure 11.9.4(b), the distance r_0 to the right of the charge at which the magnetic field of the charge is equal and opposite to the constant magnetic field is determined by

$$B_0 = \frac{\mu_0}{4\pi} \frac{qv}{r_0^2} \qquad (11.9.9)$$

The surface area of a sphere of this radius is $A = 4\pi r_0^2 = \mu_0 \, q \, v / B_0$. Now according to Eq. (11.9.7) the pressure (force per unit area) and/or tension transmitted across the surface of this sphere surrounding the charge is of the order of $P = B^2 / 2\mu_0$. Since the magnetic field on the surface of the sphere is of the order B_0, the total force transmitted by the field is of order

$$F = PA = \frac{B_0^2}{2\mu_0}(4\pi r_0^2) = \frac{B_0^2}{2\mu_0} \cdot \frac{\mu_0 qv}{B_0} \approx qvB_0 \qquad (11.9.10)$$

Of course this net force is a combination of a pressure pushing to the right on the left side of the sphere and a tension pulling to the right on the right side of the sphere. The exact expression for the force on a charge moving in a magnetic field is

$$\vec{\mathbf{F}}_B = q \, \vec{\mathbf{v}} \times \vec{\mathbf{B}} \qquad (11.9.11)$$

The rough estimate that we have just made demonstrates that the pressures and tensions transmitted across the surface of this sphere surrounding the moving charge are plausibly of the order $B^2 / 2\mu_0$. In addition, this argument gives us some insight in to why the magnetic force on a moving charge is transverse to the velocity of the charge and to the direction of the background field. This is because of the side of the charge on which the total magnetic pressure is the highest. It is this pressure that causes the deflection of the charge.

11.10 Problem-Solving Strategies

11.10.1   Calculating Self-Inductance

The self-inductance L of an inductor can be calculated using the following steps:

1. Assume a steady current I for the inductor, which may be a conducting loop, a solenoid, a toroid, or coaxial cables.

2. Choose an appropriate cross section S and compute the magnetic flux through S using

$$\Phi_B = \iint_S \vec{B} \cdot d\vec{A}$$

If the surface is bounded by N turns of wires, then the total magnetic flux through the surface would be $N\Phi_B$.

3. The inductance may be obtained as

$$L = \frac{N\Phi_B}{I}$$

11.10.2 Circuits containing inductors

Three types of single-loop circuits were examined in this chapter: RL, LC and RLC. To set up the differential equation for a circuit, we apply the Kirchhoff's loop and junction rules, as we did in Chpater 7 for the RC circuits. For circuits that contain inductors, the corresponding modified Kirchhoff's rule is schematically shown below.
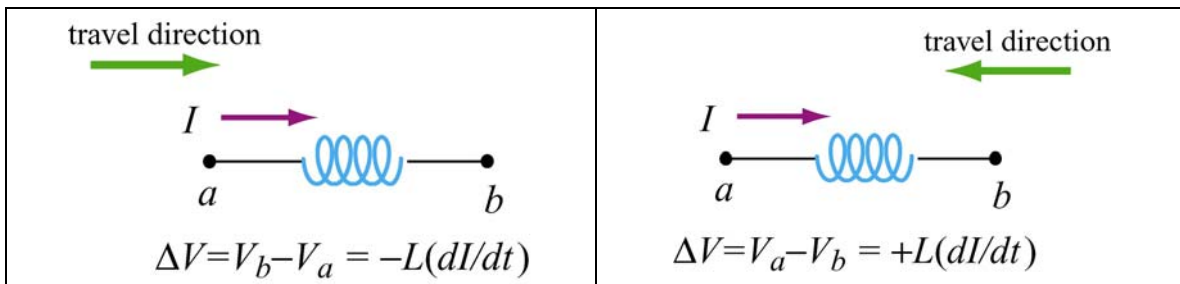
Note that the "potential difference" across the inductor is proportional to dI/dt, the rate of change of current. The situation simplifies if we are only interested in the long-term behavior of the circuit where the currents have reached their steady state and $dI/dt = 0$. In this limit, the inductor acts as a short circuit and can simply be replaced by an ideal wire.

