Module 02: Math Review

Module 02: Math Review: Outline

Vector Review (Dot, Cross Products)

Review of 1D Calculus

Scalar Functions in higher dimensions

Vector Functions

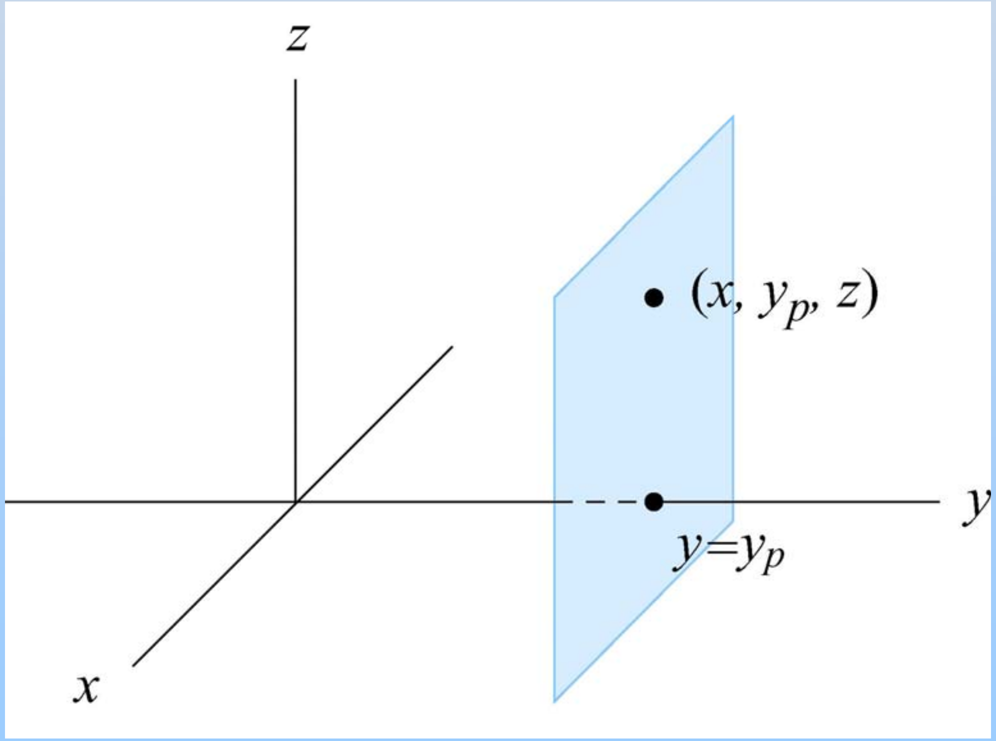
Differentials

Purpose: Provide conceptual framework NOT teach mechanics

Coordinate System

Coordinate system: used to describe the position of a point in space and consists of

1. An origin as the reference point
2. A set of coordinate axes with scales and labels
3. Choice of positive direction for each axis
4. Choice of unit vectors at each point in space



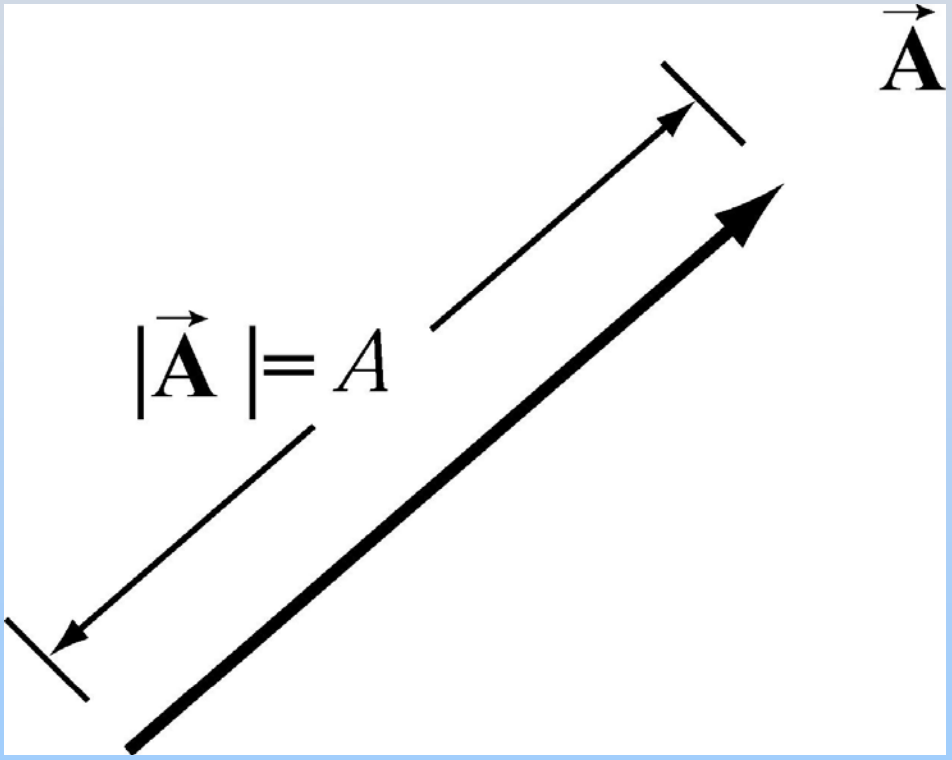
Cartesian Coordinate System

Vectors

Vector

A vector is a quantity that has both direction and magnitude. Let a vector be denoted by the symbol $\vec{\mathbf{A}}$

The magnitude of $\vec{\mathbf{A}}$ is denoted by $|\vec{\mathbf{A}}| \equiv A$



Application of Vectors

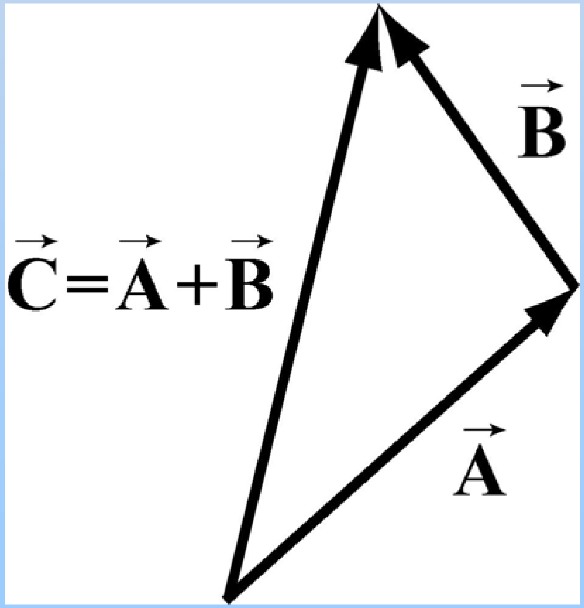
(1) Vectors can exist at any point *P* in space.

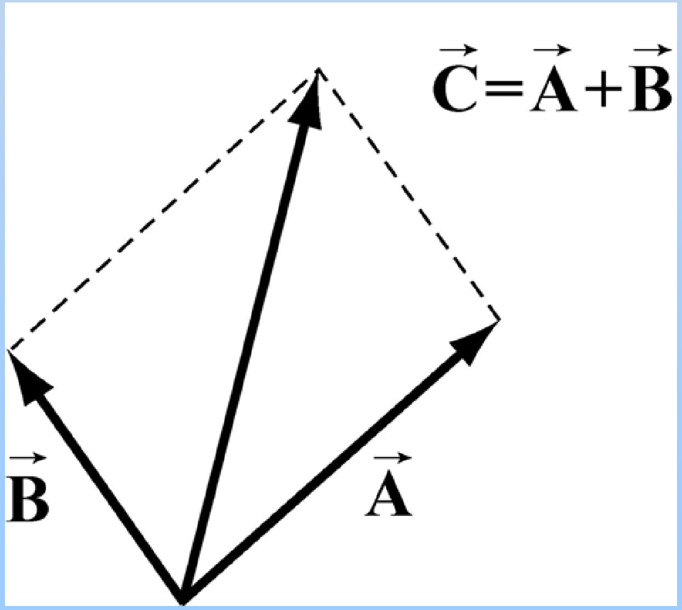
(2) Vectors have direction and magnitude.

(3) Vector Equality: Any two vectors that have the same direction and magnitude are equal no matter where in space they are located.

Vector Addition

Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors. Define a new vector $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$,the “vector addition” of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ by the geometric construction shown in either figure





Summary: Vector Properties

Addition of Vectors

1. Commutativity $\vec{\mathbf{A}}+\vec{\mathbf{B}}=\vec{\mathbf{B}}+\vec{\mathbf{A}}$

2. Associativity $(\vec{\mathbf{A}}+\vec{\mathbf{B}})+\vec{\mathbf{C}}=\vec{\mathbf{A}}+(\vec{\mathbf{B}}+\vec{\mathbf{C}})$

3. Identity Element for Vector Addition $\vec{\mathbf{0}}$ such that $\vec{\mathbf{A}}+\vec{\mathbf{0}}=\vec{\mathbf{0}}+\vec{\mathbf{A}}=\vec{\mathbf{A}}$

4. Inverse Element for Vector Addition $-\vec{\mathbf{A}}$ such that $\vec{\mathbf{A}}+\left(-\vec{\mathbf{A}}\right)=\vec{\mathbf{0}}$

Scalar Multiplication of Vectors

1. Associative Law for Scalar Multiplication $b(c\vec{\mathbf{A}})=(bc)\vec{\mathbf{A}}=(cb\,\vec{\mathbf{A}})=c\,(b\vec{\mathbf{A}})$

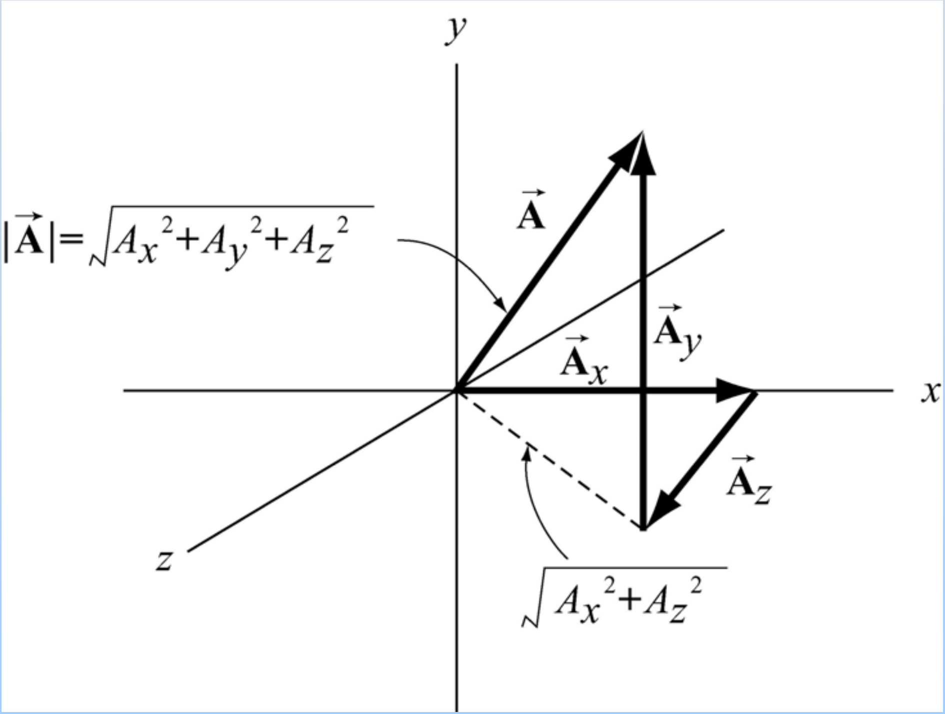
2. Distributive Law for Vector Addition $c\,(\vec{\mathbf{A}}+\vec{\mathbf{B}})=c\,\vec{\mathbf{A}}+c\,\vec{\mathbf{B}}$

