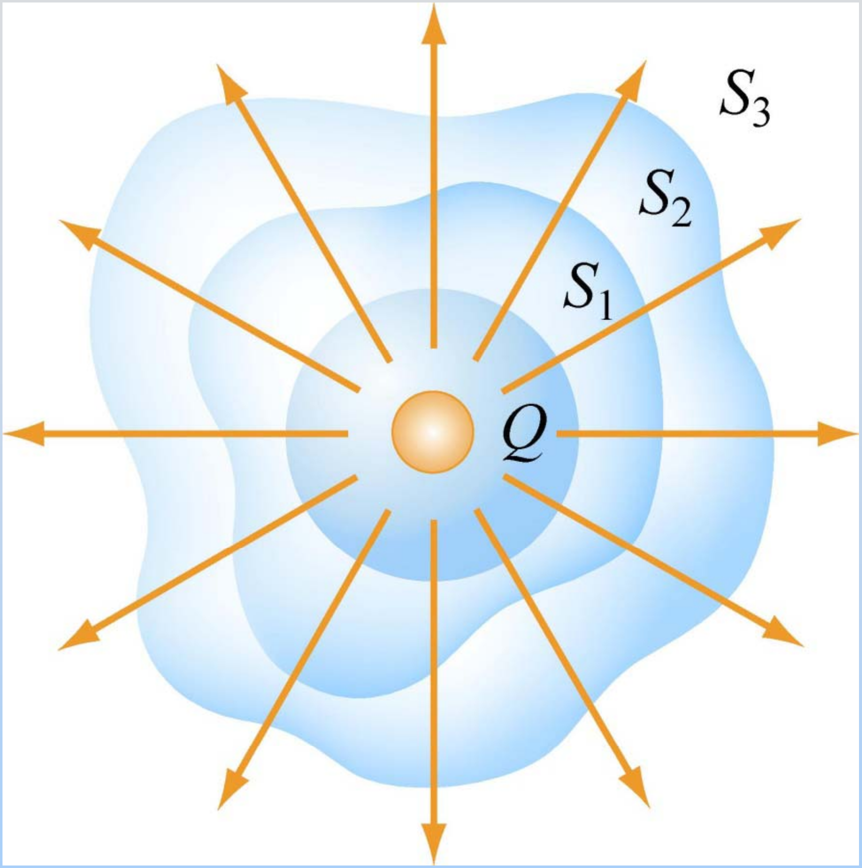
Module 05: Gauss’s Law

Gauss's Law

The first Maxwell Equation!

And a very useful computational technique to find the electric field E when the source has 'enough symmetry'.

Gauss's Law – The Idea



The total "flux" of field lines penetrating any of these closed surfaces is the same and depends only on the amount of charge inside

