Module 6: Sections 3.1 through 3.4
Module 7 and 8: Sections 3.5 through 3.9[1]

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[1] These notes are excerpted "Introduction to Electricity and Magnetism" by Sen-Ben Liao, Peter Dourmashkin, and John Belcher, Copyright 2004, ISBN 0-536-81207-1.

Electric Potential

3.1 Potential and Potential Energy

In the introductory mechanics course, we have seen that gravitational force from the Earth on a particle of mass *m* located at a distance *r* from Earth's center has an inverse-square form:

$$\vec{\mathbf{F}}_g = -G\frac{Mm}{r^2}\hat{\mathbf{r}} \tag{3.1.1}$$

where $G = 6.67\times10^{-11}\ \mathrm{N\cdot m^2/kg^2}$ is the gravitational constant and $\hat{\mathbf{r}}$ is a unit vector pointing radially outward. The Earth is assumed to be a uniform sphere of mass *M*. The corresponding gravitational field $\vec{\mathbf{g}}$, defined as the gravitational force per unit mass, is given by

$$\vec{\mathbf{g}} = \frac{\vec{\mathbf{F}}_g}{m} = -\frac{GM}{r^2}\hat{\mathbf{r}} \tag{3.1.2}$$

Notice that $\vec{\mathbf{g}}$ only depends on *M*, the mass which creates the field, and *r*, the distance from *M*.

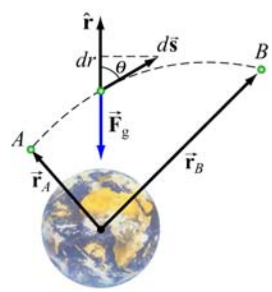


Figure 3.1.1

Consider moving a particle of mass m under the influence of gravity (Figure 3.1.1). The work done by gravity in moving m from *A* to *B* is

$$W_g = \int \vec{\mathbf{F}}_g \cdot d\,\vec{\mathbf{s}} = \int_{r_A}^{r_B}\left(-\frac{GMm}{r^2}\right)dr = \left[\frac{GMm}{r}\right]_{r_A}^{r_B} = GMm\left(\frac{1}{r_B}-\frac{1}{r_A}\right) \tag{3.1.3}$$

The result shows that W_g is independent of the path taken; it depends only on the endpoints *A* and *B*. It is important to draw distinction between W_g, the work done by the field and W_{ext}, the work done by an external agent such as you. They simply differ by a negative sign: $W_g = -W_{\text{ext}}$.

Near Earth's surface, the gravitational field $\vec{\mathbf{g}}$ is approximately constant, with a magnitude $g = GM/r_E^2 \approx 9.8\,\text{m/s}^2$, where r_E is the radius of Earth. The work done by gravity in moving an object from height y_A to y_B (Figure 3.1.2) is

$$W_g = \int \vec{\mathbf{F}}_g \cdot d\vec{\mathbf{s}} = \int_A^B mg\cos\theta\, ds = -\int_A^B mg\cos\phi\, ds = -\int_{y_A}^{y_B} mg\, dy = -mg(y_B - y_A) \qquad (3.1.4)$$

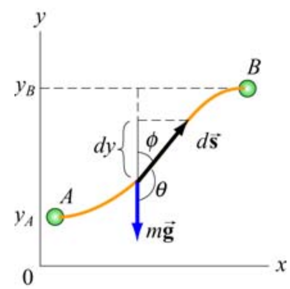


Figure 3.1.2 Moving a mass *m* from *A* to *B*.

The result again is independent of the path, and is only a function of the change in vertical height $y_B - y_A$.

In the examples above, if the path forms a closed loop, so that the object moves around and then returns to where it starts off, the net work done by the gravitational field would be zero, and we say that the gravitational force is conservative. More generally, a force $\vec{\mathbf{F}}$ is said to be *conservative* if its line integral around a closed loop vanishes:

$$\boxed{\oint \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = 0} \qquad (3.1.5)$$

When dealing with a conservative force, it is often convenient to introduce the concept of potential energy *U*. The change in potential energy associated with a conservative force $\vec{\mathbf{F}}$ acting on an object as it moves from *A* to *B* is defined as:

$$\boxed{\Delta U = U_B - U_A = -\int_A^B \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = -W} \qquad (3.1.6)$$

where W is the work done by the force on the object. In the case of gravity, $W = W_g$ and from Eq. (3.1.3), the potential energy can be written as

$$U_g = -\frac{GMm}{r} + U_0 \qquad (3.1.7)$$

where U_0 is an arbitrary constant which depends on a reference point. It is often convenient to choose a reference point where U_0 is equal to zero. In the gravitational case, we choose infinity to be the reference point, with $U_0(r=\infty)=0$. Since U_g depends on the reference point chosen, it is only the potential energy difference ΔU_g that has physical importance. Near Earth's surface where the gravitational field $\vec{\mathbf{g}}$ is approximately constant, as an object moves from the ground to a height h, the change in potential energy is $\Delta U_g = +mgh$, and the work done by gravity is $W_g = -mgh$.

A concept which is closely related to potential energy is "potential." From ΔU, the gravitational potential can be obtained as

$$\boxed{\Delta V_g = \frac{\Delta U_g}{m} = -\int_A^B (\vec{\mathbf{F}}_g / m)\cdot d\,\vec{\mathbf{s}} = -\int_A^B \vec{\mathbf{g}}\cdot d\,\vec{\mathbf{s}}} \tag{3.1.8}$$

Physically ΔV_g represents the negative of the work done per unit mass by gravity to move a particle from A to B.

Our treatment of electrostatics is remarkably similar to gravitation. The electrostatic force $\vec{\mathbf{F}}_e$ given by Coulomb's law also has an inverse-square form. In addition, it is also conservative. In the presence of an electric field $\vec{\mathbf{E}}$, in analogy to the gravitational field $\vec{\mathbf{g}}$, we define the electric potential difference between two points A and B as

$$\boxed{\Delta V = -\int_A^B (\vec{\mathbf{F}}_e / q_0)\cdot d\,\vec{\mathbf{s}} = -\int_A^B \vec{\mathbf{E}}\cdot d\,\vec{\mathbf{s}}} \tag{3.1.9}$$

where q_0 is a test charge. The potential difference ΔV represents the amount of work done per unit charge to move a test charge q_0 from point A to B, without changing its kinetic energy. Again, electric potential should not be confused with electric potential energy. The two quantities are related by

$$\Delta U = q_0 \Delta V \tag{3.1.10}$$

The SI unit of electric potential is volt (V):

$$1\,\text{volt} = 1\ \text{joule/coulomb}\ \ (1\ \text{V}=1\ \text{J/C}) \tag{3.1.11}$$

When dealing with systems at the atomic or molecular scale, a joule (J) often turns out to be too large as an energy unit. A more useful scale is electron volt (eV), which is defined as the energy an electron acquires (or loses) when moving through a potential difference of one volt:

$$1\,\text{eV} = (1.6\times10^{-19}\,\text{C})(1\,\text{V}) = 1.6\times10^{-19}\,\text{J} \tag{3.1.12}$$

3.2 Electric Potential in a Uniform Field

Consider a charge $+q$ moving in the direction of a uniform electric field $\vec{\mathbf{E}} = E_0(-\hat{\mathbf{j}})$, as shown in Figure 3.2.1(a).