3. Distributive Law for Scalar Addition $(b+c)\,\vec{\mathbf{A}}=b\,\vec{\mathbf{A}}+c\,\vec{\mathbf{A}}$

4. Identity Element for Scalar Multiplication: number 1 such that $1\,\vec{\mathbf{A}}=\vec{\mathbf{A}}$

Vector Decomposition

Choose a coordinate system with an origin and axes. We can decompose a vector into component vectors along each coordinate axis, for example along the x,y, and z-axes of a Cartesian coordinate system. A vector at *P* can be decomposed into the vector sum,



$$\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y + \vec{\mathbf{A}}_z$$

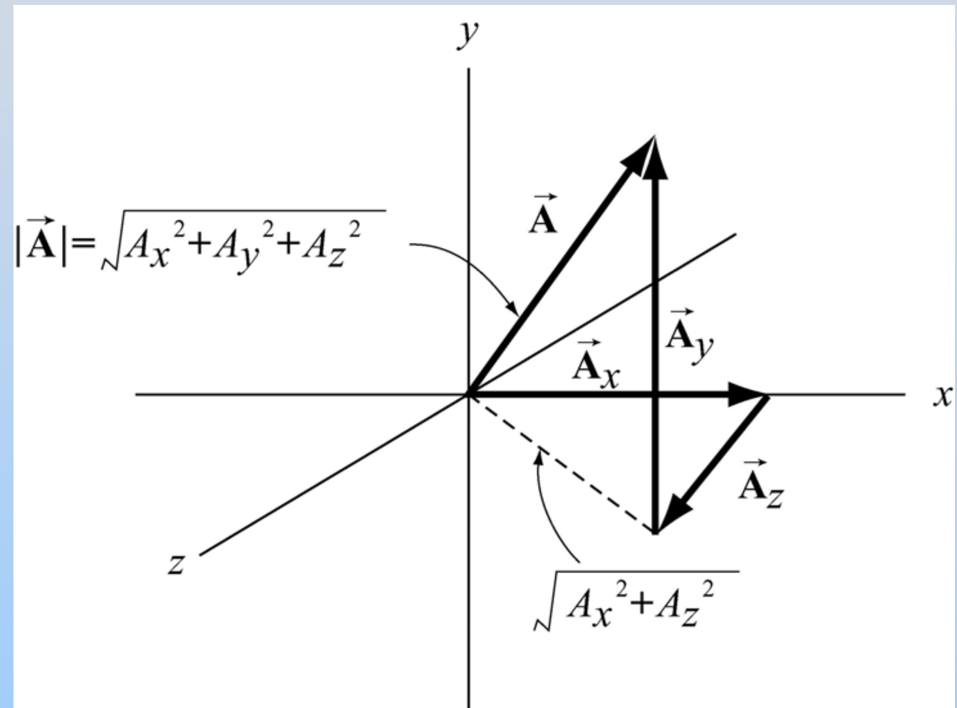
Unit Vectors and Components

The idea of multiplication by real numbers allows us to define a set of unit vectors at each point in space $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$

with $|\hat{\mathbf{i}}|=1, |\hat{\mathbf{j}}|=1, |\hat{\mathbf{k}}|=1$

Components:

$$\vec{\mathbf{A}} = (A_x, A_y, A_z)$$



$$\vec{\mathbf{A}}_x = A_x\,\hat{\mathbf{i}},\; \vec{\mathbf{A}}_y = A_y\,\hat{\mathbf{j}},\quad \vec{\mathbf{A}}_z = A_z\,\hat{\mathbf{k}}$$

$$\vec{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}$$

Vector Decomposition in Two Dimensions

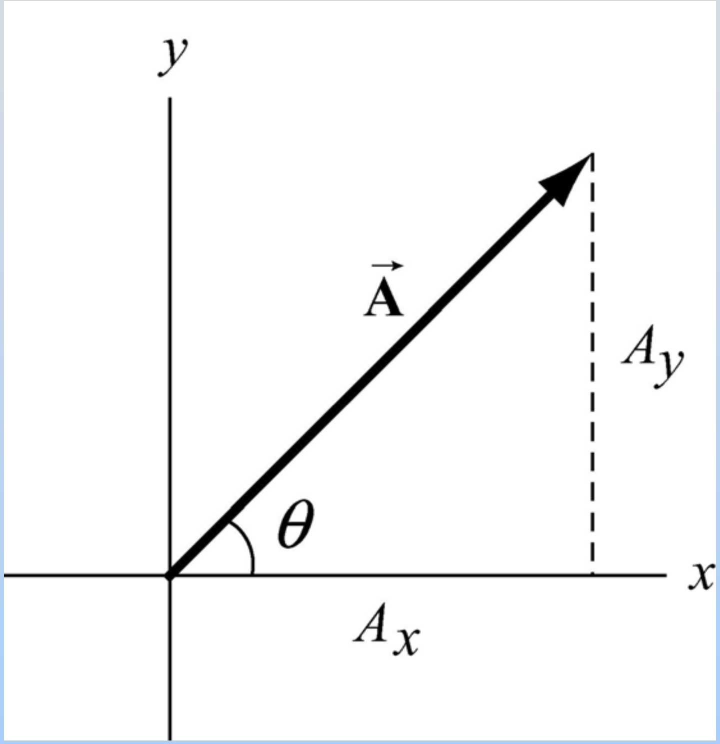
Consider a vector

$$\vec{\mathbf{A}} = (A_x, A_y, 0)$$

x- and y components:

$$A_x = A\cos(\theta), \quad A_y = A\sin(\theta)$$

Magnitude: $A = \sqrt{A_x^2 + A_y^2}$



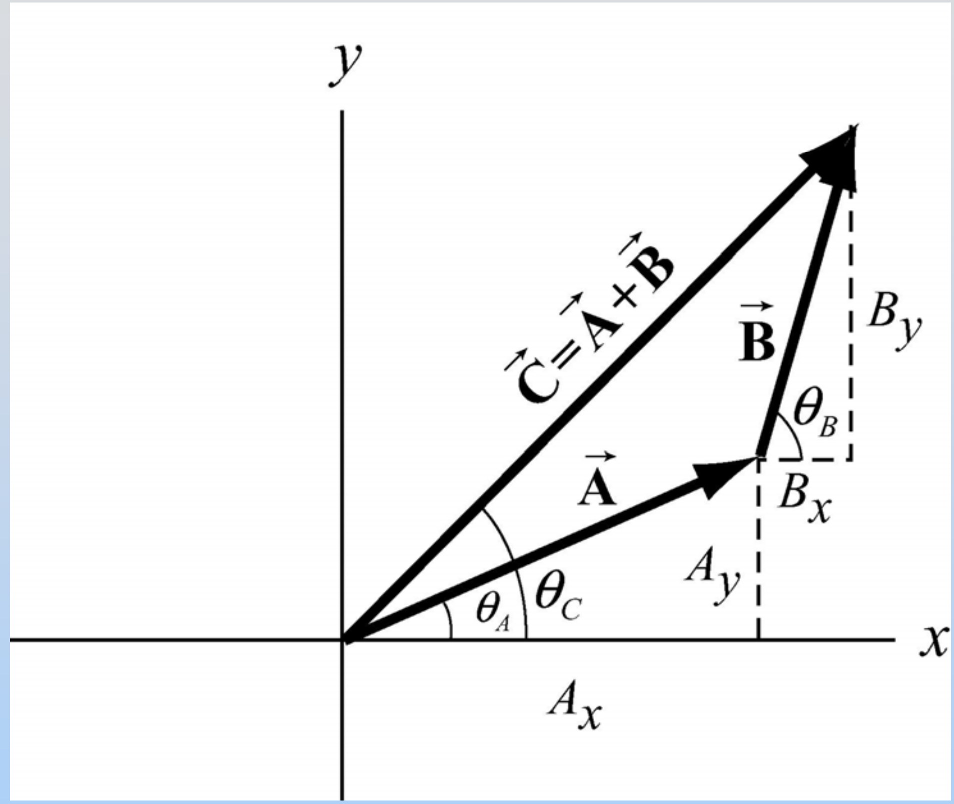
Direction: $$\frac{A_y}{A_x} = \frac{A\sin(\theta)}{A\cos(\theta)} = \tan(\theta)$$

$$\theta = \tan^{-1}(A_y / A_x)$$

Vector Addition

$$\vec{\mathbf{A}} = A\cos(\theta_A)\ \hat{\mathbf{i}} + A\sin(\theta_A)\ \hat{\mathbf{j}}$$

$$\vec{\mathbf{B}} = B\cos(\theta_B)\ \hat{\mathbf{i}} + B\sin(\theta_B)\ \hat{\mathbf{j}}$$



Vector Sum: $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$

Components

$$C_x = A_x + B_x, \quad C_y = A_y + B_y$$

$$C_x - C\cos(\theta_C) - A\cos(\theta_A) + B\cos(\theta_B)$$

$$C_y = C\sin(\theta_C) = A\sin(\theta_A) + B\sin(\theta_B)$$

$$\vec{\mathbf{C}} = (A_x + B_x)\,\hat{\mathbf{i}} + (A_y + B_y)\,\hat{\mathbf{j}} = C\cos(\theta_C)\,\hat{\mathbf{i}} + C\sin(\theta_C)\,\hat{\mathbf{j}}$$

Preview: Vector Description of Motion

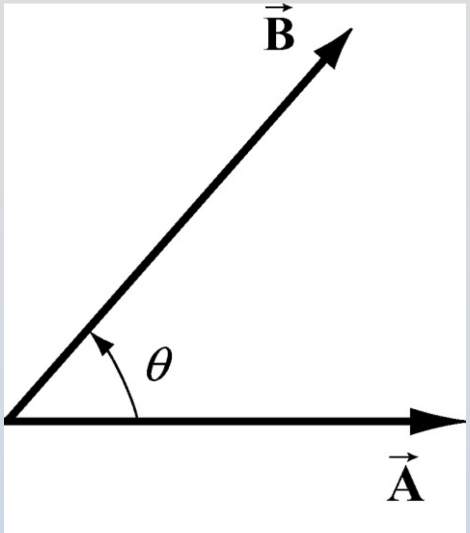
- Position $\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}}$

- Displacement $\Delta\vec{\mathbf{r}}(t) = \Delta x(t)\,\hat{\mathbf{i}} + \Delta y(t)\,\hat{\mathbf{j}}$

- Velocity $\vec{\mathbf{v}}(t) = \dfrac{dx(t)}{dt}\,\hat{\mathbf{i}} + \dfrac{dy(t)}{dt}\,\hat{\mathbf{j}} \equiv v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}}$

- Acceleration $\vec{\mathbf{a}}(t) = \dfrac{dv_x(t)}{dt}\,\hat{\mathbf{i}} + \dfrac{dv_y(t)}{dt}\,\hat{\mathbf{j}} \equiv a_x(t)\,\hat{\mathbf{i}} + a_y(t)\,\hat{\mathbf{j}}$

Scalar Product

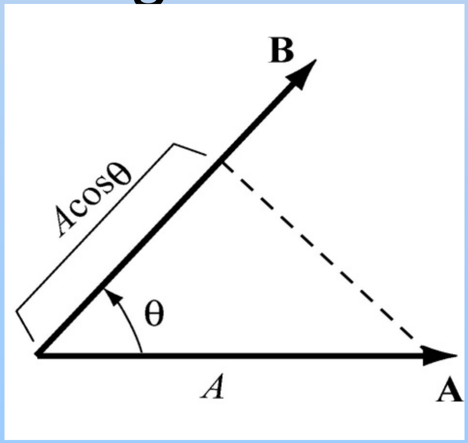


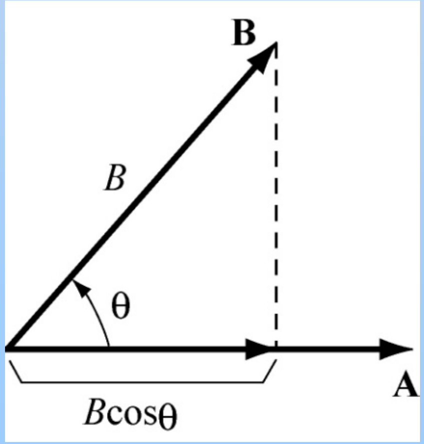
A scalar quantity

Magnitude: $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right| \cos\theta$

The scalar (dot) product can be positive, zero, or negative

Two types of projections: the scalar product is the parallel component of one vector with respect to the second vector times the magnitude of the second vector





$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \left|\vec{\mathbf{A}}\right|(\cos\theta)\left|\vec{\mathbf{B}}\right| = A_{\|}\left|\vec{\mathbf{B}}\right|$$

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \left|\vec{\mathbf{A}}\right|(\cos\theta)\left|\vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\right| B_{\|}$$

Scalar Product Properties

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} - \vec{\mathbf{B}} \cdot \vec{\mathbf{A}}$$

$$c\vec{\mathbf{A}} \quad \vec{\mathbf{B}} - c(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}})$$

$$(\vec{\mathbf{A}} + \vec{\mathbf{B}}) \cdot \vec{\mathrm{C}} = \vec{\mathbf{A}} \cdot \vec{\mathrm{C}} + \vec{\mathbf{B}} \cdot \vec{\mathrm{C}}$$

Scalar Product in Cartesian Coordinates

With unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0$$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = |\hat{\mathbf{i}}||\hat{\mathbf{i}}|\cos(0) = 1$$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = |\hat{\mathbf{i}}||\hat{\mathbf{j}}|\cos(\pi/2) = 0$$

Example:

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}, \qquad \vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$$

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_x B_x + A_y B_y + A_z B_z$$

Worked Example: Scalar Product

Given two vectors $\vec{\mathbf{A}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$

$$\vec{\mathbf{B}} - -2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

Find $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$

Solution:

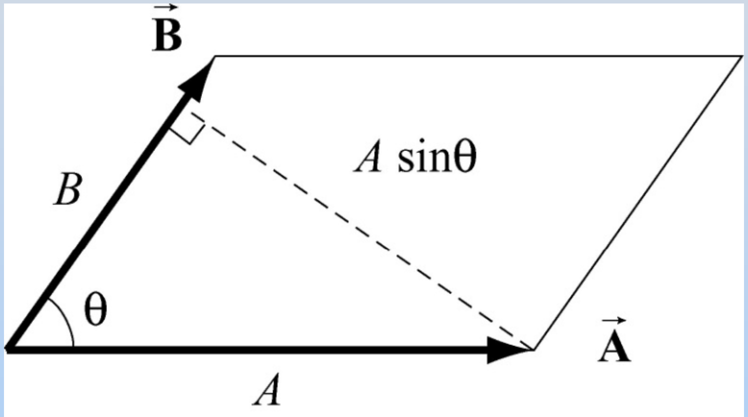
$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_x B_x + A_y B_y + A_z B_z$$

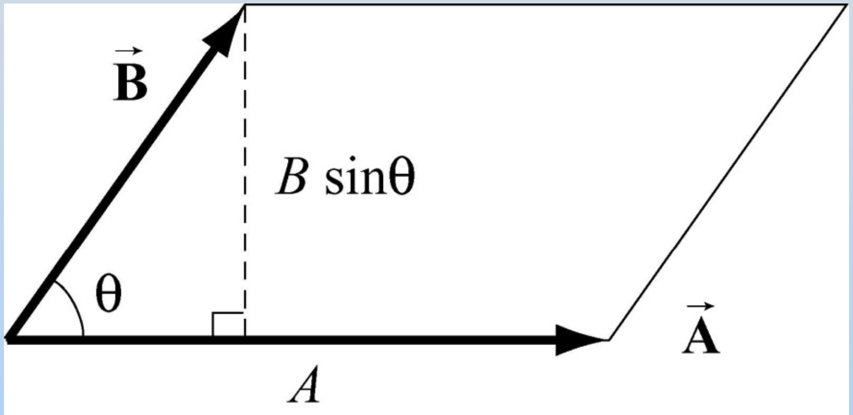
$$= (1)(-2) + (1)(-1) + (-1)(3) = -6$$

Summary: Vector Product

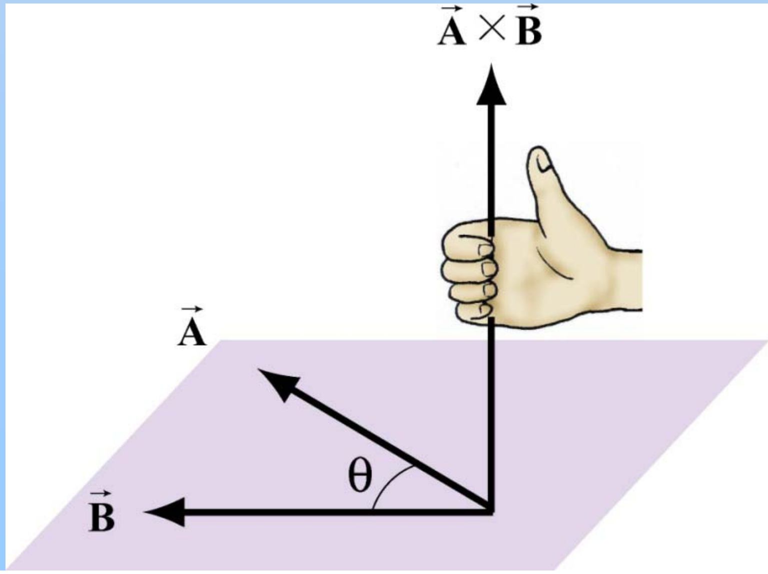
Magnitude: equal to the area of the parallelogram defined by the two vectors

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|\sin\theta = \left|\vec{\mathbf{A}}\right|\left(\left|\vec{\mathbf{B}}\right|\sin\theta\right) = \left(\left|\vec{\mathbf{A}}\right|\sin\theta\right)\left|\vec{\mathbf{B}}\right| \quad (0\le\theta\le\pi)$$





Direction: determined by the Right-Hand-Rule



Properties of Vector Products

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}}$$

$$c(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \vec{\mathbf{A}} \times c\vec{\mathbf{B}} = c\vec{\mathbf{A}} \times \vec{\mathbf{B}}$$

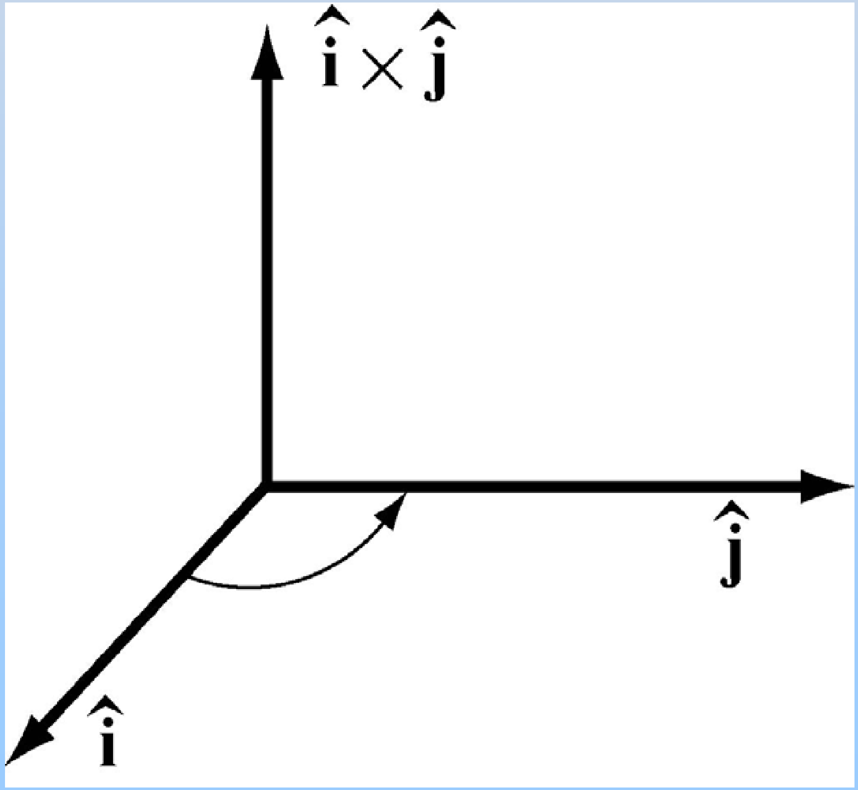
$$(\vec{\mathbf{A}} + \vec{\mathbf{B}}) \times \vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{C}} + \vec{\mathbf{B}} \times \vec{\mathbf{C}}$$

Vector Product of Unit Vectors

- Unit vectors in Cartesian coordinates

$$\left|\hat{\mathbf{i}} \times \hat{\mathbf{j}}\right| = |\hat{\mathbf{i}}||\hat{\mathbf{j}}|\sin(\pi/2) = 1$$

$$\left|\hat{\mathbf{i}} \times \hat{\mathbf{i}}\right| = |\hat{\mathbf{i}}||\hat{\mathbf{j}}|\sin(0) = 0$$



$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \qquad \hat{\mathbf{i}} \times \hat{\mathbf{i}} = \vec{\mathbf{0}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} \qquad \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \vec{\mathbf{0}}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \qquad \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \vec{\mathbf{0}}$$

Components of Vector Product

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}, \quad \vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$$

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_yB_z - A_zB_y)\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\hat{\mathbf{k}}$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Worked Example: Vector Product

Find a unit vector perpendicular to

$$\vec{\mathbf{A}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$$

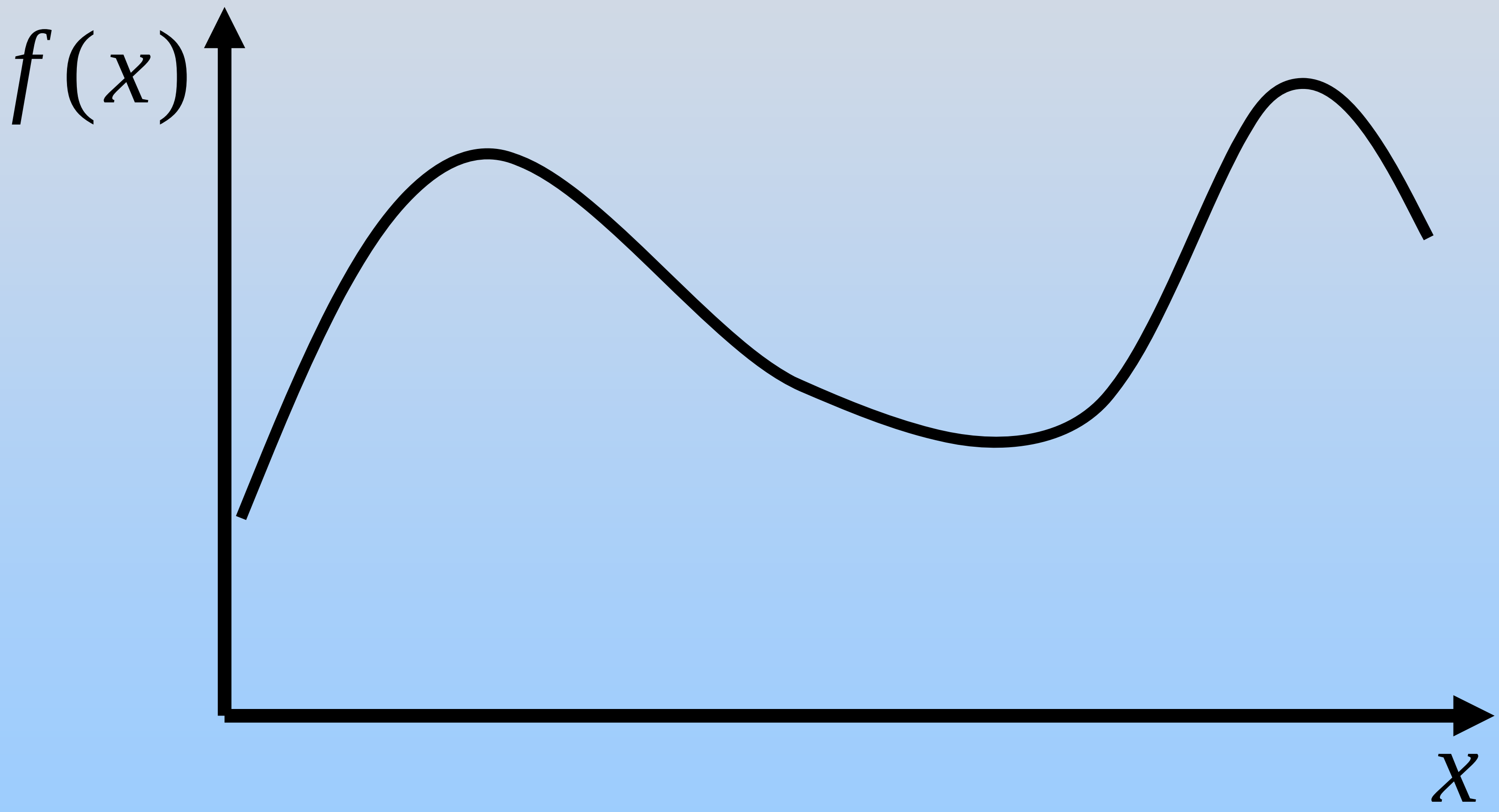
and

$$\vec{\mathbf{B}} = -2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}} \quad .$$