11.11 Solved Problems

11.11.1 Energy stored in a toroid

A toroid consists of N turns and has a rectangular cross section, with inner radius a, outer radius b and height h (see Figure 11.2.3). Find the total magnetic energy stored in the toroid.

Solution:

In Example 11.3 we showed that the self-inductance of a toroid is

$$L = \frac{N\Phi_B}{I} = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)$$

Thus, the magnetic energy stored in the toroid is simply

$$U_B = \frac{1}{2} LI^2 = \frac{\mu_0 N^2 I^2 h}{4\pi} \ln\left(\frac{b}{a}\right) \qquad (11.11.1)$$

Alternatively, the energy may be interpreted as being stored in the magnetic field. For a toroid, the magnetic field is (see Chapter 9)

$$B = \frac{\mu_0 NI}{2\pi r}$$

and the corresponding magnetic energy density is

$$u_B = \frac{1}{2} \frac{B^2}{\mu_0} = \frac{\mu_0 N^2 I^2}{8\pi^2 r^2} \qquad (11.11.2)$$

The total energy stored in the magnetic field can be found by integrating over the volume. We choose the differential volume element to be a cylinder with radius r, width dr and height h, so that $dV = 2\pi r h\, dr$. This leads to

$$U_B = \int u_B\, dV = \int_a^b \left(\frac{\mu_0 N^2 I^2}{8\pi^2 r^2}\right) 2\pi r h\, dr = \frac{\mu_0 N^2 I^2 h}{4\pi} \ln\left(\frac{b}{a}\right) \qquad (11.11.3)$$

Thus, both methods yield the same result.

11.11.2 Magnetic Energy Density

A wire of nonmagnetic material with radius R and length l carries a current I which is uniformly distributed over its cross-section. What is the magnetic energy inside the wire?

Solution:

Applying Ampere's law, the magnetic field at distance $r \le R$ can be obtained as:

$$B(2\pi r) = \mu_0 J(\pi r^2) = \mu_0 \left(\frac{I}{\pi R^2} \right)(\pi r^2) \qquad (11.11.4)$$

or

$$B = \frac{\mu_0 I r}{2\pi R^2} \qquad (11.11.5)$$

Since the magnetic energy density (energy per unit volume) is given by

$$u_B = \frac{B^2}{2\mu_0} \qquad (11.11.6)$$

the total magnetic energy stored in the system becomes

$$U_B = \int_0^R \frac{B^2}{2\mu_0}(2\pi r l\, dr) = \frac{\mu_0 I^2 l}{4\pi R^4} \int_0^R r^3\, dr = \frac{\mu_0 I^2 l}{4\pi R^4}\left(\frac{R^4}{4} \right) = \frac{\mu_0 I^2 l}{16\pi} \qquad (11.11.7)$$

11.11.3 Mutual Inductance

An infinite straight wire carrying current I is placed to the left of a rectangular loop of wire with width w and length l, as shown in the Figure 11.11.3. Determine the mutual inductance of the system.
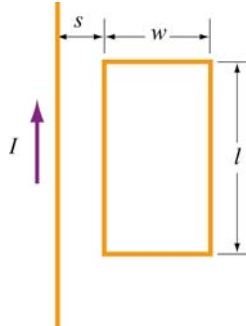

Figure 11.11.3 Rectangular loop placed near long straight current-carrying wire

Solution:

To calculate the mutual inductance M, we first need to know the magnetic flux through the rectangular loop. The magnetic field at a distance r away from the straight wire is $B = \mu_0 I / 2\pi r$, using Ampere's law. The total magnetic flux Φ_B through the loop can be obtained by summing over contributions from all differential area elements $dA = l\, dr$:

$$\Phi_B = \int d\Phi_B = \int \vec{B} \cdot d\vec{A} = \frac{\mu_0 IL}{2\pi} \int_s^{s+w} \frac{dr}{r} = \frac{\mu_0 Il}{2\pi} \ln\left(\frac{s+w}{s}\right) \qquad (11.11.8)$$