Gauss's Law – The Equation

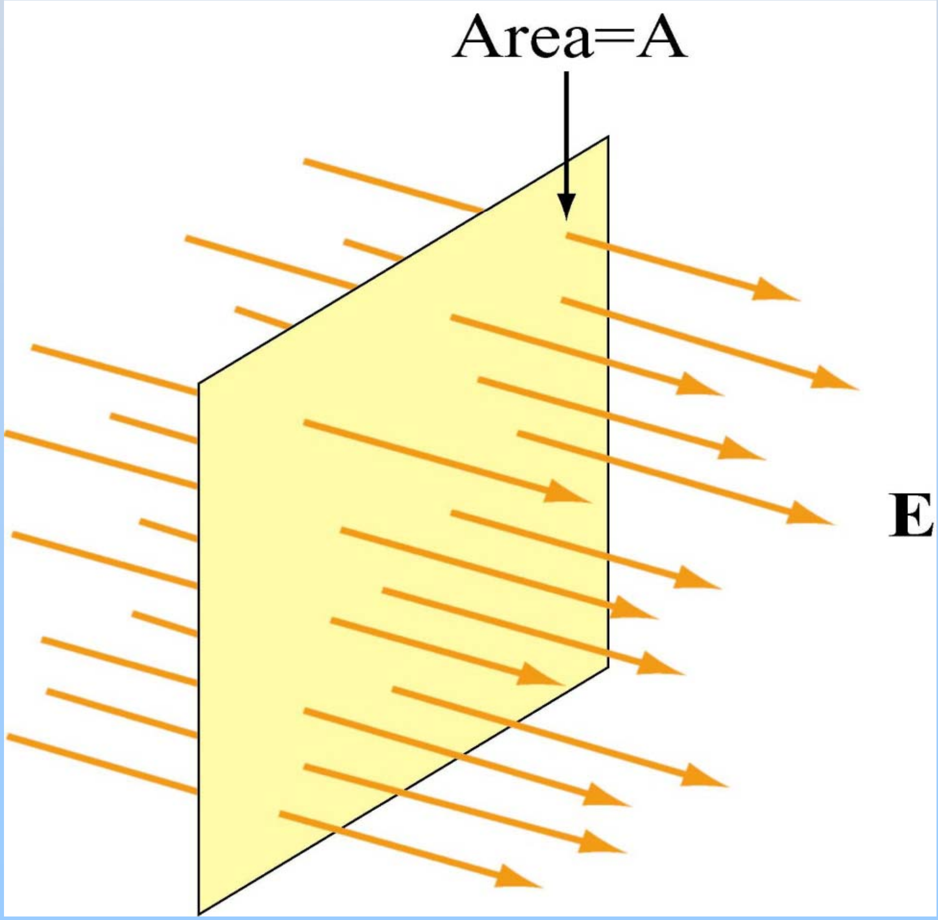
$$\Phi_E = \oint\!\!\!\oint_{\text{closed surface } S} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{in}}{\varepsilon_0}$$

Electric flux Φ_E (the surface integral of E over closed surface *S*) is proportional to charge inside the volume enclosed by *S*

Now the Details

Electric Flux Φ_E

Case I: E is constant vector field perpendicular to planar surface S of area A



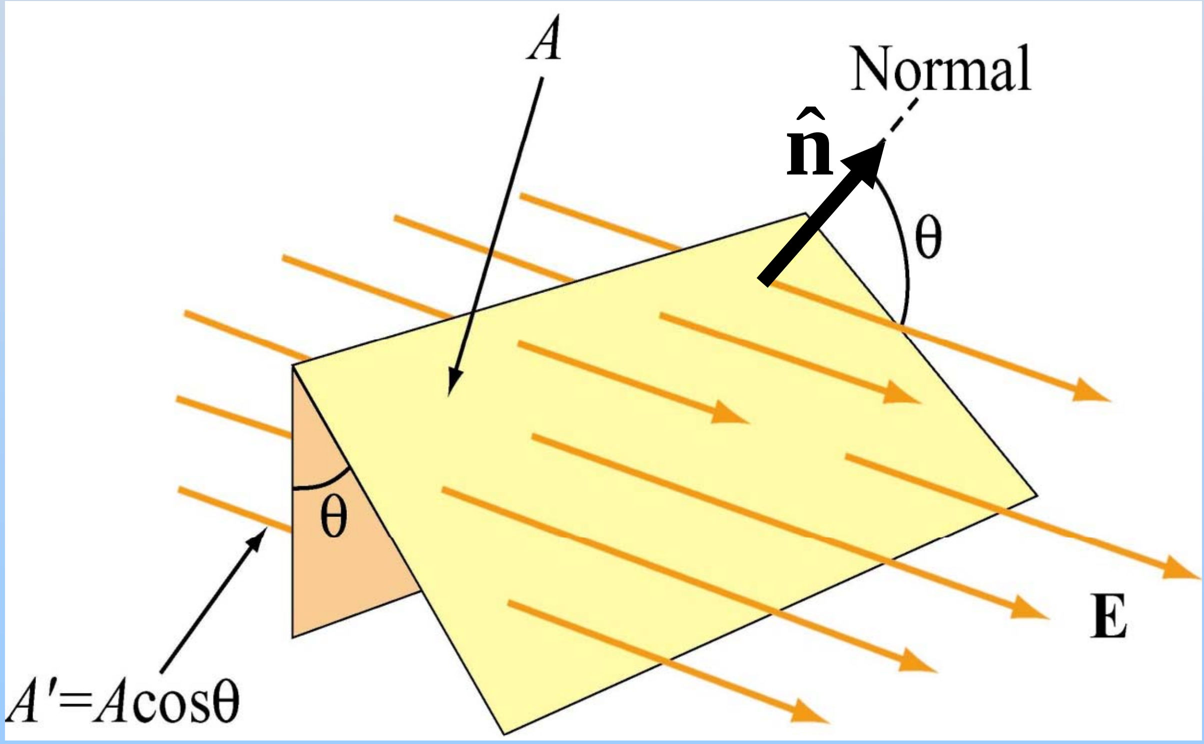
$$\Phi_E = \iint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

