Math Review
Challenge Problem Solutions

Problem 1:

Triangle Identity

Two sides of the triangle in Figure 1 form an angle θ. The sides have lengths a and b.

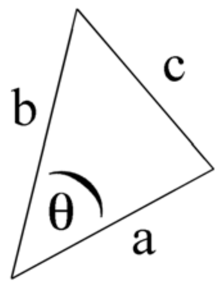


Figure 1: Law of cosines

The length of the opposite side is given by the relation triangle identity

$$c^2 = a^2 + b^2 - 2ab\cos\theta\,.$$

Suppose we describe the two given sides of the triangles by the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, with $\left|\vec{\mathbf{A}}\right| = a$ and $\left|\vec{\mathbf{B}}\right| = b$.

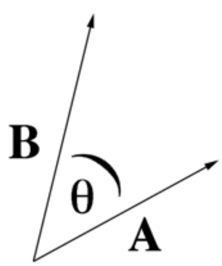


Figure 2: Vector construction

1) What is the geometric meaning of the vector $\vec{\mathbf{C}} = \vec{\mathbf{B}} - \vec{\mathbf{A}}$?
2) The square root of the dot product $\left|\vec{\mathbf{C}}\right| = \sqrt{\vec{\mathbf{C}} \cdot \vec{\mathbf{C}}}$ is the magnitude of the difference of the vectors. Show that the magnitude of the difference is the length of the opposite side of the triangle shown in figure 1, $\left|\vec{\mathbf{C}}\right| = c$.

Problem 2:

Dot and Cross products

Three vectors $\vec{\mathbf{A}}$ $\vec{\mathbf{B}}$, and $\vec{\mathbf{C}}$ form a geometric solid as shown in Figure 3. Show that the volume of the solid is equal to $\vec{\mathbf{C}}\cdot(\vec{\mathbf{A}}\times\vec{\mathbf{B}})$.

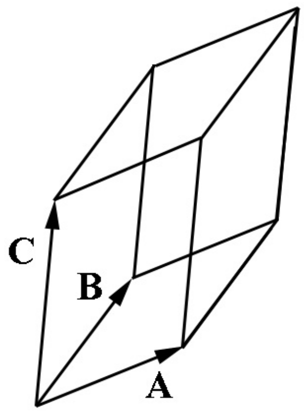
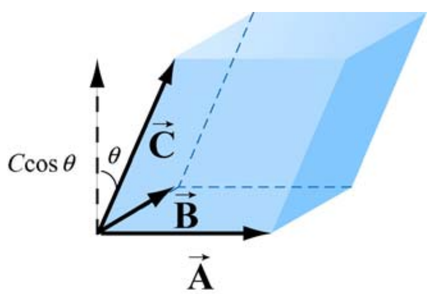


Figure 3: Volume

Problem 2 Solutions:
Volume of a Parallelepiped

(a) Consider the parallelepiped shown in the figure below:



As discussed in Review Module A, the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ form a parallelogram. The cross product $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ is a vector that points in the direction perpendicular to the parallelogram, and the magnitude $\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right|$ is equal to the area of the parallelogram. The volume, *V*, of the parallelepiped, is given by the product of the area of the parallelogram and the height of the parallelepiped. which is $C\cos\theta$ where $C=\left|\vec{\mathbf{C}}\right|$ and θ is the angle between the vector $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ and $\vec{\mathbf{C}}$. Thus, we have

$$V=\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right|\left|\vec{\mathbf{C}}\right|\cos\theta=\left(\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right)\cdot\vec{\mathbf{C}}$$

(b) By direction computation, the *triple product* $\left(\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right)\cdot\vec{\mathbf{C}}$ is

$$\begin{aligned}\left(\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right)\cdot\vec{\mathbf{C}} &= [(A_yB_z - A_zB_y)\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\hat{\mathbf{k}}]\cdot\left(C_x\hat{\mathbf{i}} + C_y\hat{\mathbf{j}} + C_z\hat{\mathbf{k}}\right)\\ &= (A_yB_z - A_zB_y)C_x + (A_zB_x - A_xB_z)C_y + (A_xB_y - A_yB_x)C_z\end{aligned}$$

On the other hand, the determinant is

$$\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = A_x(B_yC_z - B_zC_y) + A_y(B_zC_x - B_xC_z) + A_z(B_xC_y - B_yC_x)$$

With little algebra, one may show that the above two expressions are equal to each other. That is,

$$(\vec{\mathbf{A}}\times\vec{\mathbf{B}})\cdot\vec{\mathbf{C}} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Problem 3:

Two Vectors

Given two vectors, $\vec{\mathbf{A}} = (3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$ and $\vec{\mathbf{B}} = (5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$, evaluate the following:

(a) $3\vec{\mathbf{A}} + \vec{\mathbf{B}}$;
(b) $\vec{\mathbf{A}} - 4\vec{\mathbf{B}}$;
(c) $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$;
(d) $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$.
(e) What is the angle between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$?
(f) Find a unit vector perpendicular to $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$?

Problem 3 Solution:

With $\vec{\mathbf{A}} = (3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$ and $\vec{\mathbf{B}} = (5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$, we obtain the following results:

(a) $3\vec{\mathbf{A}} + \vec{\mathbf{B}} = 3\vec{A} = 3(3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) + (5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 14\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 20\hat{\mathbf{k}}$

(b) $\vec{\mathbf{A}} - 4\vec{\mathbf{B}} = (3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) - 4(5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = -17\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

(c) Since $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$ and $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$ (see Review Module A), the dot product is

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = (3)(5) + (-2)(1) + (6)(2) = 25$$

(d) With $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$, $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$ and $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$, the cross product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ of the two vectors is given by

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & -2 & 6 \\ 5 & 1 & 2 \end{vmatrix} = -10\hat{\mathbf{i}} + 24\hat{\mathbf{j}} + 13\hat{\mathbf{k}}$$

(e) The dot product of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = |\vec{\mathbf{A}}||\vec{\mathbf{B}}|\cos\theta$ where θ is the angle between the two vectors. With

$$A = |\vec{\mathbf{A}}| = \sqrt{3^2 + (-2)^2 + 6^2} = \sqrt{49} = 7$$
$$B = |\vec{\mathbf{B}}| = \sqrt{5^2 + 1^2 + 2^2} = \sqrt{30},$$

and using the result from part (c), we obtain

$$\cos\theta = \frac{\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}}{|\vec{\mathbf{A}}||\vec{\mathbf{B}}|} = \frac{25}{7\sqrt{30}} = 0.652 \quad \Rightarrow \quad \theta = 49.3°.$$

(f) The cross product $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ (or $\vec{\mathbf{B}}\times\vec{\mathbf{A}}$) is perpendicular to both $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. Therefore, from the result obtained in part (d), the unit vector may be obtained as

$$\hat{\mathbf{n}} = \pm\frac{\vec{\mathbf{A}}\times\vec{\mathbf{B}}}{|\vec{\mathbf{A}}\times\vec{\mathbf{B}}|} = \pm\frac{1}{\sqrt{(-10)^2+(24)^2+(13)^2}}\left(-10\hat{\mathbf{i}}+24\hat{\mathbf{j}}+13\hat{\mathbf{k}}\right) = \pm\frac{1}{\sqrt{845}}\left(-10\hat{\mathbf{i}}+24\hat{\mathbf{j}}+13\hat{\mathbf{k}}\right)$$

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