Thus, the mutual inductance is

$$\boxed{M = \frac{\Phi_B}{I} = \frac{\mu_0 l}{2\pi} \ln\left(\frac{s+w}{s}\right)} \qquad (11.11.9)$$

11.11.4 *RL* Circuit

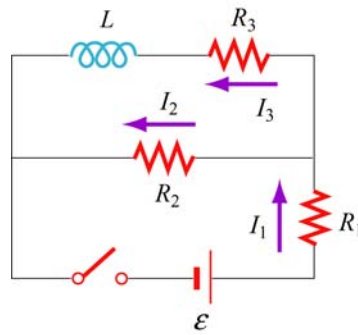Consider the circuit shown in Figure 11.11.4 below.

Figure 11.11.4 *RL* circuit

Determine the current through each resistor

(a) immediately after the switch is closed.

(b) a long time after the switch is closed.

Suppose the switch is reopened a long time after it's been closed. What is each current

(c) immediately after it is opened?

(d) after a long time?

Solution:

(a) Immediately after the switch is closed, the current through the inductor is zero because the self-induced emf prevents the current from rising abruptly. Therefore, $I_3 = 0$. Since $I_1 = I_2 + I_3$, we have $I_1 = I_2$
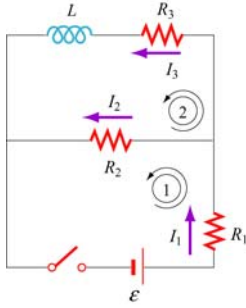
Figure 11.11.5

Applying Kirchhoff's rules to the first loop shown in Figure 11.11.5 yields

$$I_1 = I_2 = \frac{\varepsilon}{R_1 + R_2} \tag{11.11.10}$$

(b) After the switch has been closed for a long time, there is no induced emf in the inductor and the currents will be constant. Kirchhoff's loop rule gives

$$\varepsilon - I_1 R_1 - I_2 R_2 = 0 \tag{11.11.11}$$

for the first loop, and

$$I_2 R_2 - I_3 R_3 = 0 \tag{11.11.12}$$

for the second. Combining the two equations with the junction rule $I_1 = I_2 + I_3$, we obtain

$$I_1 = \frac{(R_2 + R_3)\varepsilon}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

$$I_2 = \frac{R_3 \varepsilon}{R_1 R_2 + R_1 R_3 + R_2 R_3} \tag{11.11.13}$$

$$I_3 = \frac{R_2 \varepsilon}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

(c) Immediately after the switch is opened, the current through R_1 is zero, i.e., $I_1 = 0$. This implies that $I_2 + I_3 = 0$. On the other hand, loop 2 now forms a decaying RL circuit and I_3 starts to decrease. Thus,

$$I_3 = -I_2 = \frac{R_2 \varepsilon}{R_1 R_2 + R_1 R_3 + R_2 R_3} \tag{11.11.14}$$

(d) A long time after the switch has been closed, all currents will be zero. That is, $I_1 = I_2 = I_3 = 0$.

11.11.5 *RL* Circuit

In the circuit shown in Figure 11.11.6, suppose the circuit is initially open. At time $t = 0$ it is thrown closed. What is the current in the inductor at a later time t?
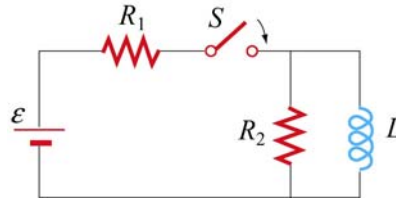

Figure 11.11.6 *RL* circuit

Solution:

Let the currents through R_1, R_2 and L be I_1, I_2 and I, respectively, as shown in Figure 11.11.7.