One Variable Calculus

Review: 1D Calculus

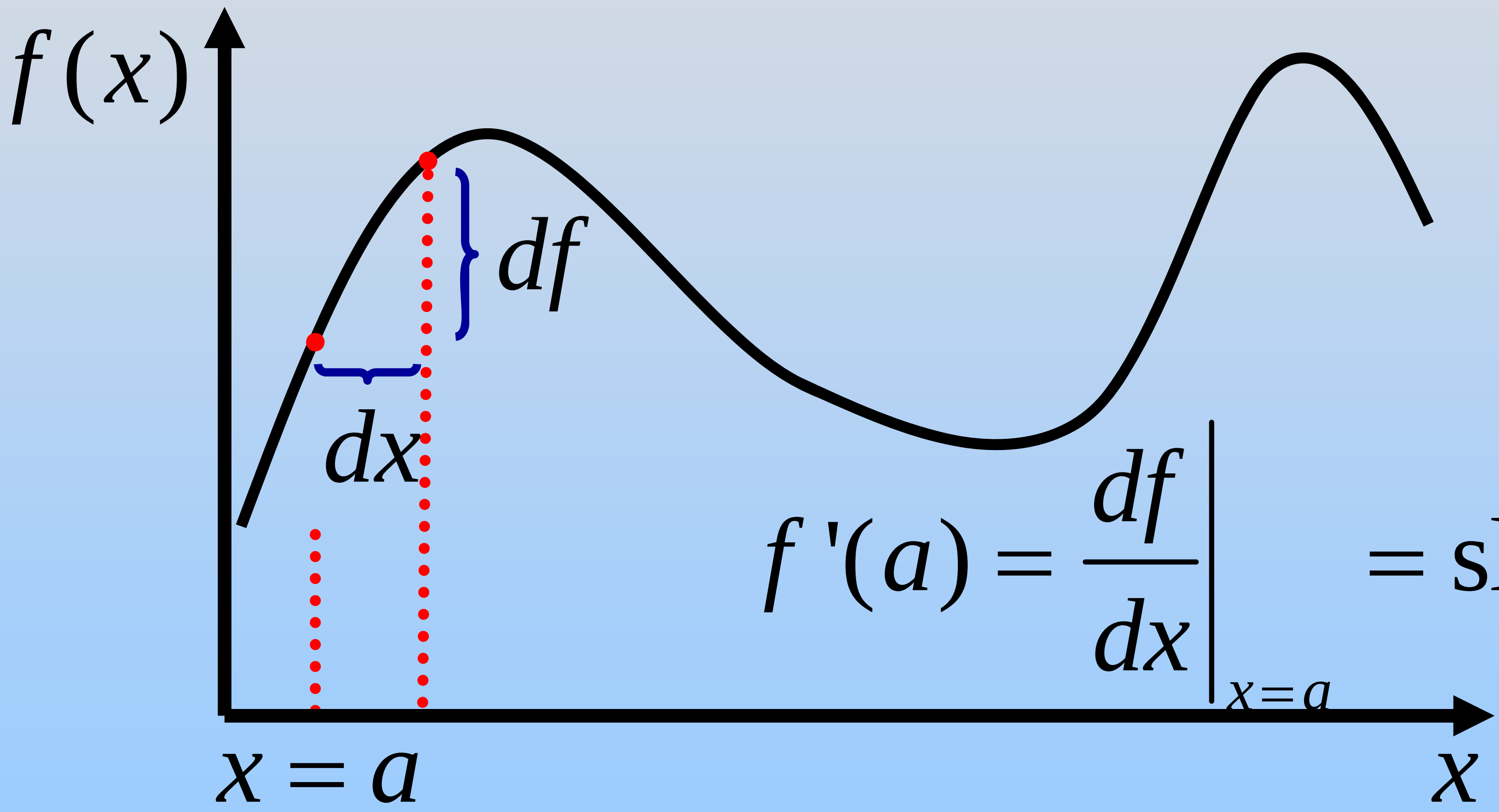
- Think about scalar functions in 1D:



Think of this as height of mountain vs position

Derivatives

How does function change with position?

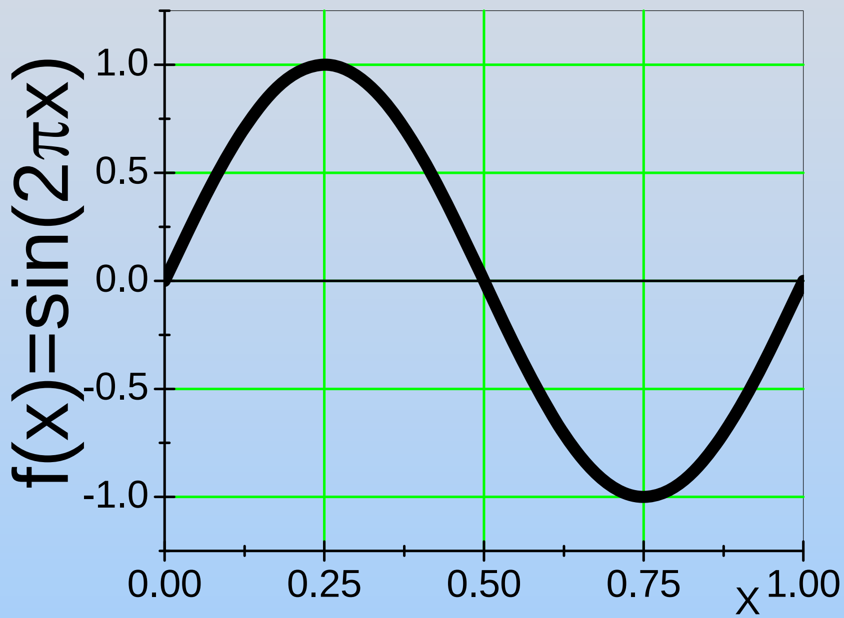


$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \text{slope}$$

Rate of change of f at $x = a$?

By the way… Taylor Series

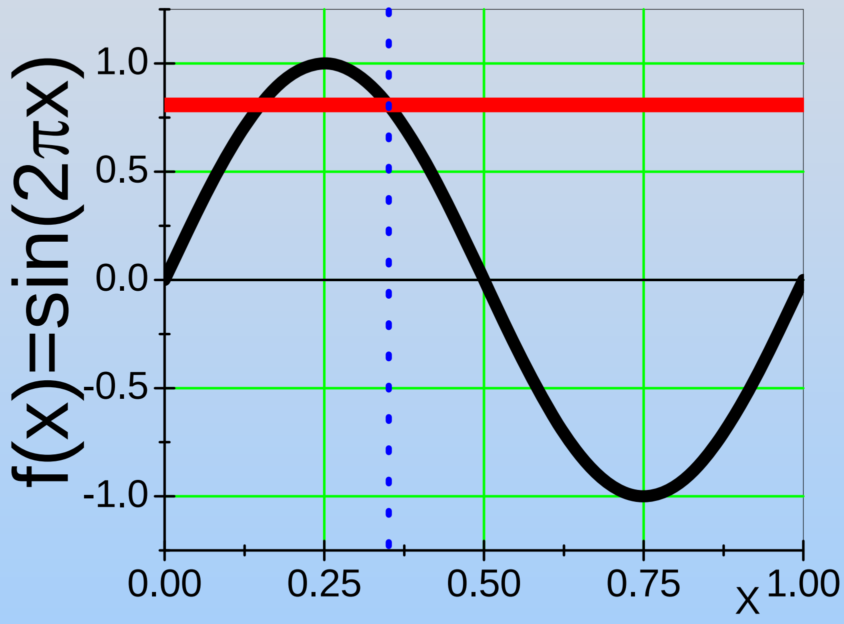
- Approximate function? Use derivatives!



What is f(x) near x=0.35?

By the way… Taylor Series

- Approximate function? Use derivatives!



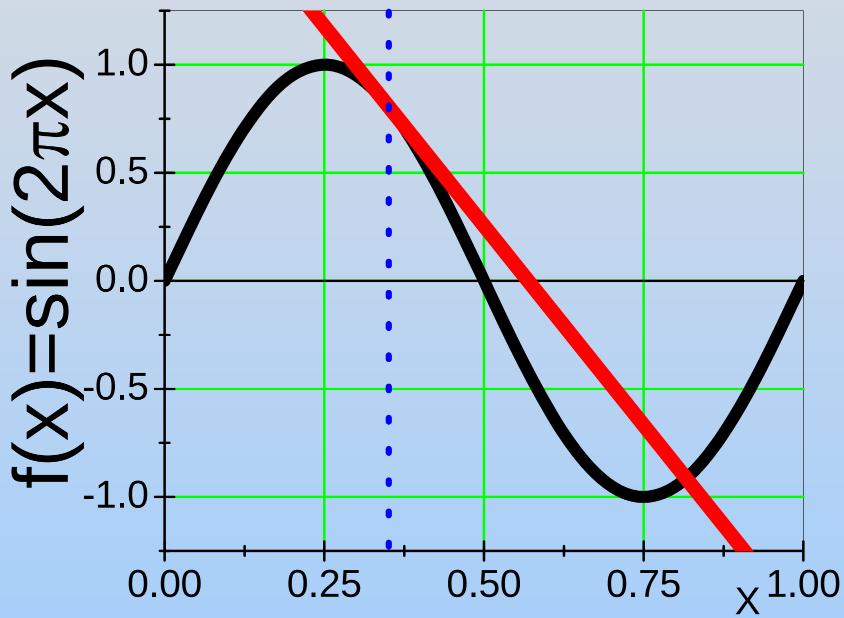
What is f(x) near x=0.35?

$$T_0(x) = f(0.35)$$

Red curve is our approximation to f(x) near x=0.35 using one term in the Taylor series

By the way… Taylor Series

- Approximate function? Use derivatives!



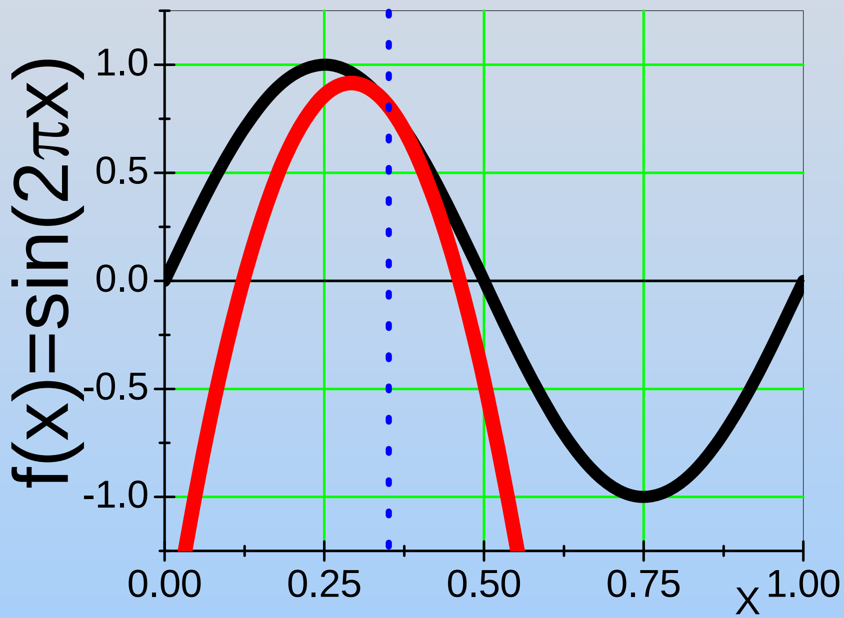
What is f(x) near x=0.35?

$$T_1(x) = f(0.35)$$
$$+ f'(0.35)\left(x - 0.35\right)$$

Red curve is our approximation to f(x) near x=0.35 using two terms in the Taylor series

By the way… Taylor Series

- Approximate function? Use derivatives!



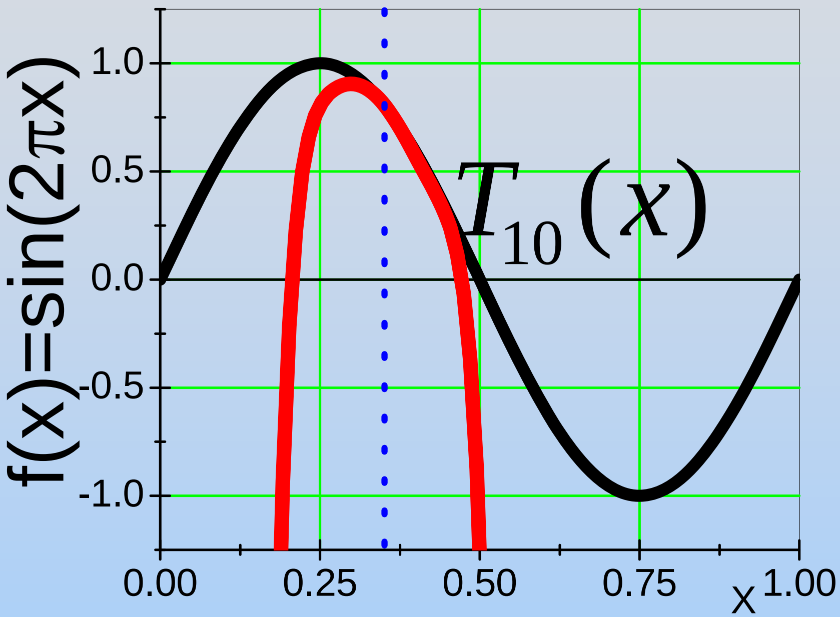
What is f(x) near x=0.35?

$$T_2(x) = f(0.35) + f'(0.35)\left(x - 0.35\right) + \frac{1}{2} f''(0.35)\left(x - 0.35\right)^2$$

Red curve is our approximation to f(x) near x=0.35 using three terms in the Taylor series

By the way… Taylor Series

- Approximate function? Use derivatives!