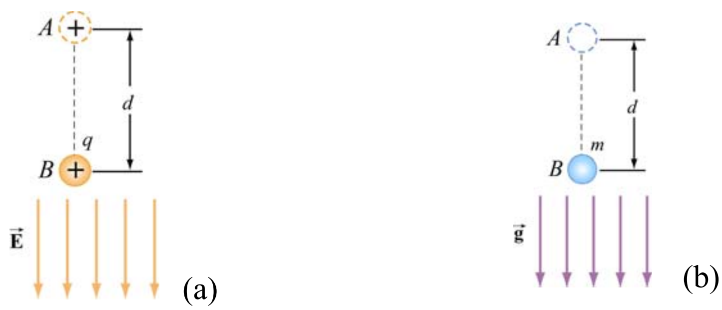


Figure 3.2.1 (a) A charge q which moves in the direction of a constant electric field $\vec{\mathbf{E}}$. (b) A mass m that moves in the direction of a constant gravitational field $\vec{\mathbf{g}}$.

Since the path taken is parallel to $\vec{\mathbf{E}}$, the potential difference between points A and B is given by

$$\Delta V = V_B - V_A = -\int_A^B \vec{\mathbf{E}}\cdot d\,\vec{\mathbf{s}} = -E_0\int_A^B ds = -E_0 d < 0 \tag{3.2.1}$$

implying that point B is at a lower potential compared to A. In fact, electric field lines always point from higher potential to lower. The change in potential energy is $\Delta U = U_B - U_A = -qE_0 d$. Since $q > 0$, we have $\Delta U < 0$, which implies that the potential energy of a positive charge decreases as it moves along the direction of the electric field. The corresponding gravitational analogy, depicted in Figure 3.2.1(b), is that a mass m loses potential energy ($\Delta U = -mgd$) as it moves in the direction of the gravitational field $\vec{\mathbf{g}}$.

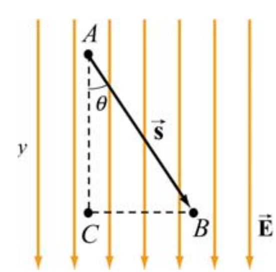


Figure 3.2.2 Potential difference due to a uniform electric field

What happens if the path from A to B is not parallel to $\vec{\mathbf{E}}$, but instead at an angle θ, as shown in Figure 3.2.2? In that case, the potential difference becomes

$$\Delta V = V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{s}} = -\vec{\mathbf{E}} \cdot \vec{\mathbf{s}} = -E_0 s\cos\theta = -E_0 y \tag{3.2.2}$$

Note that y increase downward in Figure 3.2.2. Here we see once more that moving along the direction of the electric field $\vec{\mathbf{E}}$ leads to a lower electric potential. What would the change in potential be if the path were $A \to C \to B$? In this case, the potential difference consists of two contributions, one for each segment of the path:

$$\Delta V = \Delta V_{CA} + \Delta V_{BC} \tag{3.2.3}$$

When moving from A to C, the change in potential is $\Delta V_{CA} = -E_0 y$. On the other hand, when going from C to B, $\Delta V_{BC} = 0$ since the path is perpendicular to the direction of $\vec{\mathbf{E}}$. Thus, the same result is obtained irrespective of the path taken, consistent with the fact that $\vec{\mathbf{E}}$ is conservative.

Notice that for the path $A \to C \to B$, work is done by the field only along the segment AC which is parallel to the field lines. Points B and C are at the same electric potential, i.e., $V_B = V_C$. Since $\Delta U = q\Delta V$, this means that no work is required in moving a charge from B to C. In fact, all points along the straight line connecting B and C are on the same "equipotential line." A more complete discussion of equipotential will be given in Section 3.5.

3.3 Electric Potential due to Point Charges

Next, let's compute the potential difference between two points A and B due to a charge $+Q$. The electric field produced by Q is $\vec{\mathbf{E}} = (Q/4\pi\varepsilon_0 r^2)\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector pointing toward the field point.

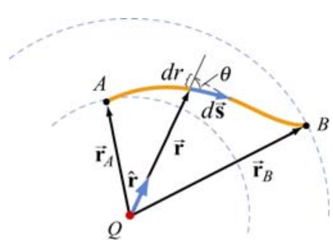


Figure 3.3.1 Potential difference between two points due to a point charge Q.

From Figure 3.3.1, we see that $\hat{\mathbf{r}} \cdot d\vec{\mathbf{s}} = ds\cos\theta = dr$, which gives

$$\Delta V = V_B - V_A = -\int_A^B \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}\cdot d\vec{\mathbf{s}} = -\int_A^B \frac{Q}{4\pi\varepsilon_0 r^2} dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right) \tag{3.3.1}$$

Once again, the potential difference ΔV depends only on the endpoints, independent of the choice of path taken.

As in the case of gravity, only the difference in electrical potential is physically meaningful, and one may choose a reference point and set the potential there to be zero. In practice, it is often convenient to choose the reference point to be at infinity, so that the electric potential at a point *P* becomes

$$V_P = -\int_\infty^P \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}} \tag{3.3.2}$$

With this reference, the electric potential at a distance *r* away from a point charge *Q* becomes

$$\boxed{V(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}} \tag{3.3.3}$$

When more than one point charge is present, by applying the superposition principle, the total electric potential is simply the sum of potentials due to individual charges:

$$V(r) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i} = k_e \sum_i \frac{q_i}{r_i} \tag{3.3.4}$$

A summary of comparison between gravitation and electrostatics is tabulated below:

Gravitation	Electrostatics
Mass m	Charge q
Gravitational force $\vec{\mathbf{F}}_g = -G\dfrac{Mm}{r^2}\hat{\mathbf{r}}$	Coulomb force $\vec{\mathbf{F}}_e = k_e\dfrac{Qq}{r^2}\hat{\mathbf{r}}$
Gravitational field $\vec{\mathbf{g}} = \vec{\mathbf{F}}_g / m$	Electric field $\vec{\mathbf{E}} = \vec{\mathbf{F}}_e / q$
Potential energy change $\Delta U = -\int_A^B \vec{\mathbf{F}}_g \cdot d\vec{\mathbf{s}}$	Potential energy change $\Delta U = -\int_A^B \vec{\mathbf{F}}_e \cdot d\vec{\mathbf{s}}$
Gravitational potential $V_g = -\int_A^B \vec{\mathbf{g}}\cdot d\vec{\mathbf{s}}$	Electric Potential $V = -\int_A^B \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}}$
For a source *M*: $V_g = -\dfrac{GM}{r}$	For a source *Q*: $V = k_e\dfrac{Q}{r}$

$\lvert \Delta U_g \rvert = mgd$ (constant $\vec{\mathbf{g}}$)	$\lvert \Delta U \rvert = qEd$ (constant $\vec{\mathbf{E}}$)

3.3.1 Potential Energy in a System of Charges

If a system of charges is assembled by an external agent, then $\Delta U = -W = +W_{\text{ext}}$. That is, the change in potential energy of the system is the work that must be put in by an external agent to assemble the configuration. A simple example is lifting a mass m through a height h. The work done by an external agent˜ you, is $+mgh$ (The gravitational field does work $-mgh$). The charges are brought in from infinity without acceleration i.e. they are at rest at the end of the process. Let's start with just two charges q_1 and q_2. Let the potential due to q_1 at a point P be V_1 (Figure 3.3.2).

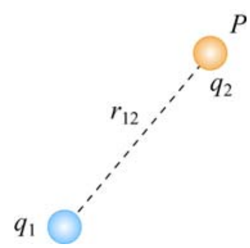


Figure 3.3.2 Two point charges separated by a distance r_{12}.

The work W_2 done by an agent in bringing the second charge q_2 from infinity to P is then $W_2 = q_2 V_1$. (No work is required to set up the first charge and $W_1 = 0$). Since $V_1 = q_1 / 4\pi\varepsilon_0 r_{12}$, where r_{12} is the distance measured from q_1 to P, we have

$$U_{12} = W_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}} \tag{3.3.5}$$

If q_1 and q_2 have the same sign, positive work must be done to overcome the electrostatic repulsion and the potential energy of the system is positive, $U_{12} > 0$. On the other hand, if the signs are opposite, then $U_{12} < 0$ due to the attractive force between the charges.

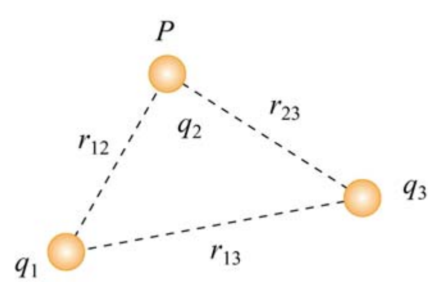


Figure 3.3.3 A system of three point charges.

To add a third charge q_3 to the system (Figure 3.3.3), the work required is

$$W_3 = q_3\left(V_1 + V_2\right) = \frac{q_3}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right) \tag{3.3.6}$$

The potential energy of this configuration is then

$$U = W_2 + W_3 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right) = U_{12} + U_{13} + U_{23} \tag{3.3.7}$$

The equation shows that the total potential energy is simply the sum of the contributions from distinct pairs. Generalizing to a system of N charges, we have

$$U = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j>i}}^{N}\frac{q_i q_j}{r_{ij}} \tag{3.3.8}$$

where the constraint $j > i$ is placed to avoid double counting each pair. Alternatively, one may count each pair twice and divide the result by 2. This leads to

$$U = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{N} q_i\left(\frac{1}{4\pi\varepsilon_0}\sum_{\substack{j=1 \\ j\neq i}}^{N}\frac{q_j}{r_{ij}}\right) = \frac{1}{2}\sum_{i=1}^{N} q_i V(r_i) \tag{3.3.9}$$

where $V(r_i)$, the quantity in the parenthesis, is the potential at $\vec{\mathbf{r}}_i$ (location of q_i) due to all the other charges.

3.4 Continuous Charge Distribution

If the charge distribution is continuous, the potential at a point P can be found by summing over the contributions from individual differential elements of charge dq.

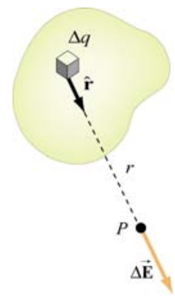


Figure 3.4.1 Continuous charge distribution

Consider the charge distribution shown in Figure 3.4.1. Taking infinity as our reference point with zero potential, the electric potential at P due to dq is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} \tag{3.4.1}$$

Summing over contributions from all differential elements, we have

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r}} \tag{3.4.2}$$

3.5 Deriving Electric Field from the Electric Potential

In Eq. (3.1.9) we established the relation between $\vec{\mathbf{E}}$ and V. If we consider two points which are separated by a small distance $d\vec{\mathbf{s}}$, the following differential form is obtained:

$$dV = -\vec{\mathbf{E}}\cdot d\vec{\mathbf{s}} \tag{3.5.1}$$

In Cartesian coordinates, $\vec{\mathbf{E}} = E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}} + E_z\hat{\mathbf{k}}$ and $d\vec{\mathbf{s}} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$, we have

$$dV = \left(E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}} + E_z\hat{\mathbf{k}}\right)\cdot\left(dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}\right) = E_x dx + E_y dy + E_z dz \tag{3.5.2}$$

which implies

$$\boxed{E_x = -\frac{\partial V}{\partial x},\quad E_y = -\frac{\partial V}{\partial y},\quad E_z = -\frac{\partial V}{\partial z}} \tag{3.5.3}$$

By introducing a differential quantity called the "del (gradient) operator"

$$\nabla \equiv \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}} \tag{3.5.4}$$

the electric field can be written as

$$\vec{\mathbf{E}} = E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}} + E_z\hat{\mathbf{k}} = -\left(\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right) = -\left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right)V = -\nabla V$$

$$\vec{\mathbf{E}} = -\nabla V \tag{3.5.5}$$

Notice that ∇ operates on a scalar quantity (electric potential) and results in a vector quantity (electric field). Mathematically, we can think of $\vec{\mathbf{E}}$ as the negative of the *gradient* of the electric potential V. Physically, the negative sign implies that if V increases as a positive charge moves along some direction, say *x*, with $\partial V/\partial x > 0$, then there is a non-vanishing component of $\vec{\mathbf{E}}$ in the opposite direction $(-E_x \neq 0)$. In the case of gravity, if the gravitational potential increases when a mass is lifted a distance *h*, the gravitational force must be downward.