$$\Phi_E = \pm EA$$

Our Goal: Always reduce problem to this

Electric Flux Φ_E

Case II: E is constant vector field directed at angle θ to planar surface S of area A



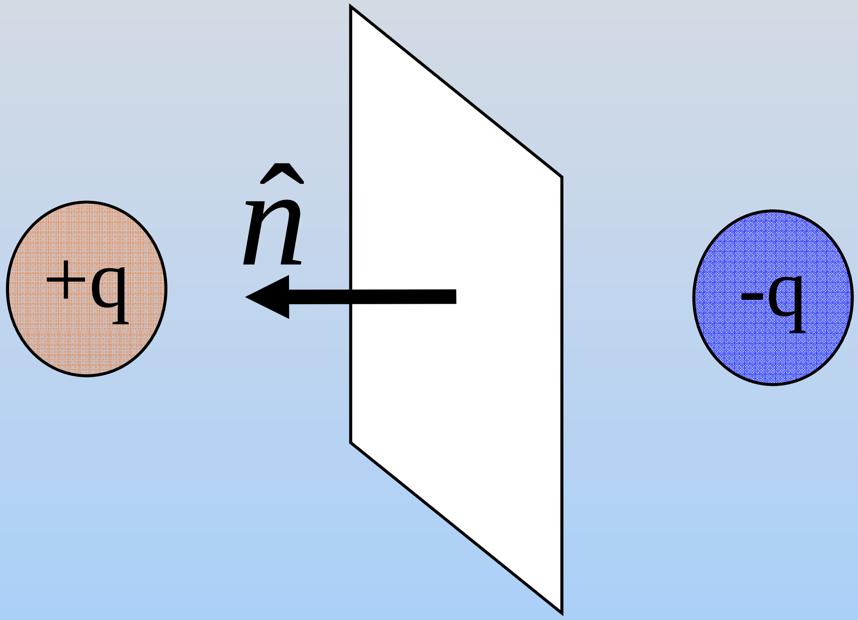
$$\Phi_E = \iint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

$$d\vec{\mathbf{A}} = dA\,\hat{\mathbf{n}}$$

$$\Phi_E = EA\cos\theta$$

Concept Question: Flux

The electric flux through the planar surface below (positive unit normal to left) is:



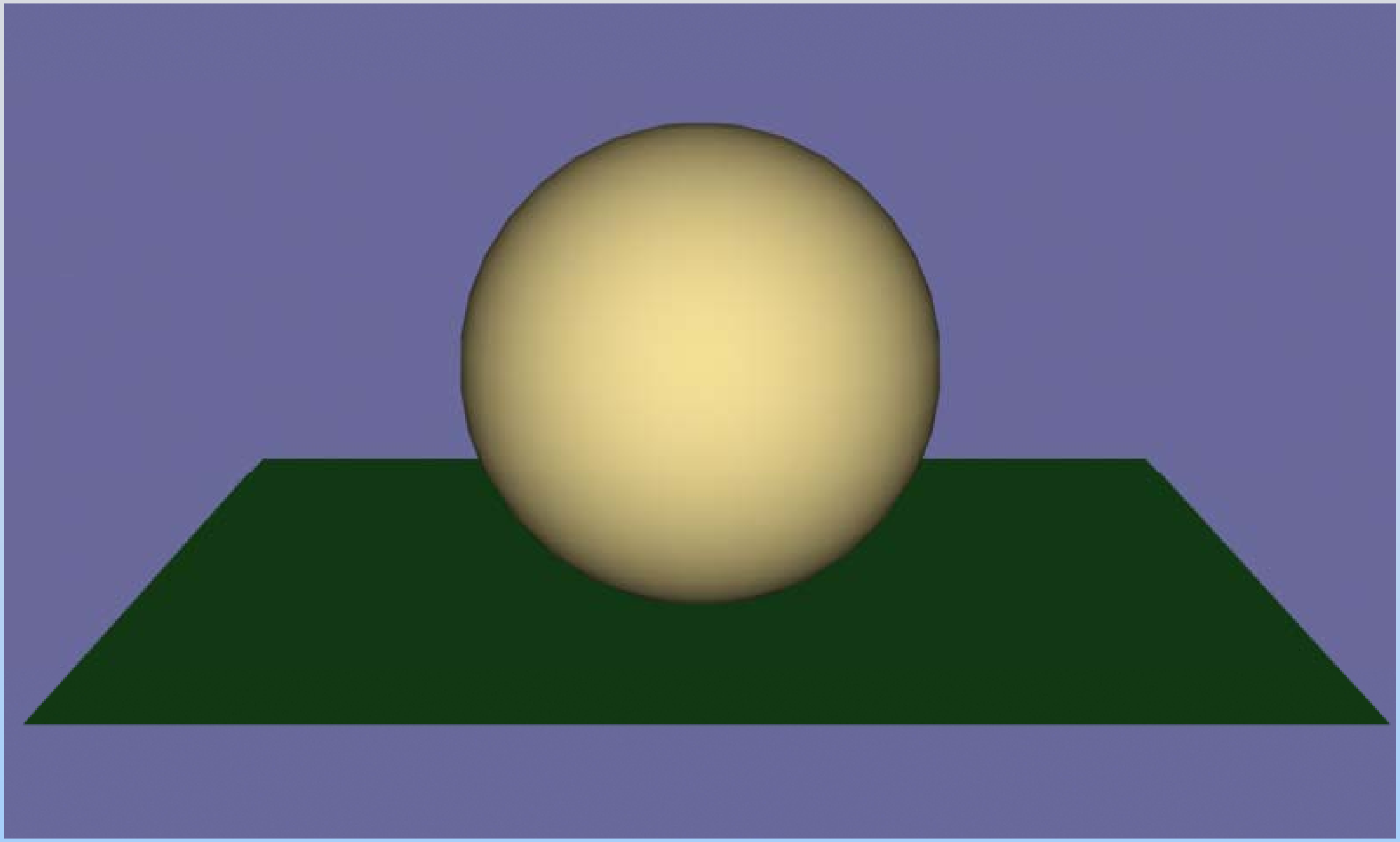
1. positive.
2. negative.
3. zero.
4. I don't know

Gauss's Law

$$\Phi_E = \underset{\text{closed surface}S}{\oiint} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{in}}{\varepsilon_0}$$