From Kirchhoff's junction rule, we have $I_1 = I_2 + I$. Similarly, applying Kirchhoff's loop rule to the left loop yields

$$\varepsilon - (I + I_2)R_1 - I_2 R_2 = 0 \tag{11.11.15}$$

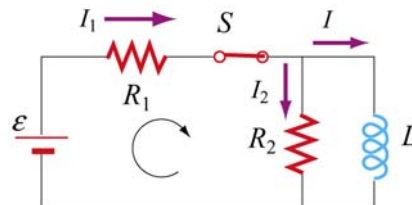**Figure 11.11.7**

Similarly, for the outer loop, the modified Kirchhoff's loop rule gives

$$\varepsilon - (I + I_2)R_1 = L\frac{dI}{dt} \tag{11.11.16}$$

The two equations can be combined to yield

$$I_2 R_2 = L\frac{dI}{dt} \quad \Rightarrow \quad I_2 = \frac{L}{R_2}\frac{dI}{dt} \tag{11.11.17}$$

Substituting into Eq. (11.11.15) the expression obtained above for I_2, we have

$$\varepsilon - \left(I + \frac{L}{R_2} \frac{dI}{dt} \right) R_1 - L \frac{dI}{dt} = \varepsilon - IR_1 - \left(\frac{R_1 + R_2}{R_2} \right) L \frac{dI}{dt} = 0 \qquad (11.11.18)$$

Dividing the equation by $(R_1 + R_2)/R_2$ leads to

$$\varepsilon' - IR' - L \frac{dI}{dt} = 0 \qquad (11.11.19)$$

where

$$R' = \frac{R_1 R_2}{R_1 + R_2}, \qquad \varepsilon' = \frac{R_2 \varepsilon}{R_1 + R_2} \qquad (11.11.20)$$

The differential equation can be solved and the solution is given by

$$I(t) = \frac{\varepsilon'}{R'} \left(1 - e^{-R't/L} \right) \qquad (11.11.21)$$

Since

$$\frac{\varepsilon'}{R'} = \frac{\varepsilon R_2 / (R_1 + R_2)}{R_1 R_2 / (R_1 + R_2)} = \frac{\varepsilon}{R_1} \qquad (11.11.22)$$

the current through the inductor may be rewritten as

$$I(t) = \frac{\varepsilon}{R_1} \left(1 - e^{-R't/L} \right) = \frac{\varepsilon}{R_1} \left(1 - e^{-t/\tau} \right) \qquad (11.11.23)$$

where $\tau = L/R'$ is the time constant.

11.11.6 *LC* Circuit

Consider the circuit shown in Figure 11.11.8. Suppose the switch which has been connected to point *a* for a long time is suddenly thrown to *b* at $t = 0$.
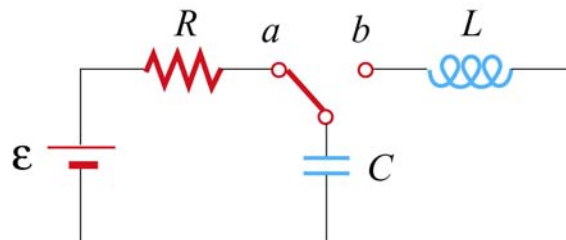

Figure 11.11.8 *LC* circuit

Find the following quantities:

(a) the frequency of oscillation of the LC circuit.

(b) the maximum charge that appears on the capacitor.

(c) the maximum current in the inductor.

(d) the total energy the circuit possesses at any time t.

Solution:

(a) The (angular) frequency of oscillation of the LC circuit is given by $\omega = 2\pi f = 1/\sqrt{LC}$. Therefore, the frequency is

$$f = \frac{1}{2\pi\sqrt{LC}} \qquad (11.11.24)$$

(b) The maximum charge stored in the capacitor before the switch is thrown to b is

$$Q = C\varepsilon \qquad (11.11.25)$$

(c) The energy stored in the capacitor before the switch is thrown is

$$U_E = \frac{1}{2}C\varepsilon^2 \qquad (11.11.26)$$

On the other hand, the magnetic energy stored in the inductor is

$$U_B = \frac{1}{2}LI^2 \qquad (11.11.27)$$

Thus, when the current is at its maximum, all the energy originally stored in the capacitor is now in the inductor:

$$\frac{1}{2}C\varepsilon^2 = \frac{1}{2}LI_0^2 \qquad (11.11.28)$$

This implies a maximum current

$$I_0 = \varepsilon\sqrt{\frac{C}{L}} \qquad (11.11.29)$$

(d) At any time, the total energy in the circuit would be equal to the initial energy that the capacitance stored, that is

$$U = U_E + U_B = \frac{1}{2}C\varepsilon^2 \qquad (11.11.30)$$

11.12 Conceptual Questions

1. How would you shape a wire of fixed length to obtain the greatest and the smallest inductance?

2. If the wire of a tightly wound solenoid is unwound and made into another tightly wound solenoid with a diameter 3 times that of the original one, by what factor does the inductance change?