If the charge distribution possesses spherical symmetry, then the resulting electric field is a function of the radial distance *r*, i.e., $\vec{\mathbf{E}} = E_r\hat{\mathbf{r}}$. In this case, $dV = -E_r\, dr$. If $V(r)$ is known, then $\vec{\mathbf{E}}$ may be obtained as

$$\boxed{\vec{\mathbf{E}} = E_r\hat{\mathbf{r}} = -\left(\frac{dV}{dr}\right)\hat{\mathbf{r}}} \tag{3.5.6}$$

For example, the electric potential due to a point charge q is $V(r) = q/4\pi\varepsilon_0 r$. Using the above formula, the electric field is simply $\vec{\mathbf{E}} = (q/4\pi\varepsilon_0 r^2)\hat{\mathbf{r}}$.

3.5.1 Gradient and Equipotentials

Suppose a system in two dimensions has an electric potential $V(x,y)$. The curves characterized by constant $V(x,y)$ are called equipotential curves. Examples of equipotential curves are depicted in Figure 3.5.1 below.

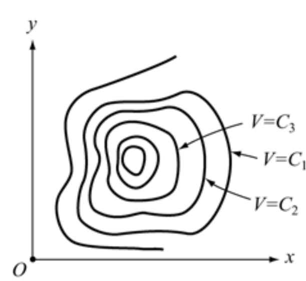


Figure 3.5.1 Equipotential curves

In three dimensions we have equipotential surfaces and they are described by $V(x,y,z)$ =constant. Since $\vec{\mathbf{E}} = -\nabla V$, we can show that the direction of $\vec{\mathbf{E}}$ is always perpendicular to the equipotential through the point. Below we give a proof in two dimensions. Generalization to three dimensions is straightforward.

Proof:

Referring to Figure 3.5.2, let the potential at a point $P(x,y)$ be $V(x,y)$. How much is V changed at a neighboring point $P(x+dx, y+dy)$? Let the difference be written as

$$\begin{aligned} dV &= V(x+dx, y+dy) - V(x,y) \\ &= \left[V(x,y) + \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \cdots \right] - V(x,y) \approx \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy \end{aligned} \tag{3.5.7}$$

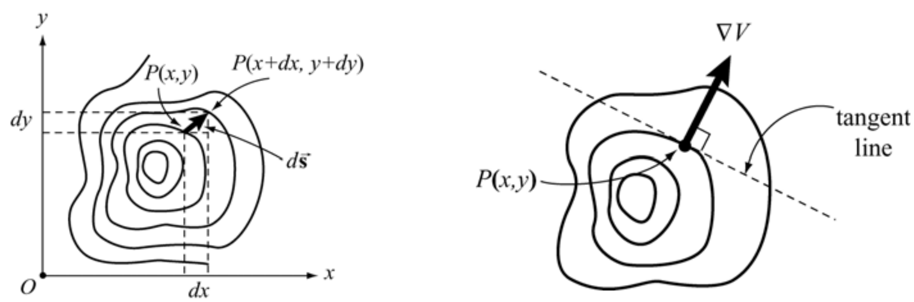


Figure 3.5.2 Change in V when moving from one equipotential curve to another

With the displacement vector given by $d\vec{\mathbf{s}} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}$, we can rewrite dV as

$$dV = \left(\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} \right) \cdot \left(dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} \right) = (\nabla V) \cdot d\mathbf{s} = -\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} \tag{3.5.8}$$

If the displacement $d\vec{\mathbf{s}}$ is along the tangent to the equipotential curve through $P(x,y)$, then $dV = 0$ because V is constant everywhere on the curve. This implies that $\vec{\mathbf{E}} \perp d\vec{\mathbf{s}}$ along the equipotential curve. That is, $\vec{\mathbf{E}}$ is perpendicular to the equipotential. In Figure 3.5.3 we illustrate some examples of equipotential curves. In three dimensions they become equipotential surfaces. From Eq. (3.5.8), we also see that the change in potential dV attains a maximum when the gradient ∇V is parallel to $d\vec{\mathbf{s}}$:

$$\max\left(\frac{dV}{ds} \right) = \left| \nabla V \right| \tag{3.5.9}$$

Physically, this means that ∇V always points in the direction of maximum rate of change of V with respect to the displacement s.

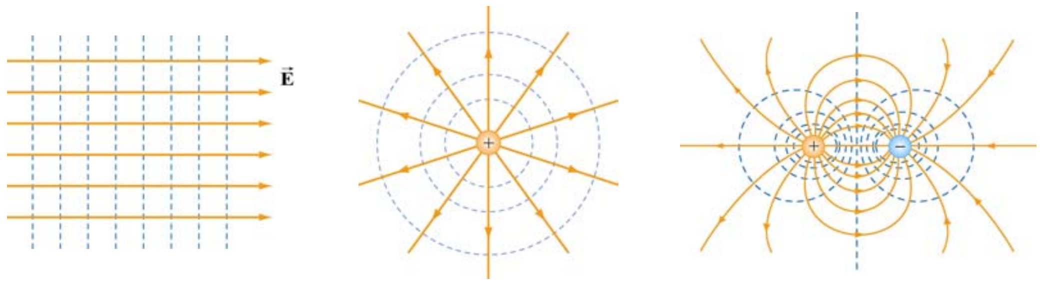


Figure 3.5.3 Equipotential curves and electric field lines for (a) a constant $\vec{\mathbf{E}}$ field, (b) a point charge, and (c) an electric dipole.

The properties of equipotential surfaces can be summarized as follows:

(i) The electric field lines are perpendicular to the equipotentials and point from higher to lower potentials.

(ii) By symmetry, the equipotential surfaces produced by a point charge form a family of concentric spheres, and for constant electric field, a family of planes perpendicular to the field lines.

(iii) The tangential component of the electric field along the equipotential surface is zero, otherwise non-vanishing work would be done to move a charge from one point on the surface to the other.

(iv) No work is required to move a particle along an equipotential surface.

A useful analogy for equipotential curves is a topographic map (Figure 3.5.4). Each contour line on the map represents a fixed elevation above sea level. Mathematically it is expressed as $z = f(x, y) = \text{constant}$. Since the gravitational potential near the surface of Earth is $V_g = gz$, these curves correspond to gravitational equipotentials.

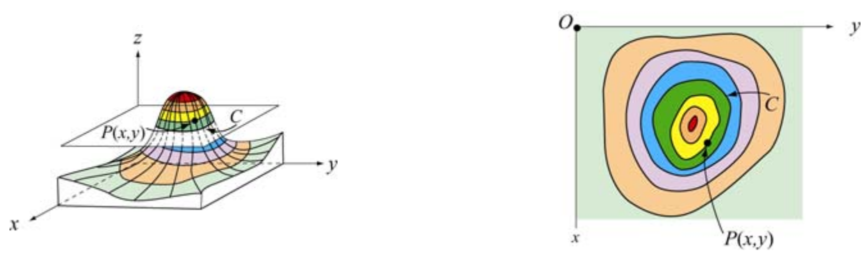


Figure 3.5.4 A topographic map

Example 3.1: Uniformly Charged Rod

Consider a non-conducting rod of length ℓ having a uniform charge density λ. Find the electric potential at P, a perpendicular distance y above the midpoint of the rod.