What is f(x) near x=0.35?

$$T_{10}(x) = f(0.35)$$

$$+ f'(0.35)\left(x-0.35\right)$$

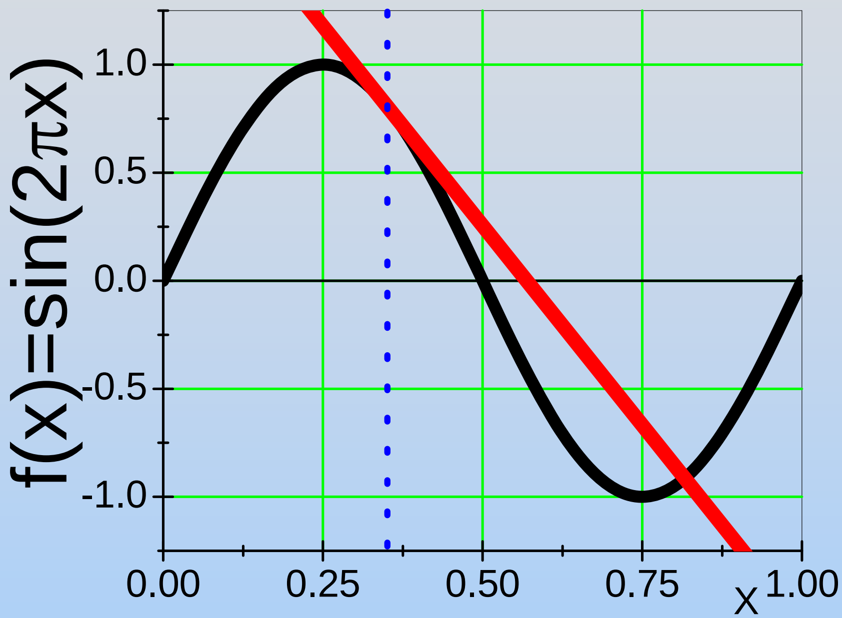
$$+\tfrac{1}{2} f''(0.35)\left(x-0.35\right)^2$$

$$+ \text{ eleven more terms!}$$

Red curve is our approximation to f(x) near x=0.35 using 11 terms in the Taylor series

$$\text{In general } T_N(x) = \sum_{n=0}^{N} \frac{\left(x-a\right)^n}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=a}$$

Taylor Series Most Commonly Used Only to 1st Order



Most Common: 1st Order

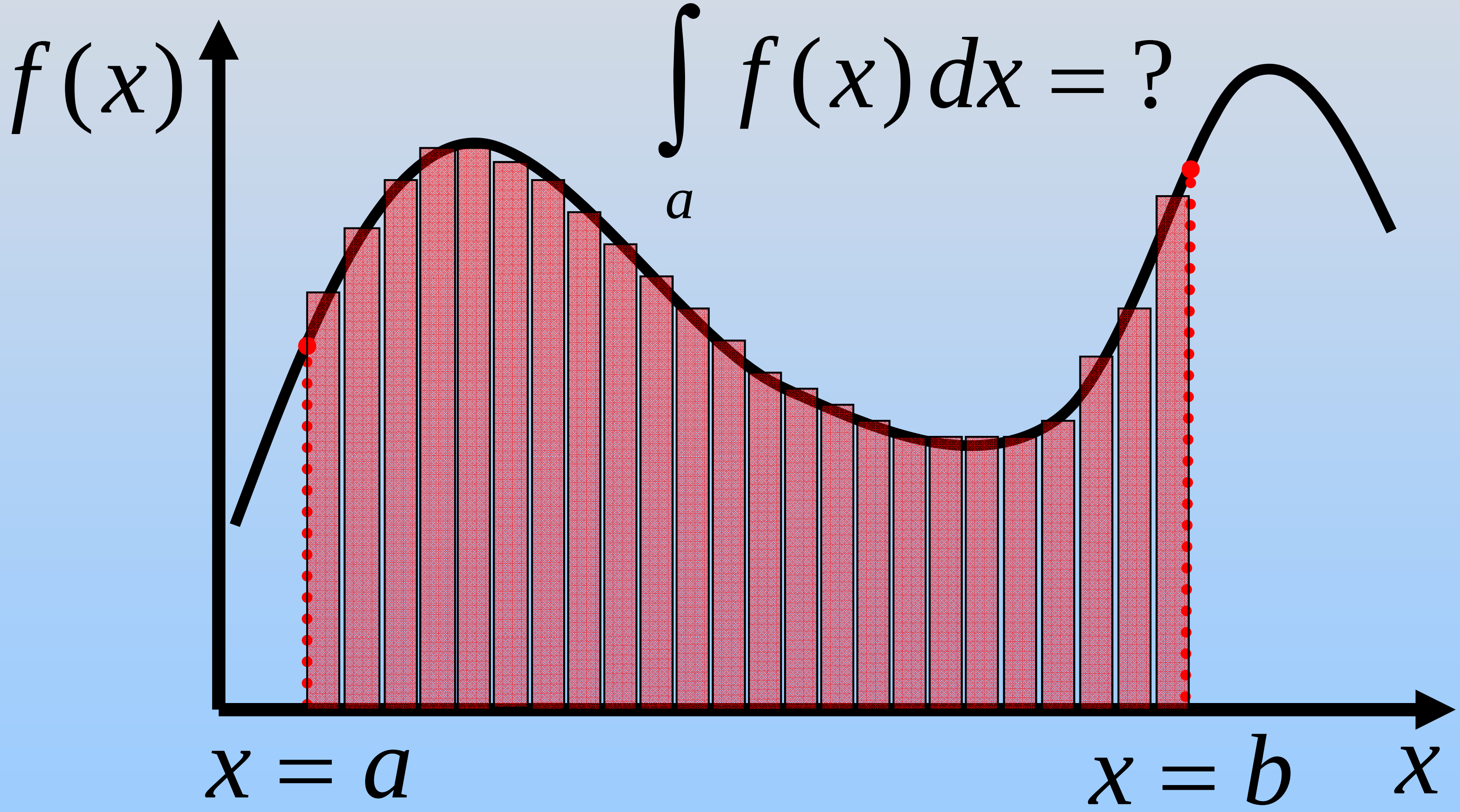
$$T_1(x) = f(a) + f'(a)\left(x - a\right)$$

- For hints as to when to use Taylor, look for "approximate" or "when x is small" or "small angle" or "close to" …

Integration

Sum function while walking along axis

$$\int_a^b f(x)\,dx = ?$$



Geometry: Find Area

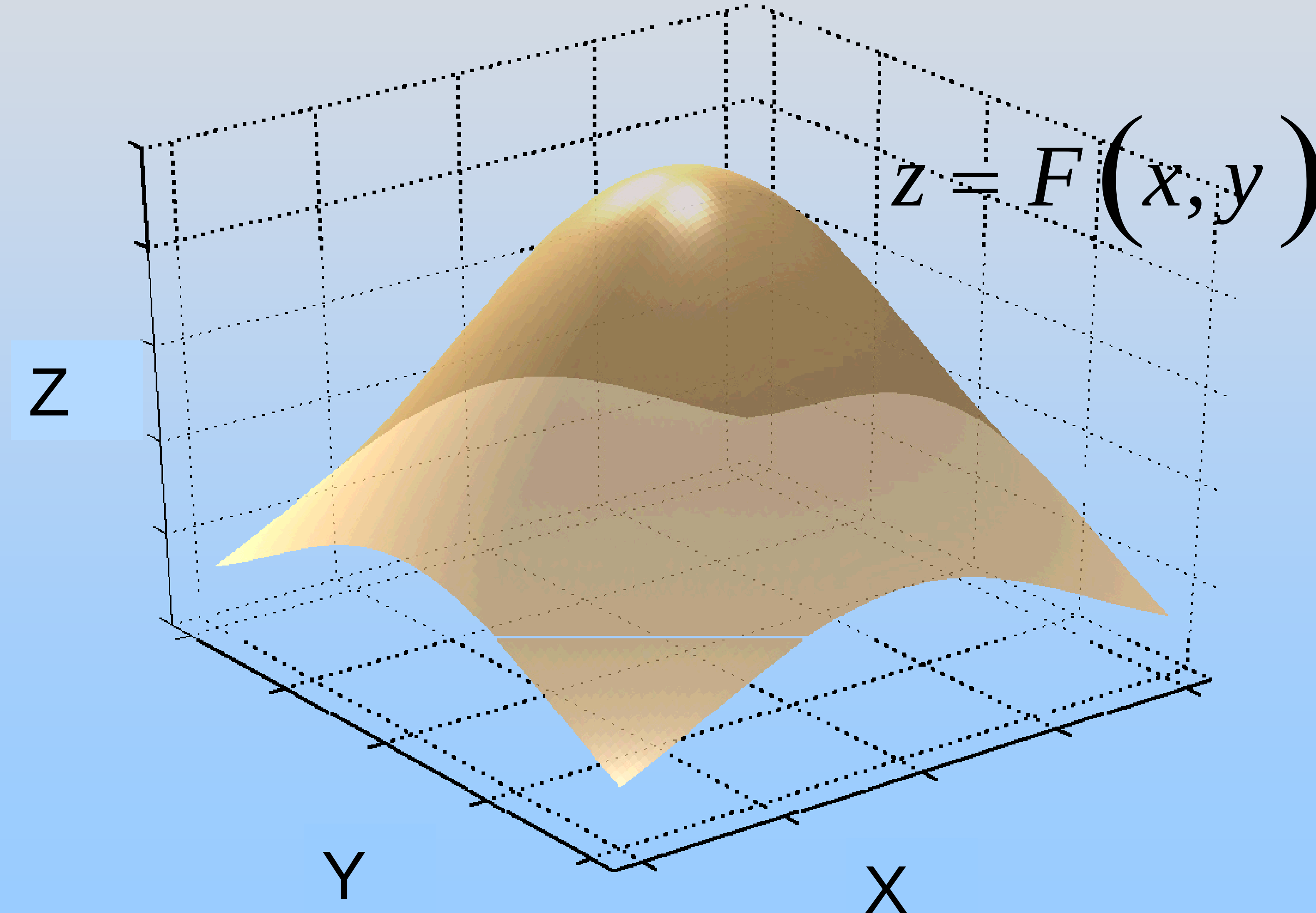
Also: Sum Contributions

Move to More Dimensions

We'll start in 2D

Scalar Functions in 2D

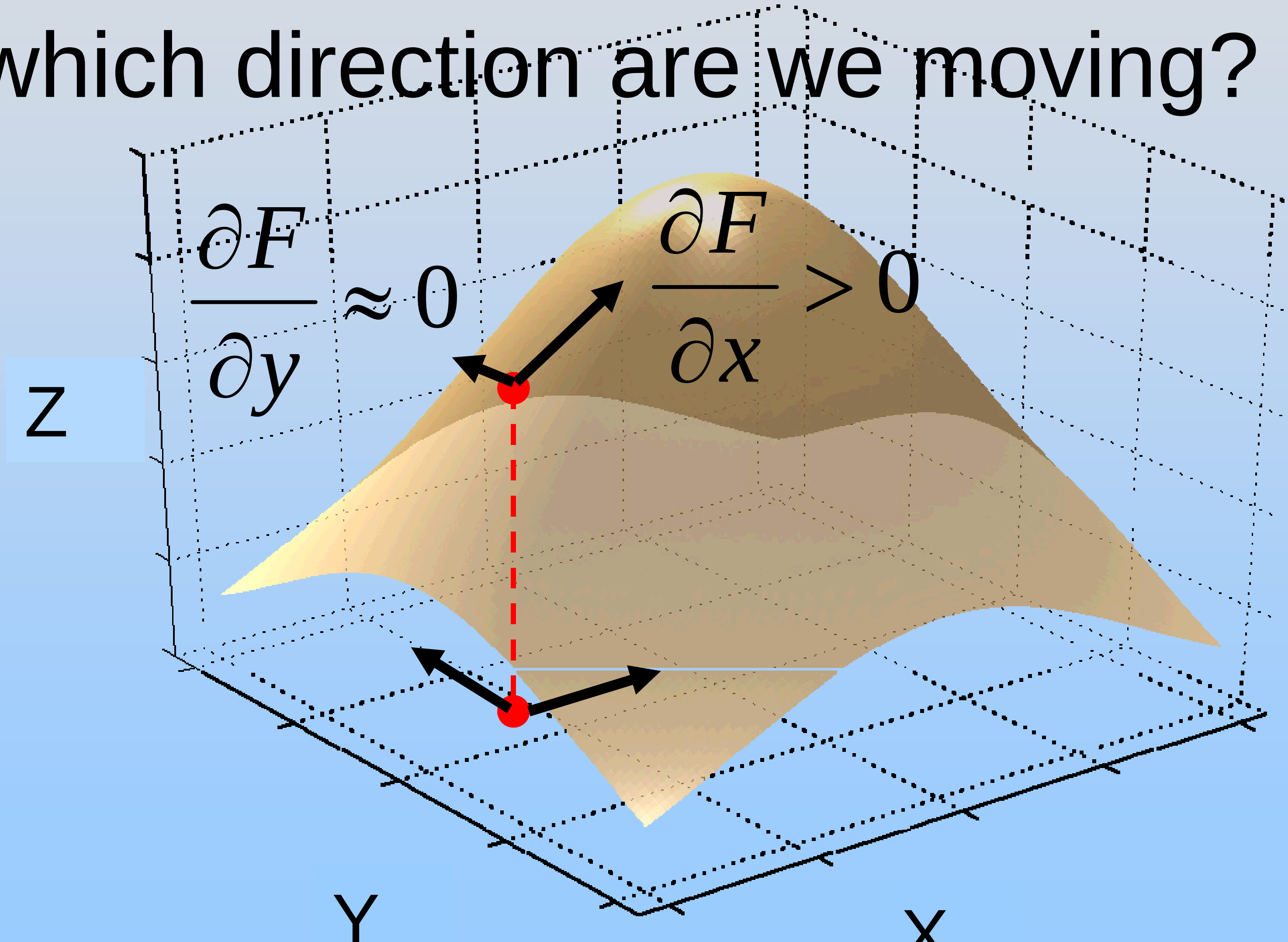
- Function is height of mountain:



Partial Derivatives

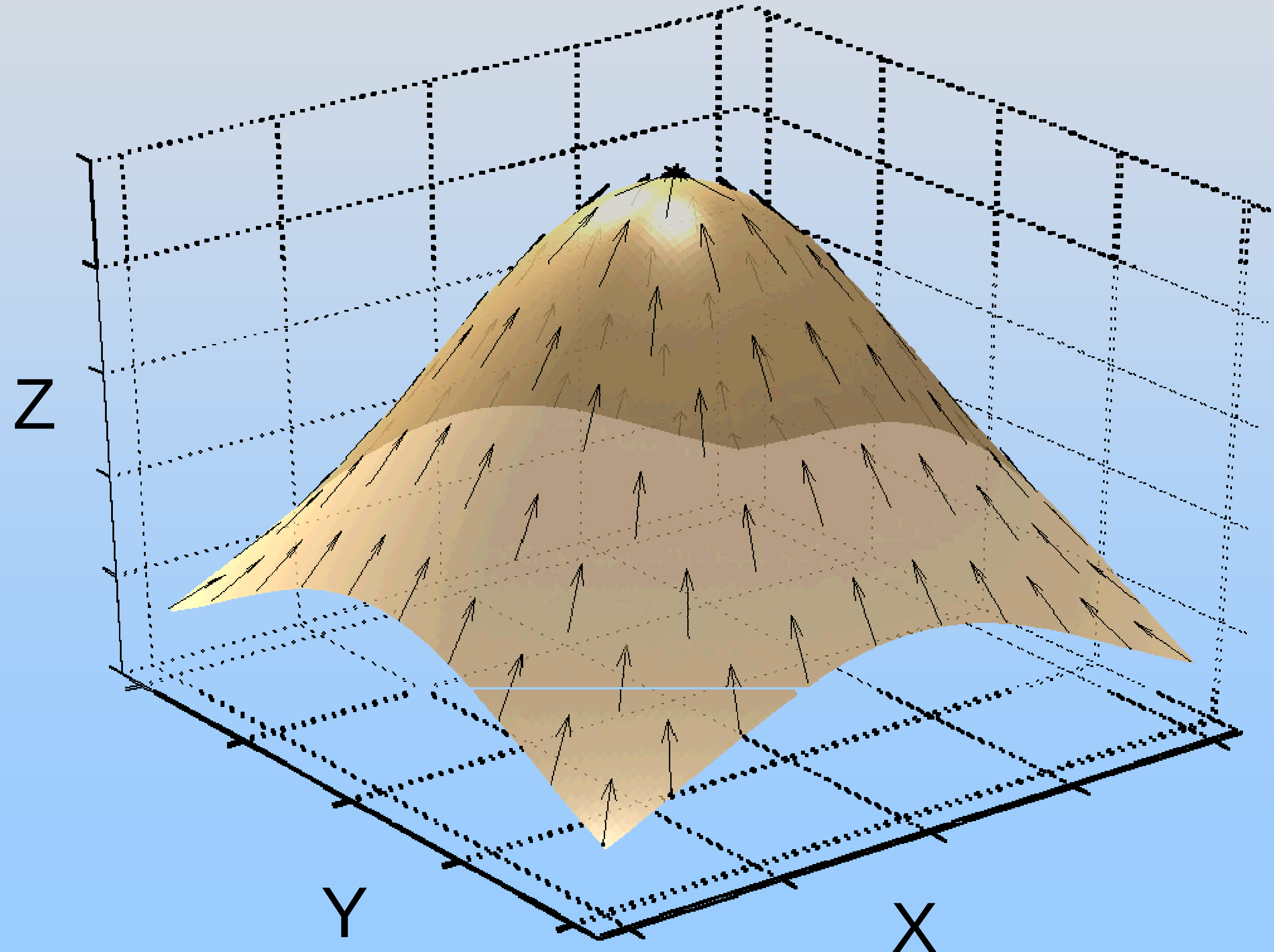
How does function change with position?

In which direction are we moving?



Gradient

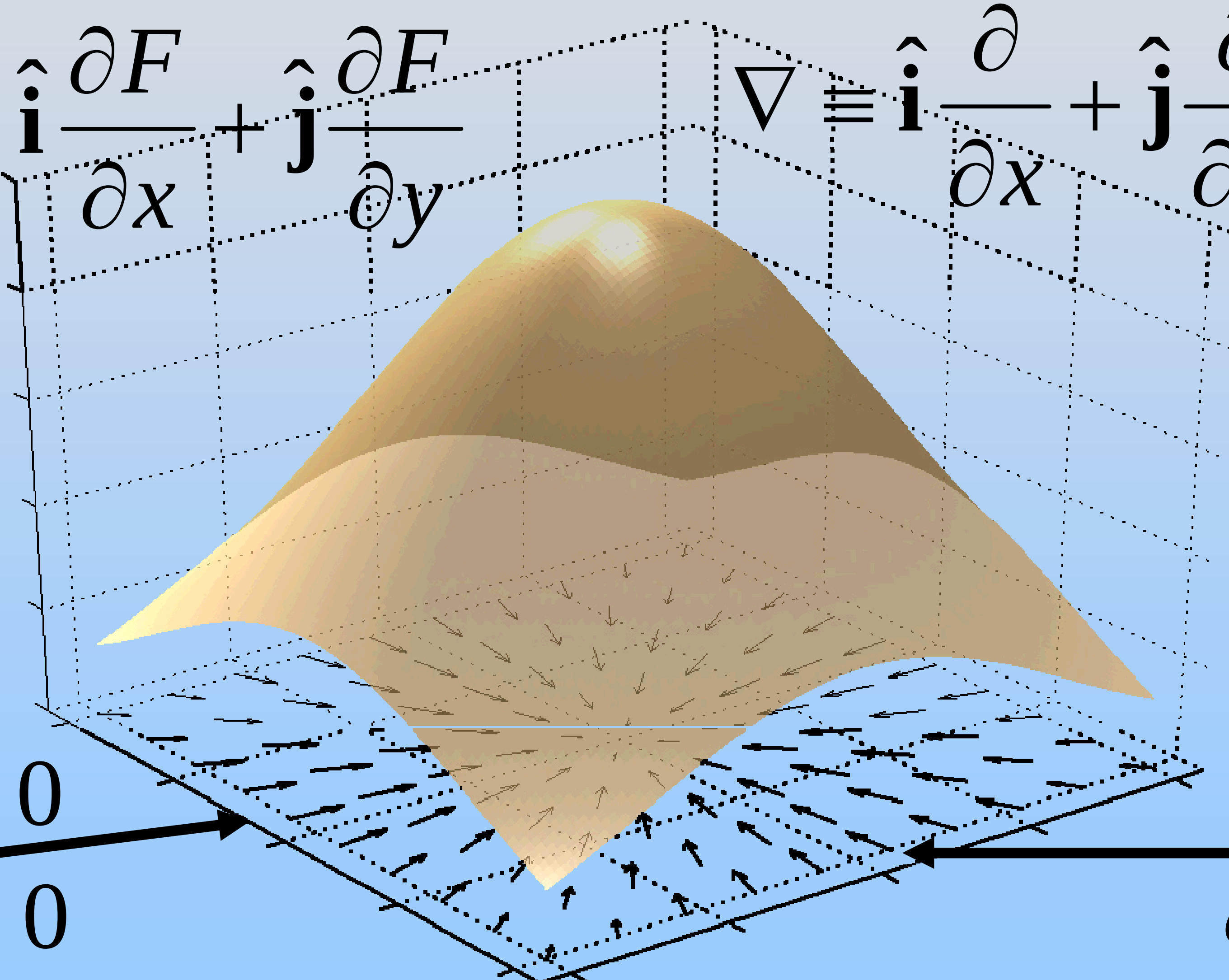
What is fastest way up the mountain?



Gradient

Gradient tells you direction to move:

$$\nabla F = \hat{\mathbf{i}}\frac{\partial F}{\partial x} + \hat{\mathbf{j}}\frac{\partial F}{\partial y} \qquad \nabla \equiv \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$$

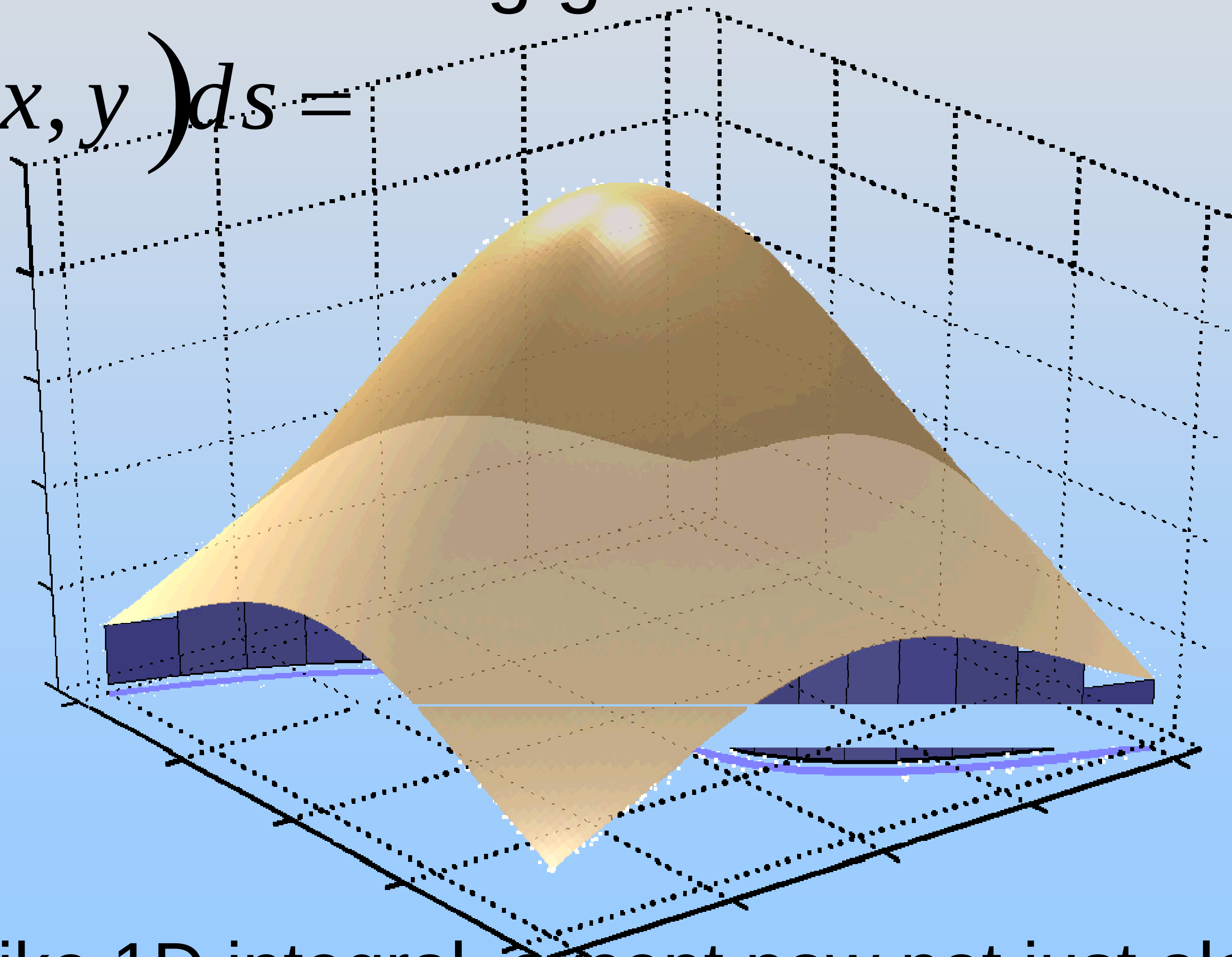


$$\partial_x F > 0 \qquad \partial_x F \approx 0$$
$$\partial_y F \approx 0 \qquad \partial_y F > 0$$

