Introduction to Electric Fields
Challenge Problem Solutions

Problem 1:

Vector fields

Make a plot of the following vector fields:

(a) $\vec{\mathbf{v}} = 3\hat{\mathbf{i}} - 5\hat{\mathbf{j}}$

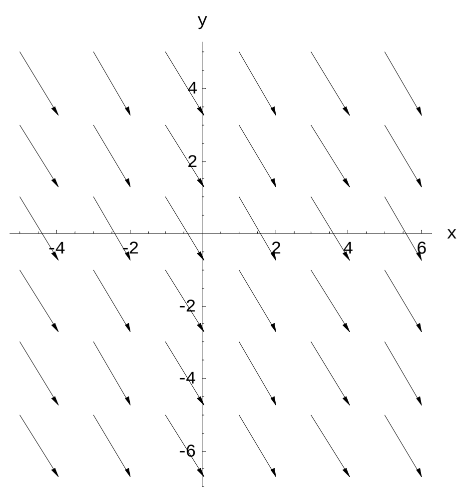
(b) $\vec{\mathbf{v}} = \vec{\mathbf{r}}$

(c) $\vec{\mathbf{v}} = \dfrac{\hat{\mathbf{r}}}{r^2}$

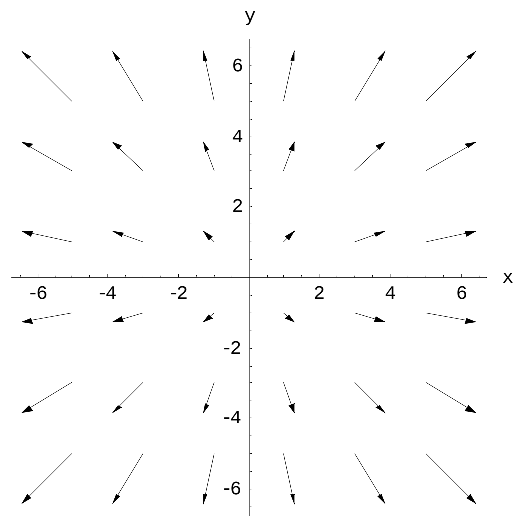
(d) $\vec{\mathbf{v}} = \dfrac{3xy}{r^5}\hat{\mathbf{i}} + \dfrac{2y^2 - x^2}{r^5}\hat{\mathbf{j}}$

Problem 1 Solutions:

(a) This is an example of a constant vector field in two dimensions. The plot is depicted below:



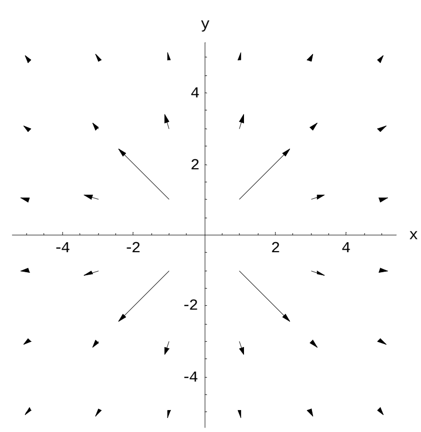
(b)



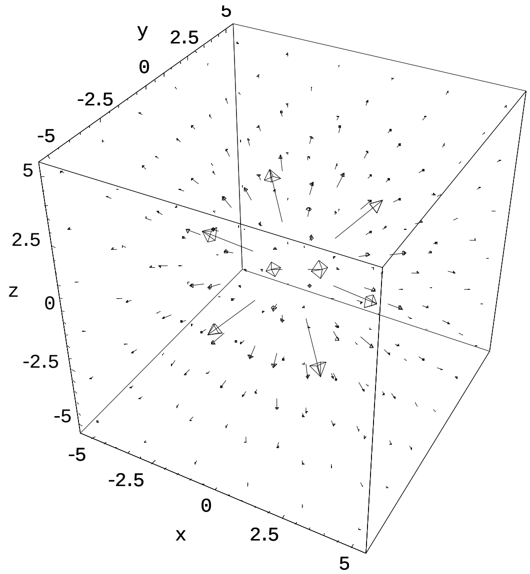
(c) In two dimensions, using the Cartesian coordinates where $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$, $\vec{\mathbf{v}}$ can be written as

$$\vec{\mathbf{v}} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\vec{\mathbf{r}}}{r^3} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{(x^2 + y^2)^{3/2}}$$