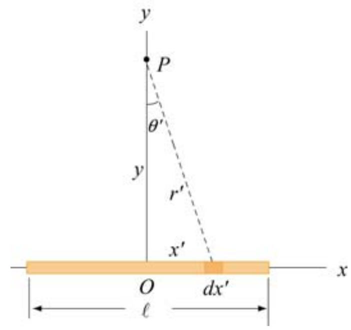


Figure 3.5.5 A non-conducting rod of length ℓ and uniform charge density λ.

Solution:

Consider a differential element of length dx' which carries a charge $dq = \lambda\, dx'$, as shown in Figure 3.5.5. The source element is located at $(x', 0)$, while the field point P is located on the y-axis at $(0, y)$. The distance from dx' to P is $r = (x'^2 + y^2)^{1/2}$. Its contribution to the potential is given by

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx'}{(x'^2 + y^2)^{1/2}}$$

Taking V to be zero at infinity, the total potential due to the entire rod is

$$\begin{aligned} V &= \frac{\lambda}{4\pi\varepsilon_0}\int_{-\ell/2}^{\ell/2}\frac{dx'}{\sqrt{x'^2+y^2}} = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[x' + \sqrt{x'^2+y^2}\right]\Bigg|_{-\ell/2}^{\ell/2} \\ &= \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(\ell/2) + \sqrt{(\ell/2)^2 + y^2}}{-(\ell/2) + \sqrt{(\ell/2)^2 + y^2}}\right] \end{aligned} \tag{3.5.10}$$

where we have used the integration formula

$$\int \frac{dx'}{\sqrt{x'^2+y^2}} = \ln\left(x' + \sqrt{x'^2+y^2}\right)$$

A plot of $V(y)/V_0$, where $V_0 = \lambda/4\pi\varepsilon_0$, as a function of y/ℓ is shown in Figure 3.5.6

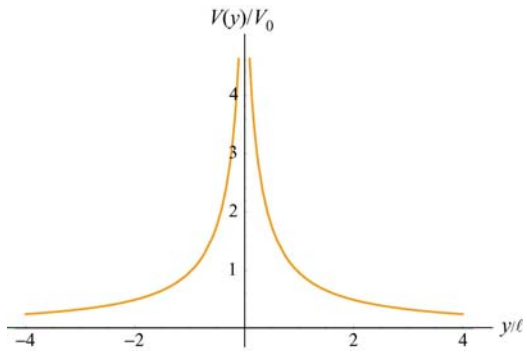


Figure 3.5.6 Electric potential along the axis that passes through the midpoint of a non-conducting rod.

In the limit ℓ ? y, the potential becomes

$$\begin{aligned}V &= \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(\ell/2)+\ell/2\sqrt{1+(2y/\ell)^2}}{-(\ell/2)+\ell/2\sqrt{1+(2y/\ell)^2}}\right] = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{1+\sqrt{1+(2y/\ell)^2}}{-1+\sqrt{1+(2y/\ell)^2}}\right] \\ &\approx \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{2}{2y^2/\ell^2}\right) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{\ell^2}{y^2}\right) \\ &= \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{\ell}{y}\right)\end{aligned} \tag{3.5.11}$$

The corresponding electric field can be obtained as

$$\boxed{E_y = -\frac{\partial V}{\partial y} = \frac{\lambda}{2\pi\varepsilon_0 y}\frac{\ell/2}{\sqrt{(\ell/2)^2+y^2}}}$$

in complete agreement with the result obtained in Eq. (2.10.9).

Example 3.2: Uniformly Charged Ring

Consider a uniformly charged ring of radius R and charge density λ (Figure 3.5.7). What is the electric potential at a distance z from the central axis?

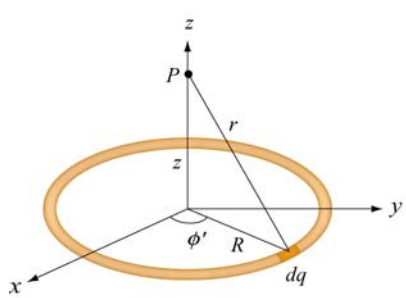


Figure 3.5.7 A non-conducting ring of radius *R* with uniform charge density λ.

Solution:

Consider a small differential element $d\ell = R\,d\phi'$ on the ring. The element carries a charge $dq = \lambda\,d\ell = \lambda R\,d\phi'$, and its contribution to the electric potential at *P* is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R\,d\phi'}{\sqrt{R^2+z^2}}$$

The electric potential at *P* due to the entire ring is

$$V = \int dV = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R}{\sqrt{R^2+z^2}}\oint d\phi' = \frac{1}{4\pi\varepsilon_0}\frac{2\pi\lambda R}{\sqrt{R^2+z^2}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{R^2+z^2}} \qquad (3.5.12)$$

where we have substituted $Q = 2\pi R\lambda$ for the total charge on the ring. In the limit $z \gg R$, the potential approaches its "point-charge" limit:

$$V \approx \frac{1}{4\pi\varepsilon_0}\frac{Q}{z}$$

From Eq. (3.5.12), the *z*-component of the electric field may be obtained as

$$E_z = -\frac{\partial V}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{R^2+z^2}}\right) = \frac{1}{4\pi\varepsilon_0}\frac{Qz}{(R^2+z^2)^{3/2}} \qquad (3.5.13)$$

in agreement with Eq. (2.10.14).

Example 3.3: Uniformly Charged Disk

Consider a uniformly charged disk of radius R and charge density σ lying in the *xy*-plane. What is the electric potential at a distance z from the central axis?