Line Integral

Sum function while walking under surface along given curve

$$\int_C f\left(x,y\right)ds =$$

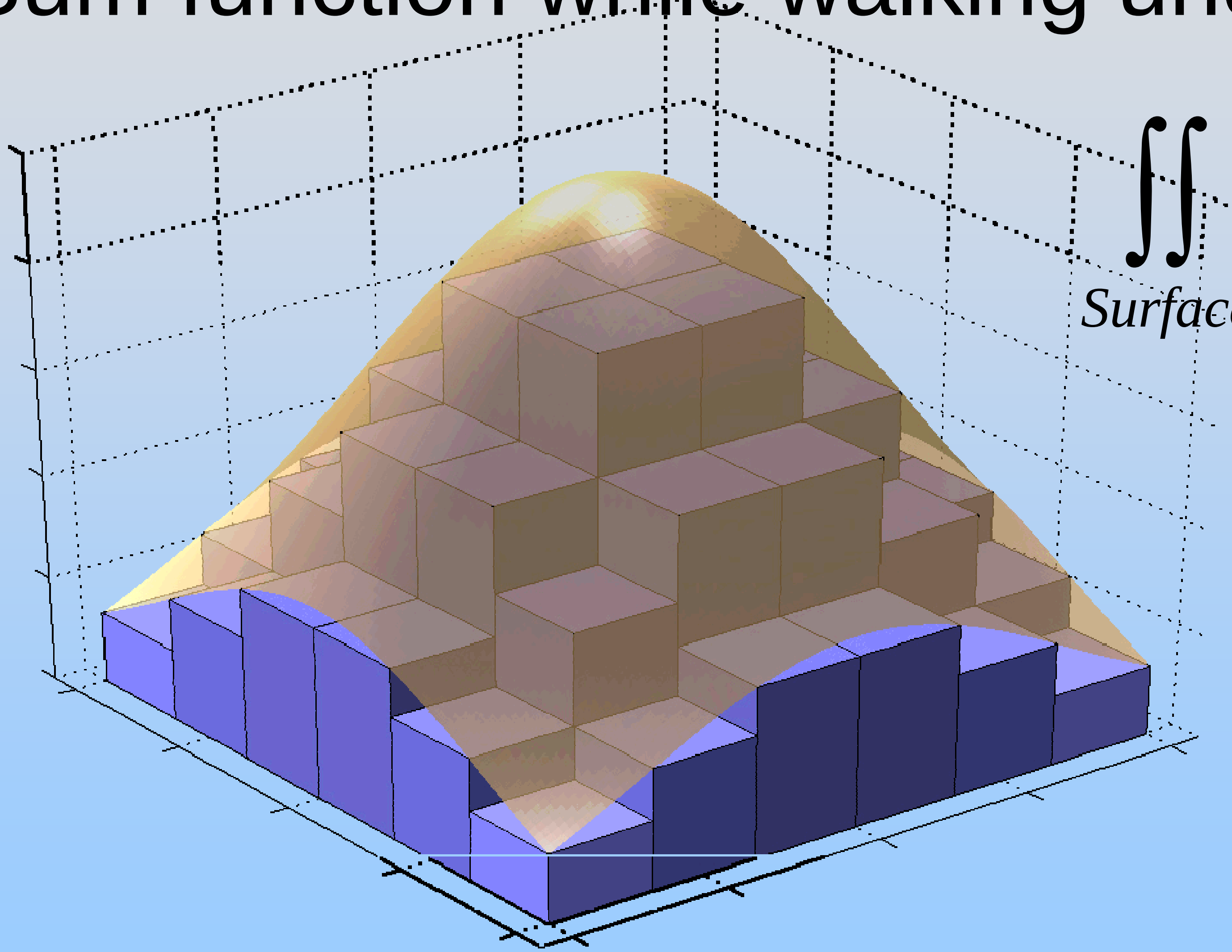


Just like 1D integral, except now not just along x

2D Integration

Sum function while walking under surface

$$\iint_{Surface} F(x,y)\,dA$$



Just Geometry: Finding Volume Under Surface

N-D Integration in General

Now think “contribution” from each piece

Find area of surface? $$\iint_{Surface} dA$$

Volume of object? $$\iiint_{Object} dV$$

Mass Density

Mass of object? $$\iiint_{Object} dM = \iiint_{Object} \rho \, dV$$

IDEA: Break object into small pieces, visit each, asking “What is contribution?”

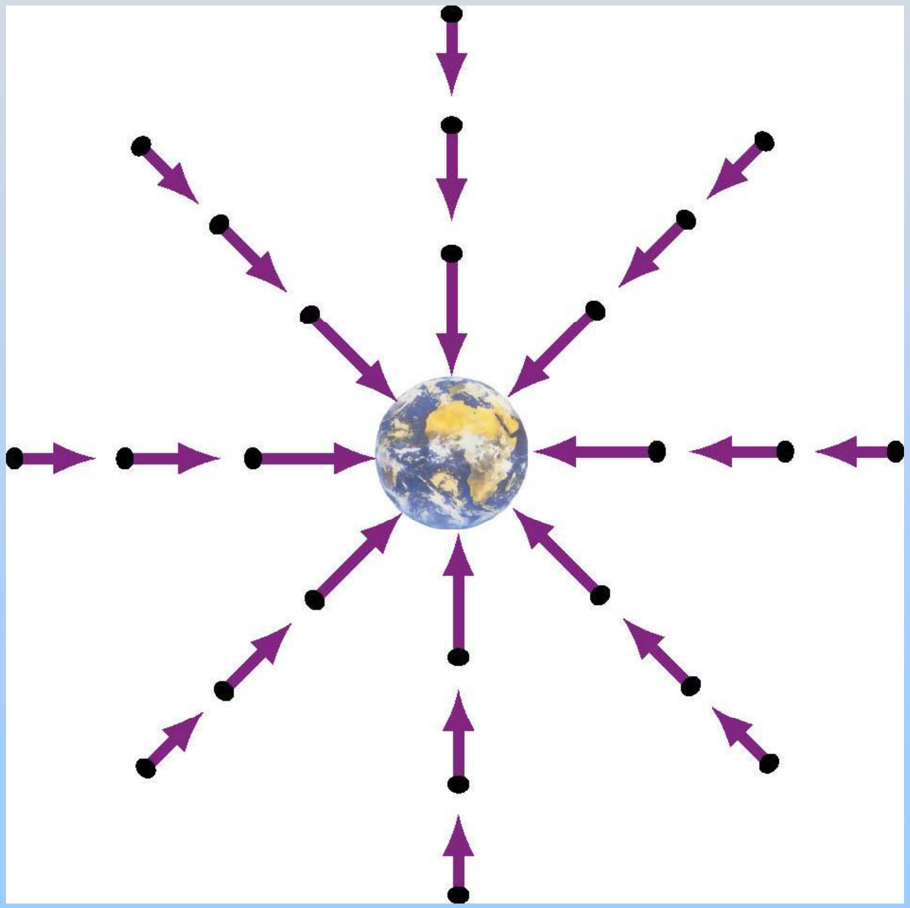
You Now Know It All

Small Extension to

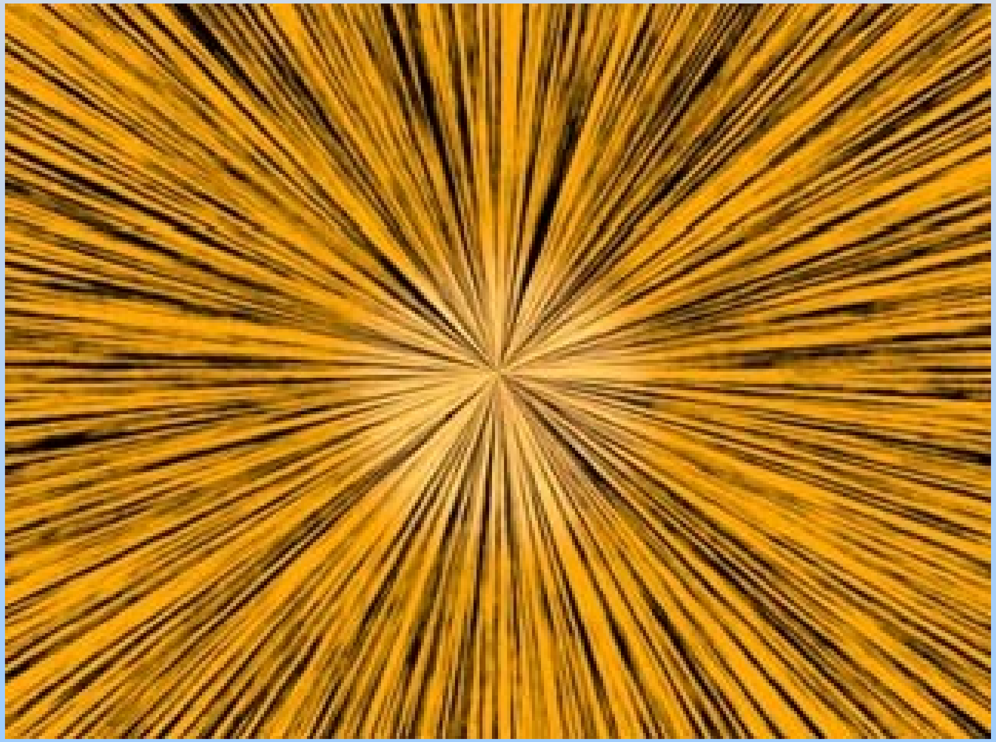
Vector Functions

Can't Easily Draw Multidimensional Vector Functions

In 2D various representations:



Vector Field Diagram



"Grass Seeds" / "Iron Filings"

Integrating Vector Functions

Two types of questions generally asked:

1) Integral of vector function yielding vector

Ex.: Mass Distribution $\vec{\mathbf{g}} = -G \iiint_{object} \frac{dM}{r^2} \hat{\mathbf{r}}$

IDEA: Use Components - Just like scalar

$$\iint \vec{\mathbf{F}}(\vec{\mathbf{r}}) dA =$$

$$\hat{\mathbf{i}} \iint F_x(\vec{\mathbf{r}}) dA + \hat{\mathbf{j}} \iint F_y(\vec{\mathbf{r}}) dA + \hat{\mathbf{k}} \iint F_z(\vec{\mathbf{r}}) dA$$

Integrating Vector Functions

Two types of questions generally asked:

2) Integral of vector function yielding scalar

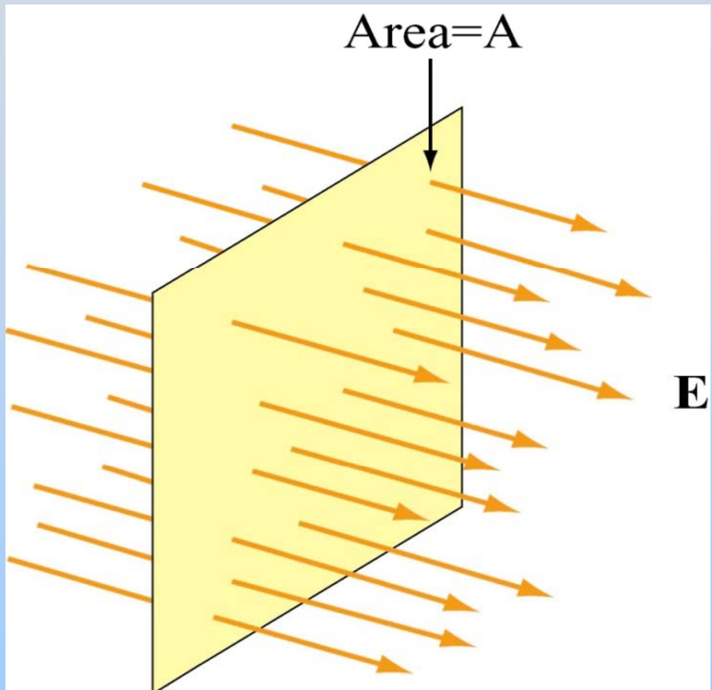
Line Integral Ex.: Work $W - \int_{Curve} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}$

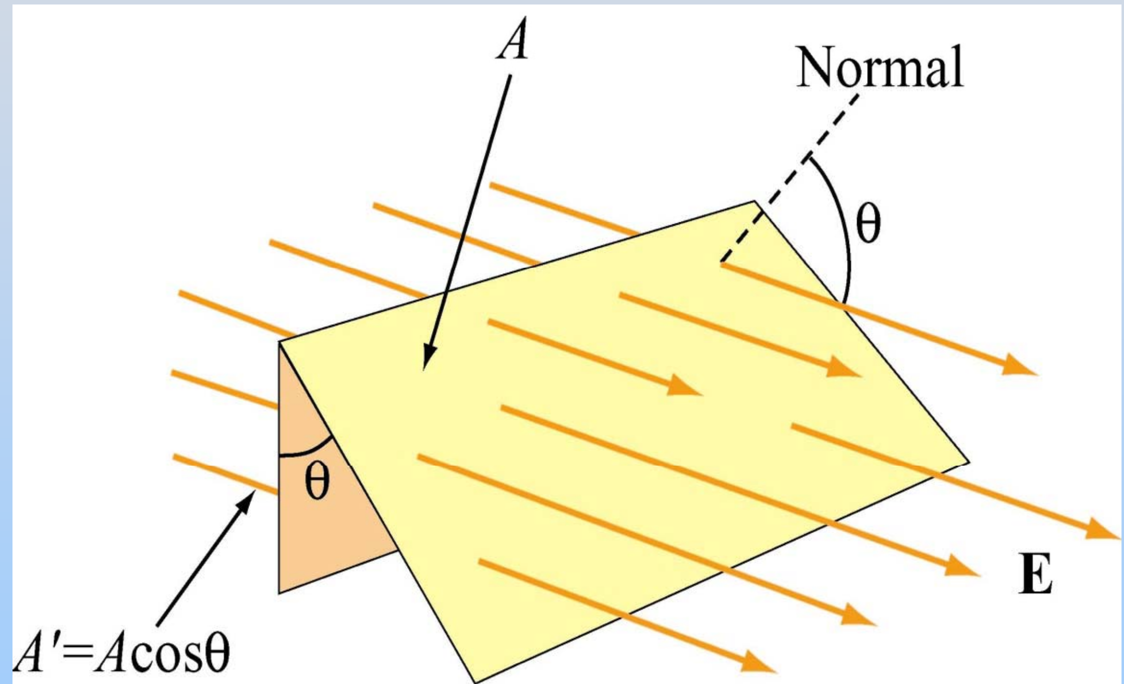
IDEA: While walking along the curve how much of the function lies *along* our path

Integrating Vector Functions

One last example: Flux

Q: How much does field **E** penetrate the surface?