3. What analogies can you draw between an ideal solenoid and a parallel-plate capacitor?

4. In the RL circuit show in Figure 11.12.1, can the self-induced emf ever be greater than the emf supplied by the battery?
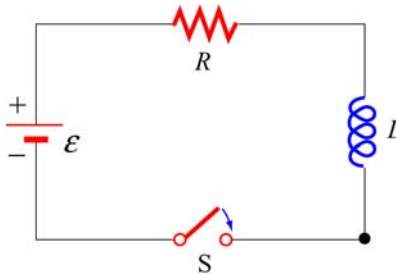

Figure 11.12.1

5. The magnetic energy density $u_B = B^2/2\mu_0$ may also be interpreted as the magnetic pressure. Using the magnetic pressure concept, explain the attractive (repulsive) force between two coils carrying currents in the same (opposite) direction.

6. Explain why the LC oscillation continues even after the capacitor has been completely discharged.

7. Explain physically why the time constant $\tau = L/R$ in an RL circuit is proportional to L and inversely proportional to R.

11.13 Additional Problems

11.13.1 Solenoid

A solenoid with a length of 30 cm, a radius of 1.0 cm and 500 turns carries a steady current $I = 2.0$ A .

(a) What is the magnetic field at the center of the solenoid along the axis of symmetry?

(b) Find the magnetic flux through the solenoid, assuming the magnetic field to be uniform.

(c) What is the self-inductance of the solenoid?

(d) What is the induced emf in the solenoid if the rate of change of current is $dI/dt = 100$ A/s ?

11.13.2 Self-Inductance

Suppose you try to wind a wire of length d and radius a into an inductor which has the shape of a cylinder with a circular cross section of radius r. The windings are tight without wires overlapping. Show that the self-inductance of this inductor is

$$L = \mu_0 \frac{rd}{4a}$$

11.13.3 Coupled Inductors

(a) If two inductors with inductances L_1 and L_2 are connected in series, show that the equivalent inductance is

$$L_{eq} = L_1 + L_2 \pm 2M$$

where M is their mutual inductance. How is the sign chosen for M ? Under what condition can M be ignored?

(b) If the inductors are instead connected in parallel, show that, if their mutual inductance can be ignored, the equivalent inductance is given by

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2}$$

How would you take the effect of *M* into consideration?

11.13.4 *RL* Circuit

The *LR* circuit shown in Figure 11.13.1 contains a resistor R_1 and an inductance L in series with a battery of emf ε_0. The switch S is initially closed. At $t = 0$, the switch S is opened, so that an additional very large resistance R_2 (with $R_2 \gg R_1$) is now in series with the other elements.
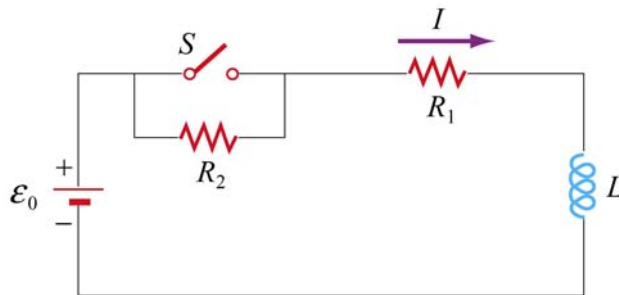

Figure 11.13.1 *RL* circuit

(a) If the switch has been *closed* for a long time before $t = 0$, what is the steady current I_0 in the circuit?

(b) While this current I_0 is flowing, at time $t = 0$, the switch S is opened. Write the differential equation for $I(t)$ that describes the behavior of the circuit at times $t \geq 0$. Solve this equation (by integration) for $I(t)$ under the approximation that $\varepsilon_0 = 0$. (Assume that the battery emf is negligible compared to the total emf around the circuit for times just after the switch is opened.) Express your answer in terms of the initial current I_0, and R_1, R_2, and L.