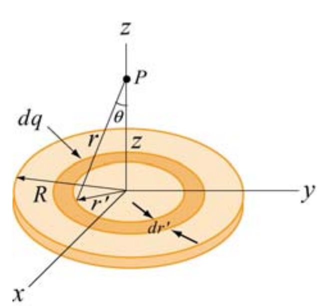


Figure 3.4.3 A non-conducting disk of radius *R* and uniform charge density σ

Solution:

Consider a circular ring of radius r' and width dr'. The charge on the ring is $dq' = \sigma dA' = \sigma(2\pi r' dr')$. The field point *P* is located along the z-axis a distance z from the plane of the disk. From the figure, we also see that the distance from a point on the ring to *P* is $r = (r'^2 + z^2)^{1/2}$. Therefore, the contribution to the electric potential at *P* is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma(2\pi r' dr')}{\sqrt{r'^2 + z^2}}$$

By summing over all the rings that make up the disk, we have

$$V = \frac{\sigma}{4\pi\varepsilon_0}\int_0^R \frac{2\pi r' dr'}{\sqrt{r'^2 + z^2}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{r'^2 + z^2}\right]\Bigg|_0^R = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2 + z^2} - |z|\right] \qquad (3.5.14)$$

In the limit $|z| \gg R$,

$$\sqrt{R^2 + z^2} = |z|\left(1 + \frac{R^2}{z^2}\right)^{1/2} = |z|\left(1 + \frac{R^2}{2z^2} + \cdots\right),$$

and the potential simplifies to the point-charge limit:

$$V \approx \frac{\sigma}{2\varepsilon_0}\cdot\frac{R^2}{2|z|} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma(\pi R^2)}{|z|} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|z|}$$

As expected, at large distance, the potential due to a non-conducting charged disk is the same as that of a point charge *Q*. A comparison of the electric potentials of the disk and a point charge is shown in Figure 3.4.4.

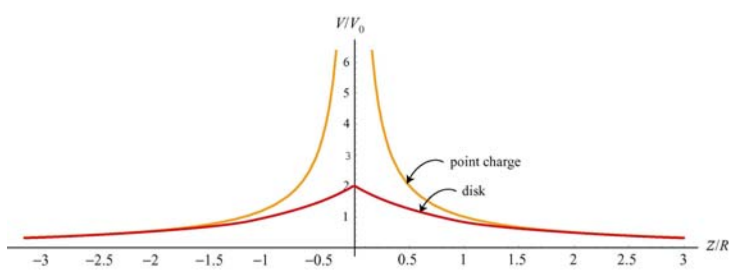


Figure 3.4.4 Comparison of the electric potentials of a non-conducting disk and a point charge. The electric potential is measured in terms of $V_0 = Q/4\pi\varepsilon_0 R$.

Note that the electric potential at the center of the disk ($z = 0$) is finite, and its value is

$$V_c = \frac{\sigma R}{2\varepsilon_0} = \frac{Q}{\pi R^2} \cdot \frac{R}{2\varepsilon_0} = \frac{1}{4\pi\varepsilon_0} \frac{2Q}{R} = 2V_0 \tag{3.5.15}$$

This is the amount of work that needs to be done to bring a unit charge from infinity and place it at the center of the disk.

The corresponding electric field at P can be obtained as:

$$E_z = -\frac{\partial V}{\partial z} = \frac{\sigma}{2\varepsilon_0}\left[\frac{z}{|z|} - \frac{z}{\sqrt{R^2+z^2}}\right] \tag{3.5.16}$$

which agrees with Eq. (2.10.18). In the limit R ? z, the above equation becomes $E_z = \sigma / 2\varepsilon_0$, which is the electric field for an infinitely large non-conducting sheet.

Example 3.4: Calculating Electric Field from Electric Potential

Suppose the electric potential due to a certain charge distribution can be written in Cartesian Coordinates as

$$V(x,y,z) = Ax^2y^2 + Bxyz$$

where A, B and C are constants. What is the associated electric field?

Solution:

The electric field can be found by using Eq. (3.5.3):

$$E_x = -\frac{\partial V}{\partial x} = -2Axy^2 - Byz$$
$$E_y = -\frac{\partial V}{\partial y} = -2Ax^2y - Bxz$$
$$E_z = -\frac{\partial V}{\partial z} = -Bxy$$

Therefore, the electric field is $\vec{\mathbf{E}} = (-2Axy^2 - Byz)\,\hat{\mathbf{i}} - (2Ax^2y + Bxz)\,\hat{\mathbf{j}} - Bxy\,\hat{\mathbf{k}}$.

3.6 Summary

- A force $\vec{\mathbf{F}}$ is **conservative** if the line integral of the force around a closed loop vanishes:

$$\oint \vec{\mathbf{F}} \cdot d\,\vec{\mathbf{s}} = 0$$

- The change in potential energy associated with a conservative force $\vec{\mathbf{F}}$ acting on an object as it moves from *A* to *B* is

$$\Delta U = U_B - U_A = -\int_A^B \vec{\mathbf{F}} \cdot d\,\vec{\mathbf{s}}$$

- The **electric potential difference** ΔV between points *A* and *B* in an electric field $\vec{\mathbf{E}}$ is given by

$$\Delta V = V_B - V_A = \frac{\Delta U}{q_0} = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$$

  The quantity represents the amount of work done per unit charge to move a test charge q_0 from point *A* to *B*, without changing its kinetic energy.

- The electric potential due to a point charge Q at a distance *r* away from the charge is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

  For a collection of charges, using the superposition principle, the electric potential is

$$V = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{Q_i}{r_i}$$

- The **potential energy** associated with two point charges q_1 and q_2 separated by a distance r_{12} is

$$U = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$$

- From the electric potential V, the electric field may be obtained by taking the **gradient** of V:

$$\vec{\mathbf{E}} = -\nabla V$$

In Cartesian coordinates, the components may be written as

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}$$

- The electric potential due to a continuous charge distribution is

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r}$$

3.7 Problem-Solving Strategy: Calculating Electric Potential

In this chapter, we showed how electric potential can be calculated for both the discrete and continuous charge distributions. Unlike electric field, electric potential is a scalar quantity. For the discrete distribution, we apply the superposition principle and sum over individual contributions:

$$V = k_e \sum_i \frac{q_i}{r_i}$$

For the continuous distribution, we must evaluate the integral

$$V = k_e \int \frac{dq}{r}$$

In analogy to the case of computing the electric field, we use the following steps to complete the integration:

(1) Start with $dV = k_e \dfrac{dq}{r}$.

(2) Rewrite the charge element dq as

$$dq = \begin{cases} \lambda\, dl & \text{(length)} \\ \sigma\, dA & \text{(area)} \\ \rho\, dV & \text{(volume)} \end{cases}$$

depending on whether the charge is distributed over a length, an area, or a volume.

(3) Substitute dq into the expression for dV.

(4) Specify an appropriate coordinate system and express the differential element (*dl*, *dA* or dV) and r in terms of the coordinates (see Table 2.1.)