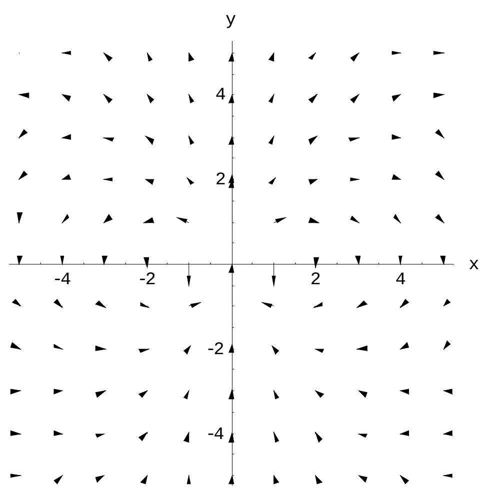
The plot is shown below:



Both the gravitational field of the Earth $\vec{\mathbf{g}}$ and the electric field $\vec{\mathbf{E}}$ due to a point charge have the same characteristic behavior as $\vec{\mathbf{v}}$. Note that in three dimensions, we would have $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, and the plot would look like



(d)



The plot is characteristic of the electric field due to a point electric dipole located at the origin.

Problem 2:

Scalar fields

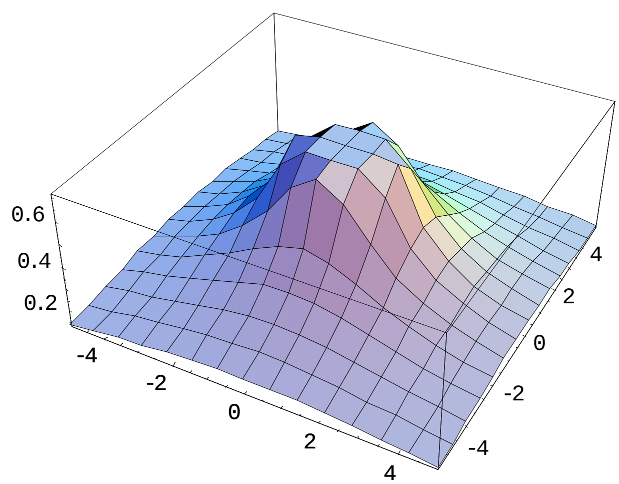
Make a plot of the following scalar functions in two dimensions:

(a) $f(r) = \dfrac{1}{r}$

(b) $f(x, y) = \dfrac{1}{\sqrt{x^2 + (y-1)^2}} - \dfrac{1}{\sqrt{x^2 + (y+1)^2}}$

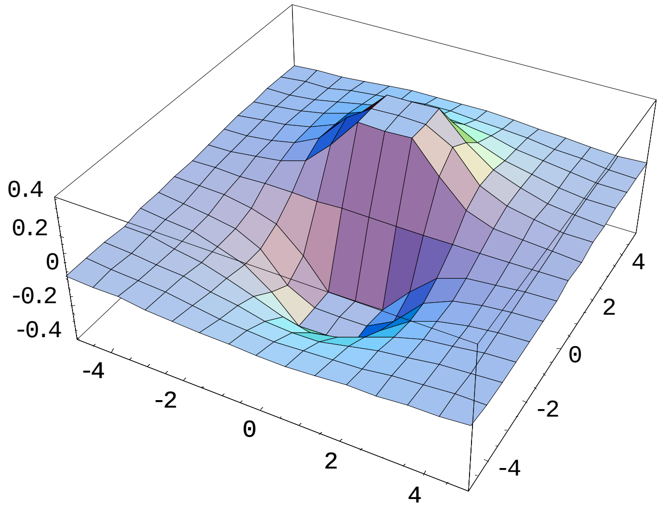
Problem 2 Solutions:

(a) In two dimensions, $r = \sqrt{x^2 + y^2}$



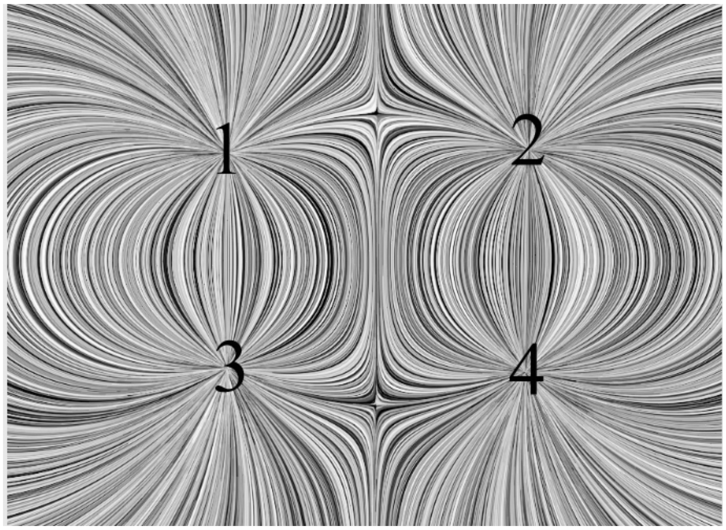
The plot represents the electric potential due to a point charge located at the origin.

(b)



This plot represents the potential due to a dipole with the positive charge located at $y = 1$ and the negative charge at $y = -1$.

Problem 3:



Above we show the grass seeds representation of the field of four point charges, located at the positions indicated by the numbers. Which statement is true about the signs of these charges:

a) All four charges have the same sign.

b) Charges 1 and 2 have the same sign, and that sign is opposite the sign of 3 and 4.

c) Charges 1 and 3 have the same sign, and that sign is opposite the sign of 2 and 4.

d) Charges 1 and 4 have the same sign, and that sign is opposite the sign of 2 and 3.

e) None of the above.

Problem 3 Solution:

b. Field lines continuously connect charges 1 and 3, and 2 and 4 respectively, indicating that the charge of those pairs are opposite in sign. The field is a zero between charges 1 and 2 indicating that they repel and hence are of the same sign. A smilar argument holds for charges 3 and 4.

Problem 4:

Equations of Field Lines

The "grass seeds" representation of the vector function $\vec{\mathbf{F}}(x,y) = y\,\hat{\mathbf{i}} + x^2\,\hat{\mathbf{j}}$ is shown below. We also show one of the field lines for this vector field[1].

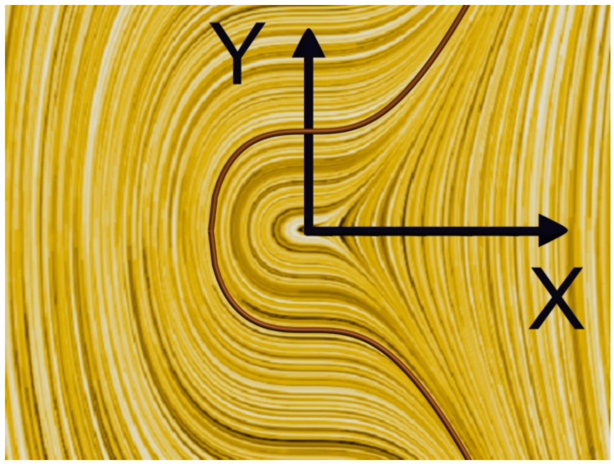


Figure 1: A grass seeds representation of the vector field given above and one of the field lines of the same field. See visualizations link for the grass seeds applet. Click on the figure above to get a movie of the family of field lines for this vector field.

Find by explicit construction the equation of the field lines for this vector field. That is, integrate the equation

$$\frac{dy}{dx} = \frac{F_y(x,y)}{F_x(x,y)}$$

to find the functions *y(x)* that satisfy this differential equation. To do this, set up the differential equation and then isolate all the variables depending on *y* on the left side of the equation with the differential *dy*, and all the variables depending on *x* on the right side of the equation, with the differential *dx*, and integrate (for example, the integral of (*y dy*) is ½ y^2, and so on).

We only show one of the field lines in the above figure. If you click on this figure you will download a movie that shows that field line as the constant of integration varies, thereby sweeping out the family of possible field lines.

Problem 4 Solutions:

From $\vec{\mathbf{F}}(x,y) = y^2\hat{\mathbf{i}} + 6\sin x\,\hat{\mathbf{j}}$, the *x*- and the *y*-components of the field are $F_x(x,y) = y^2$ and $F_y(x,y) = 6\sin x$. Therefore, we obtain the following differential equation:

$$\frac{dy}{dx} = \frac{F_y(x, y)}{F_x(x, y)} = \frac{6\sin x}{y^2}$$

or

$y^2\,dy = 6\sin x\,dx$

Integrating both sides, we have

$$\frac{y^3}{3} = -6\cos x + C$$

where C is the constant of integration.