(c) Using your results from (b), find the value of the total emf around the circuit (which from Faraday's law is $-L\,dI/dt$) just after the switch is opened. Is your assumption in (b) that ε_0 could be ignored for times just after the switch is opened OK?

(d) What is the magnitude of the potential drop across the resistor R_2 at times $t > 0$, just after the switch is opened? Express your answers in terms of ε_0, R_1, and R_2. How does the potential drop across R_2 just after $t = 0$ compare to the battery emf ε_0, if $R_2 = 100R_1$?

11.13.5 *RL* Circuit

In the circuit shown in Figure 11.13.2, $\varepsilon = 100$ V, $R_1 = 10 \ \Omega$, $R_2 = 20 \ \Omega$, $R_3 = 30 \ \Omega$, and the inductance L in the right loop of the circuit is 2.0 H. The inductance in the left loop of the circuit is zero.
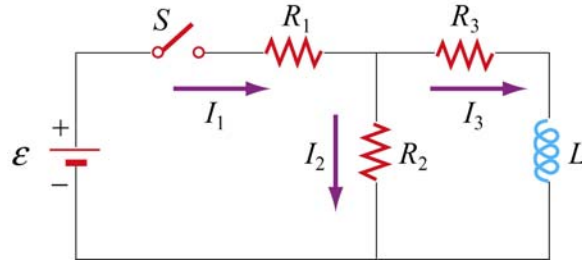

Figure 11.13.2 *RL* circuit

(a) Find I_1 and I_2 immediately after switch S is closed.

(b) Find I_1 and I_2 a long time later. What is the energy stored in the inductor a long time later?

(c) A long, long time later, switch S is opened again. Find I_1 and I_2 immediately after switch S is opened again.

(d) Find I_1 and I_2 a long time after switch S is opened. How much energy is dissipated in resistors R_2 and R_3 between the time immediately after switch S is opened again, and a long time after that?

(e) Give a crude estimate of what "a long time" is in this problem.

11.13.6 Inductance of a Solenoid With and Without Iron Core

(a) A long solenoid consists of N turns of wire, has length l, and cross-sectional area A. Show that the self-inductance can be written as $L = \mu_0 N^2 A / l$. Note that L increases as N^2, and has dimensions of μ_0 times a length (as must always be true).

(b) A solenoid has a length of 126 cm and a diameter of 5.45 cm, with 1870 windings. What is its inductance if its interior is vacuum?

(c) If we now fill the interior with iron with an effective permeability constant $\kappa_m = 968$, what is its inductance?

(d) Suppose we connect this iron core inductor up in series with a battery and resistor, and that the total resistance in the circuit, including that of the battery and inductor, is 10 Ω. How long does it take after the circuit is established for the current to reach 50% of its final value? [Ans: (b) 8.1 mH; (c) 7.88 H; (d) 0.55 s].

11.13.7 *RLC* Circuit

An *RLC* circuit with battery is set up as shown in Figure 11.13.3. There is no current flowing in the circuit until time $t = 0$, when the switch S_1 is closed.

Figure 11.13.3
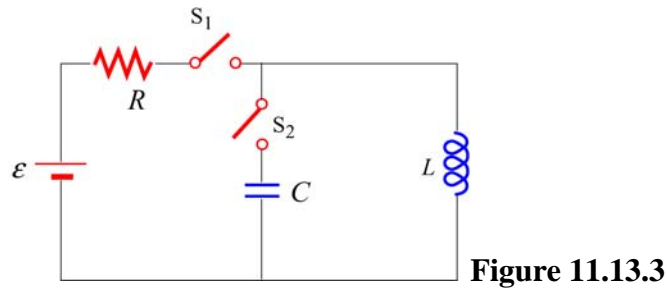
(a) What is the current I in the circuit at a time $t > 0$ after the switch S_1 is closed?

(b) What is the current I in the circuit a very long time ($t >> L/R$) after the switch S_1 is closed?