Note: Integral must be over closed surface

Open and Closed Surfaces



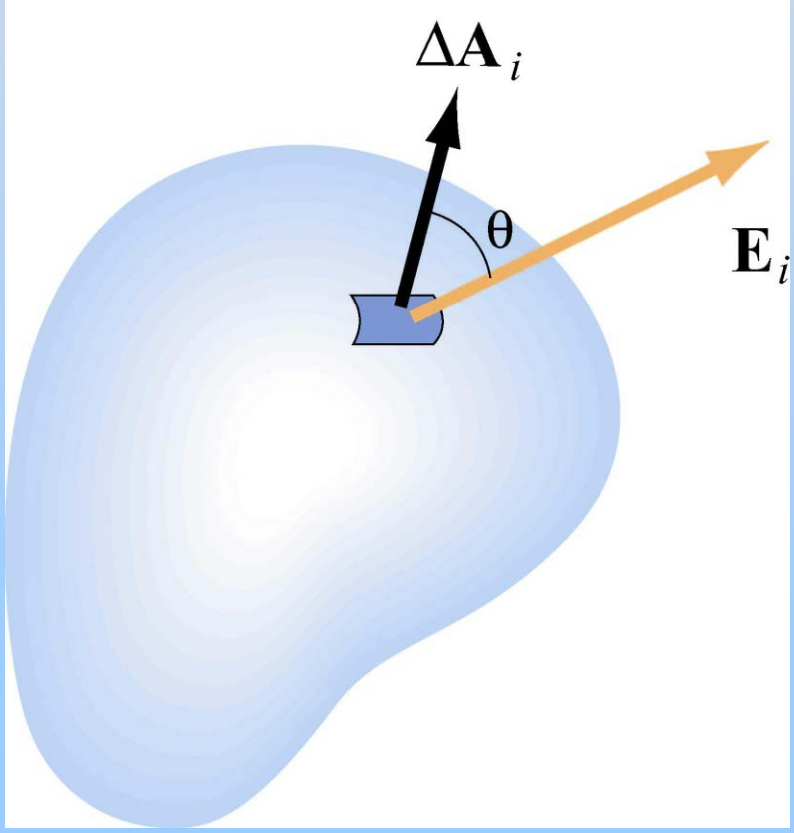
A rectangle is an open surface — it does NOT contain a volume

A sphere is a closed surface — it DOES contain a volume

Area Element dA: Closed Surface

For closed surface, d**A** is normal to surface and points outward

(from inside to outside)