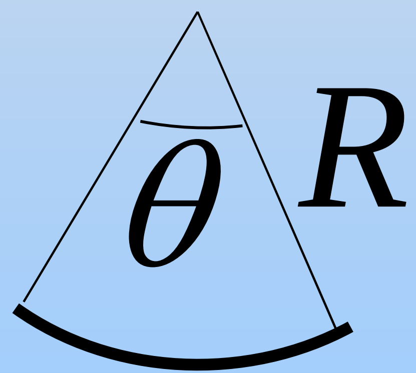


$$\text{Flux } \Phi_E = \iint_{\text{Surface}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

Arc Length on Circle

One Important Geometry Fact:

Relation between arc length on circle and included angle



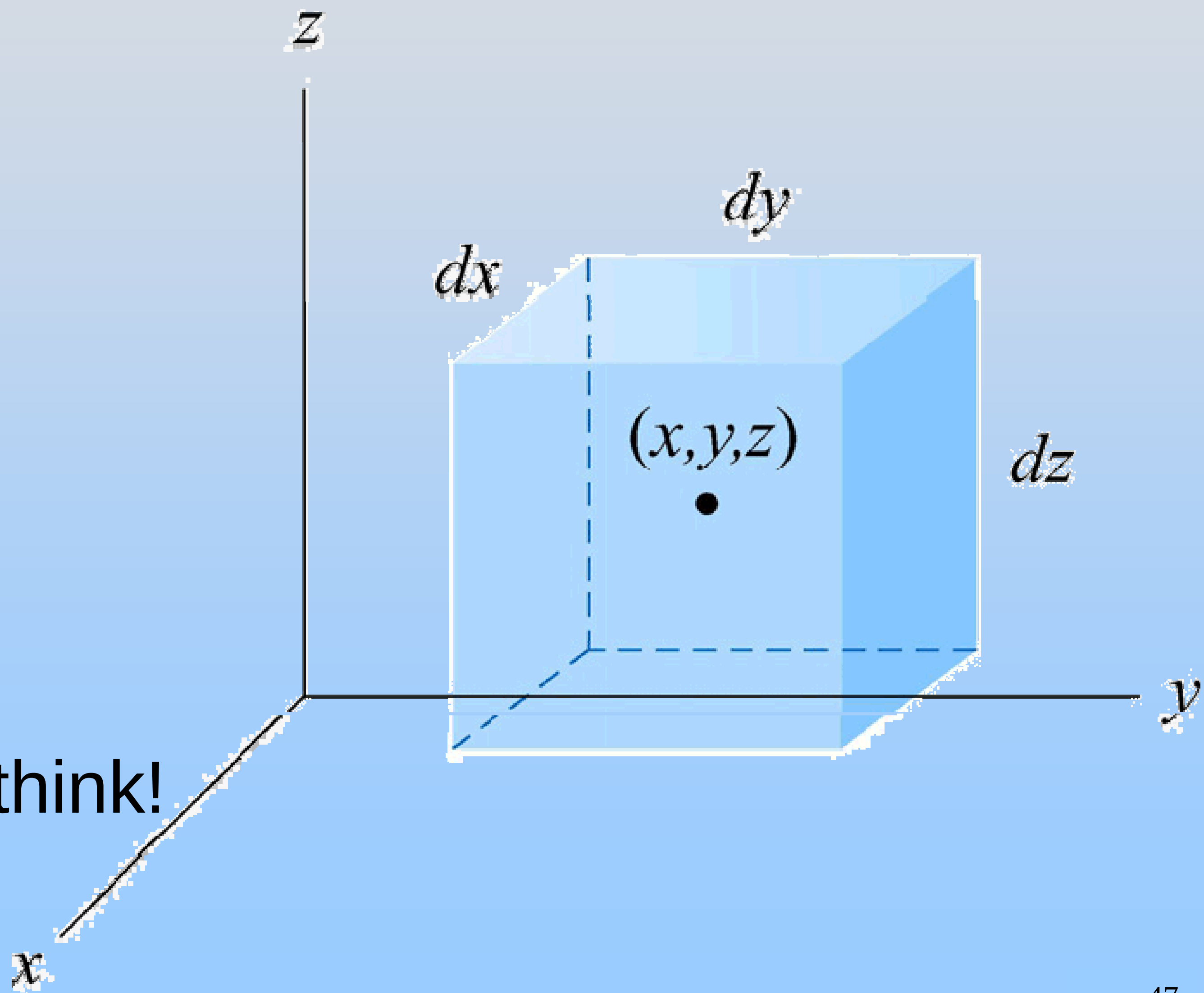
$$L = R\theta$$

Differentials

Rectangular Coordinates

$$dV - dx\,dy\,dz$$

$$dA = dx\,dy$$
$$dA = dx\,dz$$
$$dA = dy\,dz$$



Draw picture and think!

Differentials

<u>Cylindrical Coordinates</u>

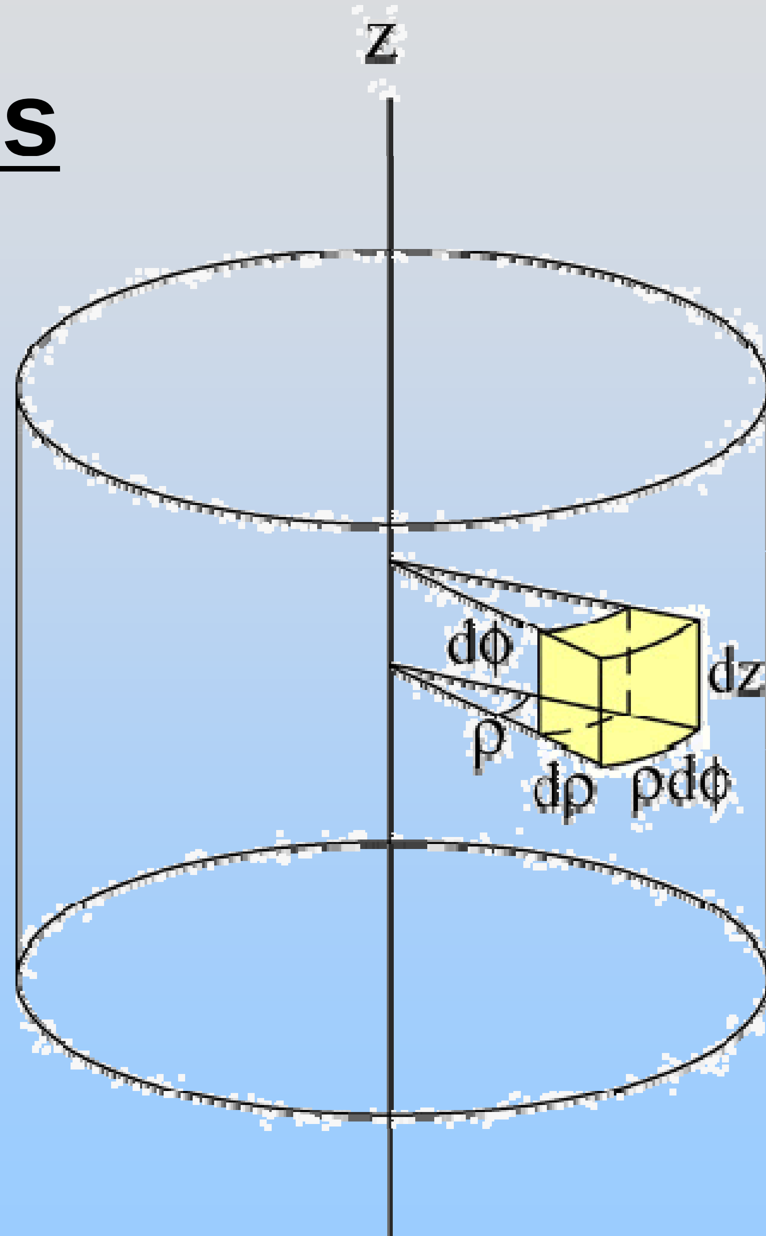
$$dV - \rho d\varphi \; d\rho \; dz$$

$$dA = \rho d\varphi \; dz$$

$$dA = \rho d\varphi \; d\rho$$

$$dA = d\rho \; dz$$

Draw picture and think!

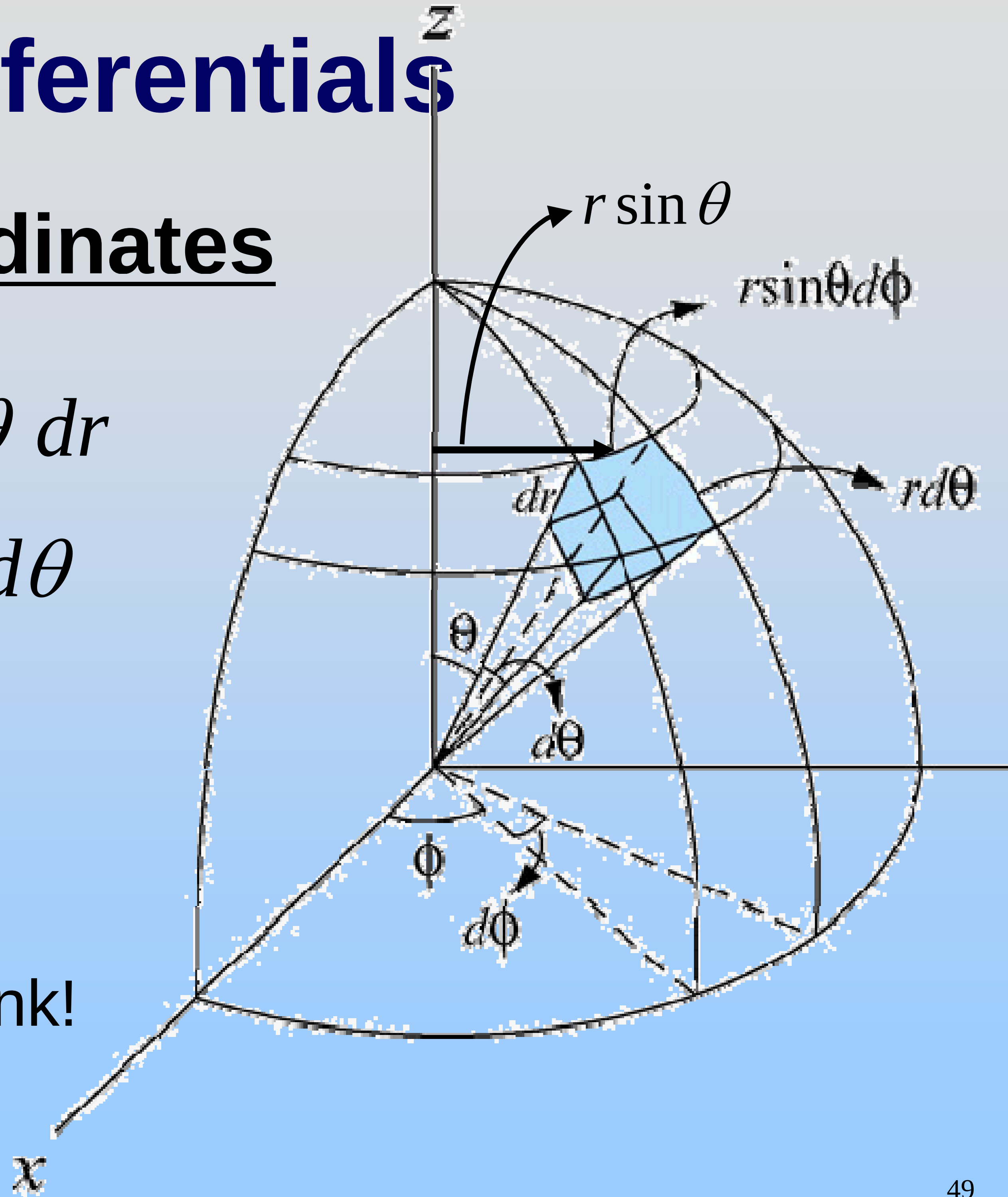


Differentials

Spherical Coordinates

$$dV - r\sin\theta d\varphi \; rd\theta \; dr$$

$$dA = r\sin\theta d\varphi \; rd\theta$$



Draw picture and think!

Electricity and Magnetism: Math Review

Vectors:

- Dot Product: How parallel?
- Cross Product: How perpendicular?

Derivatives:

- Rate of change (slope) of function
- Gradient tells you how to go up fast

Integrals:

- Visit each piece and ask contribution

8.02SC Physics II: Electricity and Magnetism
Fall 2010