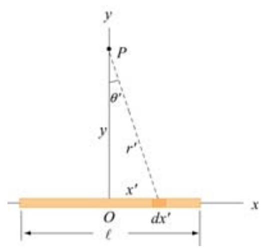
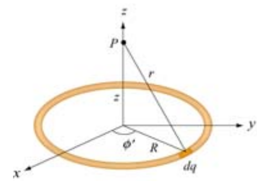
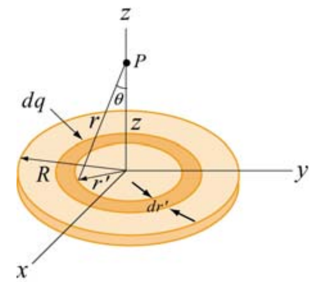
(5) Rewrite dV in terms of the integration variable.

(6) Complete the integration to obtain *V*.

Using the result obtained for V, one may calculate the electric field by $\vec{\mathbf{E}} = -\nabla V$. Furthermore, the accuracy of the result can be readily checked by choosing a point *P* which lies sufficiently far away from the charge distribution. In this limit, if the charge distribution is of finite extent, the field should behave as if the distribution were a point charge, and falls off as $1/r^2$.

Below we illustrate how the above methodologies can be employed to compute the electric potential for a line of charge, a ring of charge and a uniformly charged disk.

3.8 Solved Problems

	Charged Rod	Charged Ring	Charged disk
Figure			
(2) Express dq in terms of charge density	$dq = \lambda\, dx'$	$dq = \lambda\, dl$	$dq = \sigma\, dA$
(3) Substitute dq into expression for dV	$dV = k_e \dfrac{\lambda\, dx'}{r}$	$dV = k_e \dfrac{\lambda\, dl}{r}$	$dV = k_e \dfrac{\sigma dA}{r}$
(4) Rewrite r and the differential element in terms of the appropriate coordinates	dx' $r = \sqrt{x'^2 + y^2}$	$dl = R\, d\phi'$ $r = \sqrt{R^2 + z^2}$	$dA = 2\pi r'\, dr'$ $r = \sqrt{r'^2 + z^2}$
(5) Rewrite dV	$dV = k_e \dfrac{\lambda\, dx'}{(x'^2 + y^2)^{1/2}}$	$dV = k_e \dfrac{\lambda R\, d\phi'}{(R^2 + z^2)^{1/2}}$	$dV = k_e \dfrac{2\pi\sigma r'\, dr'}{(r'^2 + z^2)^{1/2}}$
(6) Integrate to get V	$V = \dfrac{\lambda}{4\pi\varepsilon_0} \int_{-\ell/2}^{\ell/2} \dfrac{dx'}{\sqrt{x'^2 + y^2}}$ $= \dfrac{\lambda}{4\pi\varepsilon_0} \ln\left[\dfrac{(\ell/2) + \sqrt{(\ell/2)^2 + y^2}}{-(\ell/2) + \sqrt{(\ell/2)^2 + y^2}}\right]$	$V = k_e \dfrac{R\lambda}{(R^2 + z^2)^{1/2}} \oint d\phi'$ $= k_e \dfrac{(2\pi R\lambda)}{\sqrt{R^2 + z^2}}$ $= k_e \dfrac{Q}{\sqrt{R^2 + z^2}}$	$V = k_e 2\pi\sigma \int_0^R \dfrac{r'\, dr'}{(r'^2 + z^2)^{1/2}}$ $= 2k_e \pi\sigma \left(\sqrt{z^2 + R^2} - \|z\|\right)$ $= \dfrac{2k_e Q}{R^2} \left(\sqrt{z^2 + R^2} - \|z\|\right)$
Derive E from V	$E_y = -\dfrac{\partial V}{\partial y}$ $= \dfrac{\lambda}{2\pi\varepsilon_0 y} \dfrac{\ell/2}{\sqrt{(\ell/2)^2 + y^2}}$	$E_z = -\dfrac{\partial V}{\partial z} = \dfrac{k_e Q z}{(R^2 + z^2)^{3/2}}$	$E_z = -\dfrac{\partial V}{\partial z} = \dfrac{2k_e Q}{R^2}\left(\dfrac{z}{\|z\|} - \dfrac{z}{\sqrt{z^2 + R^2}}\right)$
Point-charge limit for E	$E_y \approx \dfrac{k_e Q}{y^2} \quad y \gg \ell$	$E_z \approx \dfrac{k_e Q}{z^2} \quad z \gg R$	$E_z \approx \dfrac{k_e Q}{z^2} \quad z \gg R$

3.8.1 Electric Potential Due to a System of Two Charges

Consider a system of two charges shown in Figure 3.8.1.

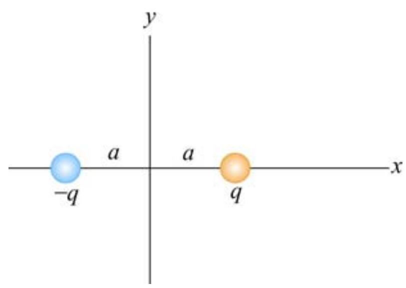


Figure 3.8.1 Electric dipole

Find the electric potential at an arbitrary point on the x axis and make a plot.

Solution:

The electric potential can be found by the superposition principle. At a point on the x axis, we have

$$V(x)=\frac{1}{4\pi\varepsilon_0}\frac{q}{|x-a|}+\frac{1}{4\pi\varepsilon_0}\frac{(-q)}{|x+a|}=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|x-a|}-\frac{1}{|x+a|}\right]$$

The above expression may be rewritten as

$$\frac{V(x)}{V_0}=\frac{1}{|x/a-1|}-\frac{1}{|x/a+1|}$$

where $V_0=q/4\pi\varepsilon_0 a$. The plot of the dimensionless electric potential as a function of x/a. is depicted in Figure 3.8.2.

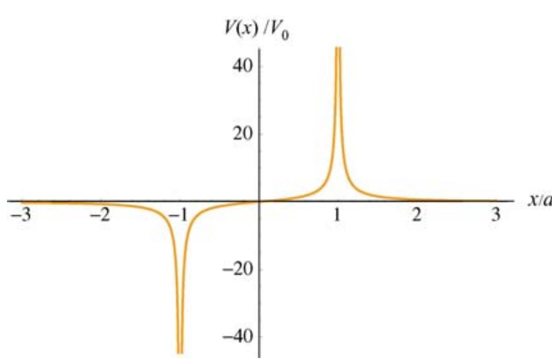


Figure 3.8.2

As can be seen from the graph, $V(x)$ diverges at $x/a=\pm 1$, where the charges are located.

3.8.2 Electric Dipole Potential

Consider an electric dipole along the *y*-axis, as shown in the Figure 3.8.3. Find the electric potential V at a point *P* in the *x*-*y* plane, and use V to derive the corresponding electric field.