Problem 5:

Ratio of Electric and Gravitational Forces **(10 points)**

What is the ratio of the magnitudes of the electrostatic force and the gravitational force between two protons if the protons are separated by a distance $r = 1.0\times10^{-15}\, m$? In SI units the magnitude of the charge of the proton is $e = 1.6\times10^{-19}\, C$ and the mass of the proton is $m_p = 1.67\times10^{-27}\, kg$.

Problem 5 Solution:

The ratio of the forces is given by

$$\frac{\left|\vec{\mathbf{F}}_{elec}\right|}{\left|\vec{\mathbf{F}}_{grav}\right|} = \frac{ke^2 / r^2}{Gm_p^2 \big/ r^2} = \frac{ke^2}{Gm_p^2} = \frac{(9.0\times10^9\ \mathrm{N}\times\mathrm{m}^2\times\mathrm{C}^{-2})(1.6\times10^{-19}\ C)^2}{(6.67\times10^{-11}\ \mathrm{N}\times\mathrm{m}^2\times\mathrm{kg}^{-2})(1.67\times10^{-27}\ kg)^2} = 1.2\times10^{36}$$

Problem 6:

Bohr Theory

In the Bohr theory of the hydrogen atom, an electron moves in a circular orbit about a proton, where the radius of the orbit is $0.529\times10^{-10}\,\text{m}$.

(a) Find the electric force between the two.

(b) If this force causes the centripetal acceleration of the electron, what is the speed of the electron?

Problem 6 Solutions:

(a) Using Coulomb's law, the force between the proton and the electron is

$$F=\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}=\left(8.99\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\frac{\left(1.60\times10^{-19}\ \text{C}\right)^2}{\left(0.529\times10^{-10}\ \text{m}\right)^2}=8.22\times10^{-8}\ \text{N}$$

(b) Using $F=\dfrac{mv^2}{r}$, the speed of the electron is

$$v=\sqrt{\frac{Fr}{m}}=\sqrt{\frac{8.22\times10^{-8}\ \text{N}\left(0.529\times10^{-10}\ \text{m}\right)}{9.11\times10^{-31}\ \text{kg}}}=2.19\times10^{6}\ \text{m/s}$$

Problem 7:

Two objects with charges $-q$ and $+3q$ are placed on a line as shown in the figure below.

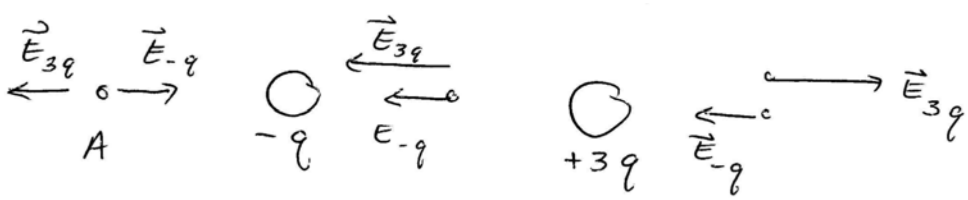
Besides an infinite distance away from the charges, where else can the electric field possibly be zero?

1. Between the two charges.
2. To the right of the charge on the right.
3. To the left of the charge on the left.
4. The electric field is only zero an infinite distance away from the charges.

Explain your reasoning

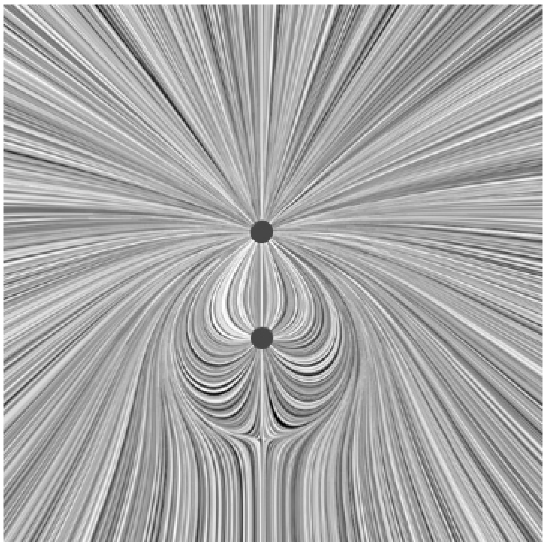
Problem 7 Solution:

3. The electric field is the vector sum of the electric fields due to each charged object. There are two properties that determine the strength of the electric field, distance from the source (the strength of the field is proportional to $1/r^2$), and the magnitude of the charge (the strength of the field is proportional to q). In the figure below the electric fields of the two objects are shown at several points. At the point A to the left of the charged object on the left, the vectors point in opposite directions. Since the point A is closer to the object with charge $-q$ than the object with charge $+3q$, these two properties can balance and the vectors can add to zero. Whereas on the right, both properties contribute to making the field due to the object with charge $+3q$ larger than the field due to the object with charge $-q$, and then cannot possibly sum to zero. In the region between the objects the electric vectors both point to the left so they cannot sum to zero.