(c) How much energy is stored in the magnetic field of the solenoid a very long time $(t >> L/R)$ after the switch is closed?

For the next two questions, assume that a very long time $(t >> L/R)$ after the switch S_1 was closed, the voltage source is disconnected from the circuit by opening the switch S_1 and that the solenoid is simultaneously connected to a capacitor by closing the switch S_2. Assume there is negligible resistance in this new circuit.

Figure 11.13.4
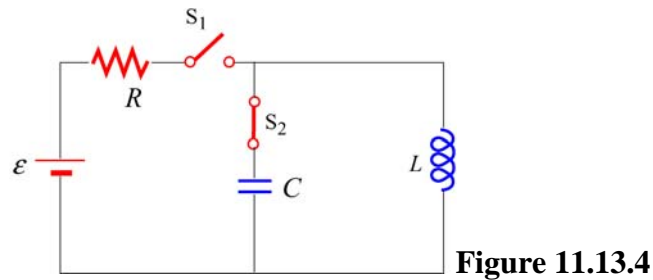
(d) What is the maximum amount of charge that will appear on the capacitor, in terms of the quantities given?

(e) How long will it take for the capacitor to first reach a maximal charge after the switch S_2 has been closed?

11.13.8 Spinning Cylinder

Two concentric, conducting cylindrical shells are charged up by moving $+Q$ from the outer to the inner conductor, so that the inner conductor has a charge of $+Q$ spread uniformly over its area, and the outer conductor is left with $-Q$ uniformly distributed. The radius of the inner conductor is a; the radius of the outer conductor is b; the length of both is l; and you may assume that $l \gg a, b$.

(a) What is the electric field for $r < a$, $a < r < b$, and $r > b$? Give both magnitude and direction.

(b) What is the total amount of energy in the electric field? (Hint: you may use a variety of ways to calculate this, such as using the energy density, or the capacitance, or the potential as a function of Q. It never hurts to check by doing it two different ways.)

(c) If the cylinders are now both spun counterclockwise (looking down the z axis) at the same angular velocity ω (so that the period of revolution is $T = 2\pi / \omega$), what is the total current (magnitude and sign) carried by each of the cylinders? Give your answer in terms of ω and the quantities from the first paragraph, and consider a current to be positive if it is in the same direction as ω.

(d) What is the magnetic field created when the cylinders are spinning at angular velocity ω? You should give magnitude and direction of $\vec{\mathbf{B}}$ in each of the three regions: $r < a$, $a < r < b$, $r > b$. (Hint: it's easiest to do this by calculating $\vec{\mathbf{B}}$ from each cylinder independently and then getting the net magnetic field as the vector sum.)

(e) What is the total energy in the magnetic field when the cylinders are spinning at ω?

11.13.9 Spinning Loop

A circular, conducting loop of radius a has resistance R and is spun about its diameter which lies along the y-axis, perpendicular to an external, uniform magnetic field $\vec{\mathbf{B}} = B\hat{\mathbf{k}}$. The angle between the normal to the loop and the magnetic field is θ, where $\theta = \omega t$. You may ignore the self-inductance of the loop.

(a) What is the magnetic flux through the loop as a function of time?

(b) What is the emf induced around the loop as a function of time?

(c) What is the current flowing in the loop as a function of time?

(d) At an instant that the normal to the loop aligns with the x-axis, the top of the loop lies on the $+z$ axis. At this moment is the current in this piece of loop in the $+\hat{\mathbf{j}}$ or $-\hat{\mathbf{j}}$ direction?

(e) What is the magnitude of the new magnetic field B_{ind} (as a function of time) created at the center of the loop by the induced current?

(f) Estimate the self-inductance L of the loop, using approximation that the magnetic field B_{ind} is uniform over the area of the loop and has the value calculated in part (e).

(g) At what angular speed ω will the maximum induced magnetic field B_{ind} equal the external field B (therefore thoroughly contradicting the assumption of negligible self-inductance that went into the original calculation of B_{ind})? Express your answer in terms of R and L.

8.02SC Physics II: Electricity and Magnetism
Fall 2010