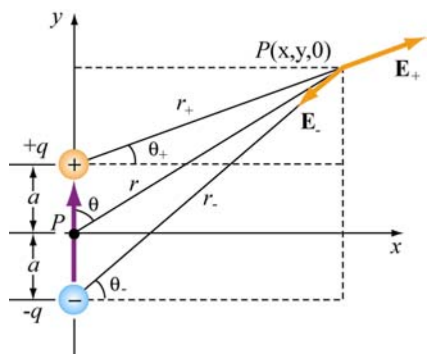


Figure 3.8.3

By superposition principle, the potential at *P* is given by

$$V = \sum_i V_i = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

where $r_\pm^{\ 2} = r^2 + a^2 \mp 2ra\cos\theta$. If we take the limit where $r \gg a$, then

$$\frac{1}{r_\pm} = \frac{1}{r}\left[1 + (a/r)^2 \mp 2(a/r)\cos\theta\right]^{-1/2} = \frac{1}{r}\left[1 - \frac{1}{2}(a/r)^2 \pm (a/r)\cos\theta + \cdots\right]$$

and the dipole potential can be approximated as

$$\begin{aligned} V &= \frac{q}{4\pi\varepsilon_0 r}\left[1 - \frac{1}{2}(a/r)^2 + (a/r)\cos\theta - 1 + \frac{1}{2}(a/r)^2 + (a/r)\cos\theta + \cdots\right] \\ &\approx \frac{q}{4\pi\varepsilon_0 r}\cdot\frac{2a\cos\theta}{r} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\vec{\mathbf{p}}\cdot\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2} \end{aligned}$$

where $\vec{\mathbf{p}} = 2aq\,\hat{\mathbf{j}}$ is the electric dipole moment. In spherical polar coordinates, the gradient operator is

$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\boldsymbol{\varphi}}$$

Since the potential is now a function of both r and θ, the electric field will have components along the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ directions. Using $\vec{\mathbf{E}} = -\nabla V$, we have

$$\boxed{E_r = -\frac{\partial V}{\partial r} = \frac{p\cos\theta}{2\pi\varepsilon_0 r^3}, \quad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}, \quad E_\phi = 0}$$

3.8.3 Electric Potential of an Annulus

Consider an annulus of uniform charge density σ, as shown in Figure 3.8.4. Find the electric potential at a point P along the symmetric axis.

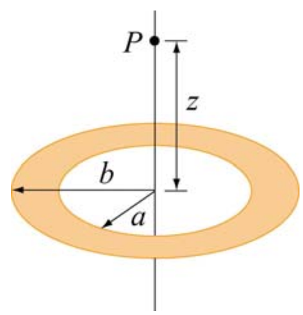


Figure 3.8.4 An annulus of uniform charge density.

Solution:

Consider a small differential element dA at a distance r away from point P. The amount of charge contained in dA is given by

$$dq = \sigma dA = \sigma(r'd\theta)dr'$$

Its contribution to the electric potential at P is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma r'dr'd\theta}{\sqrt{r'^2+z^2}}$$

Integrating over the entire annulus, we obtain

$$V = \frac{\sigma}{4\pi\varepsilon_0}\int_a^b\int_0^{2\pi}\frac{r'dr'd\theta}{\sqrt{r'^2+z^2}} = \frac{2\pi\sigma}{4\pi\varepsilon_0}\int_a^b\frac{r'ds}{\sqrt{r'^2+z^2}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{b^2+z^2}-\sqrt{a^2+z^2}\right]$$

where we have made used of the integral

$$\int \frac{ds\, s}{\sqrt{s^2+z^2}} = \sqrt{s^2+z^2}$$

Notice that in the limit $a \to 0$ and $b \to R$, the potential becomes

$$V = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+z^2} - |z|\right]$$

which coincides with the result of a non-conducting disk of radius R shown in Eq. (3.5.14).

3.8.4 Charge Moving Near a Charged Wire

A thin rod extends along the z-axis from $z = -d$ to $z = d$. The rod carries a positive charge Q uniformly distributed along its length $2d$ with charge density $\lambda = Q/2d$.

(a) Calculate the electric potential at a point $z > d$ along the z-axis.

(b) What is the change in potential energy if an electron moves from $z = 4d$ to $z = 3d$?

(c) If the electron started out at rest at the point $z = 4d$, what is its velocity at $z = 3d$?

Solutions:

(a) For simplicity, let's set the potential to be zero at infinity, $V(\infty) = 0$. Consider an infinitesimal charge element $dq = \lambda\, dz'$ located at a distance z' along the z-axis. Its contribution to the electric potential at a point $z > d$ is

$$dV = \frac{\lambda}{4\pi\varepsilon_0}\frac{dz'}{z - z'}$$

Integrating over the entire length of the rod, we obtain

$$V(z) = \frac{\lambda}{4\pi\varepsilon_0}\int_{z+d}^{z-d}\frac{dz'}{z-z'} = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{z+d}{z-d}\right)$$

(b) Using the result derived in (a), the electrical potential at $z = 4d$ is

$$V(z=4d) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{4d+d}{4d-d}\right) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{5}{3}\right)$$

Similarly, the electrical potential at $z = 3d$ is

$$V(z=3d)=\frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{3d+d}{3d-d}\right)=\frac{\lambda}{4\pi\varepsilon_0}\ln 2$$

The electric potential difference between the two points is

$$\Delta V = V(z=3d)-V(z=4d)=\frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{6}{5}\right)>0$$

Using the fact that the electric potential difference ΔV is equal to the change in potential energy per unit charge, we have

$$\Delta U = q\Delta V = -\frac{|e|\lambda}{4\pi\varepsilon_0}\ln\left(\frac{6}{5}\right)<0$$

where $q=-|e|$ is the charge of the electron.

(c) If the electron starts out at rest at $z=4d$ then the change in kinetic energy is

$$\Delta K = \frac{1}{2}mv_f^{\,2}$$

By conservation of energy, the change in kinetic energy is

$$\Delta K = -\Delta U = \frac{|e|\lambda}{4\pi\varepsilon_0}\ln\left(\frac{6}{5}\right)>0$$

Thus, the magnitude of the velocity at $z=3d$ is

$$v_f = \sqrt{\frac{2|e|}{4\pi\varepsilon_0}\frac{\lambda}{m}\ln\left(\frac{6}{5}\right)}$$

3.9 Conceptual Questions

1. What is the difference between electric potential and electric potential energy?

2. A uniform electric field is parallel to the *x*-axis. In what direction can a charge be displaced in this field without any external work being done on the charge?

3. Is it safe to stay in an automobile with a metal body during severe thunderstorm? Explain.

4. Why are equipotential surfaces always perpendicular to electric field lines?

5. The electric field inside a hollow, uniformly charged sphere is zero. Does this imply that the potential is zero inside the sphere?

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