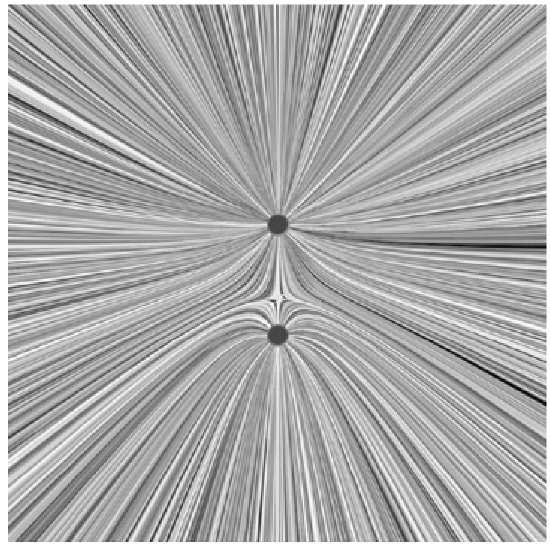


Problem 8:

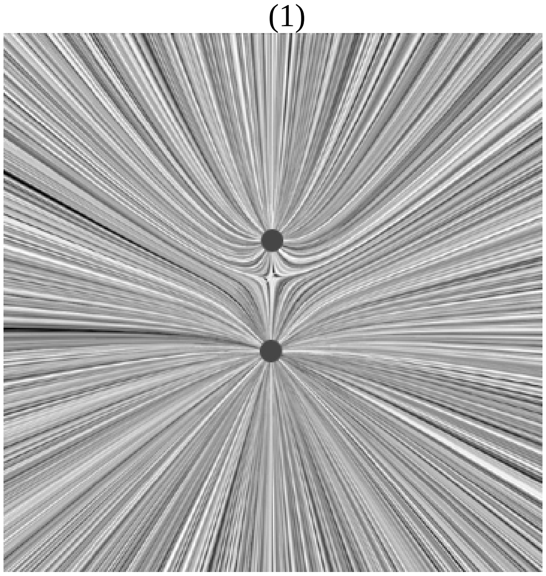
Two objects with charges $-4Q$ and $-Q$ lie on the y-axis. The object with the charge $-4Q$ is ***above*** the object with charge $-Q$. Below are four possible "grass seed" representations of the electric field of the two charges. Which of these representations is most nearly right for the two charges in this problem?



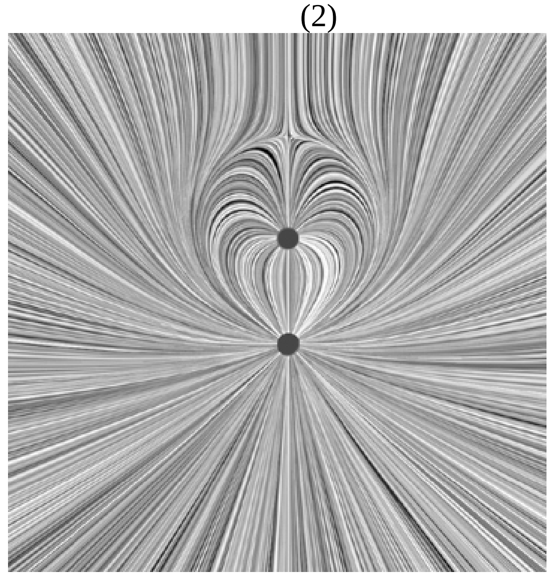
(1)



(2)



(3)



(4)

Explain your reasoning.

Problem 8 Solution:

(2) Both sources have negative charge so the field lines very near each source must point towards that source. Therefore there must be a point between the sources where the field is zero. (This eliminates figures (1) and (4).) The zero of the field must be closer to the

weaker source in order to cancel the field from the stronger source that is further away. The weaker source is below the stronger source, so the figure (2) is the correct ‘grass seed field’ representation of the electric field of both sources.