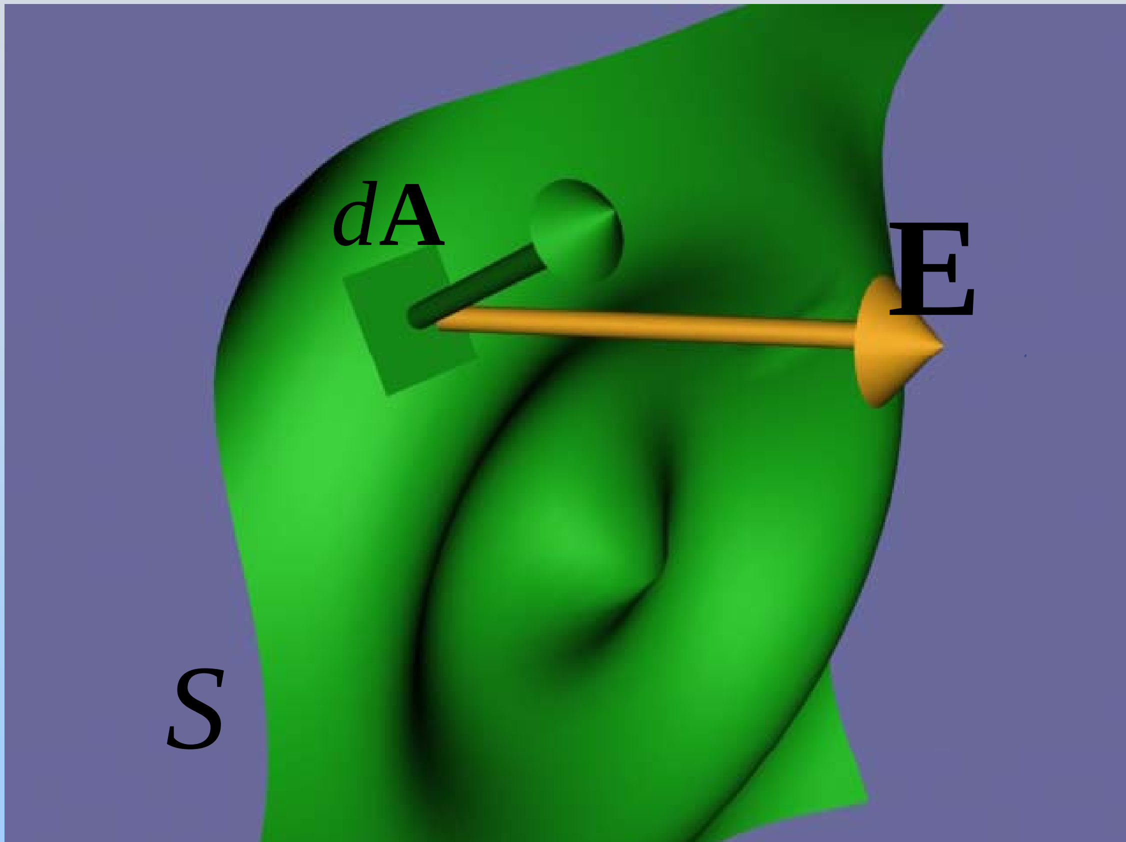


$\Phi_E > 0$ if E points out

$\Phi_E < 0$ if E points in

Electric Flux Φ_E

Case III: E not constant, surface curved

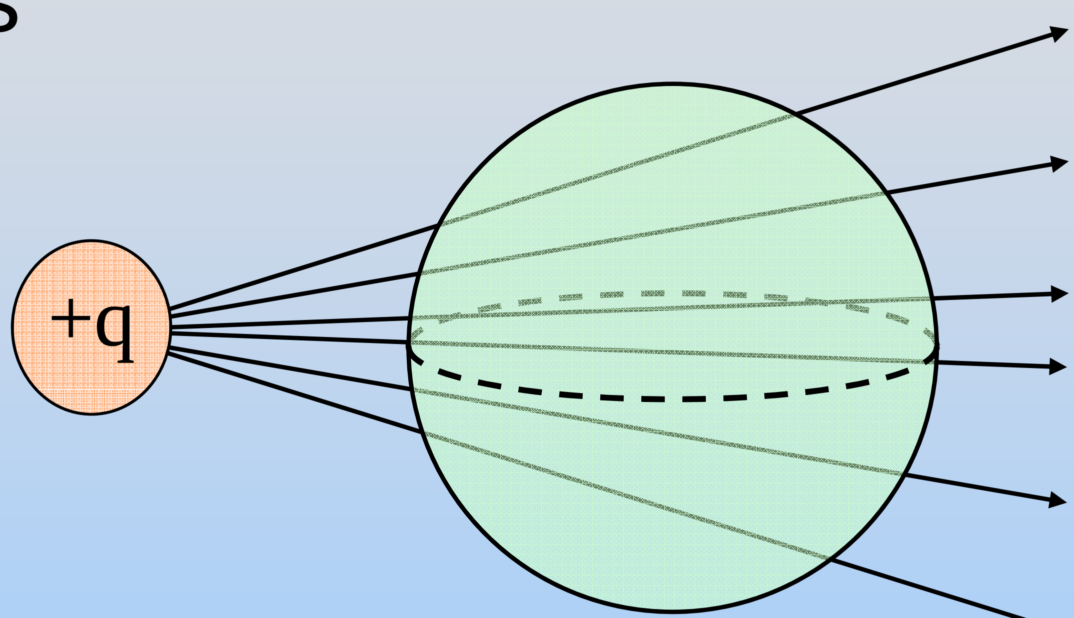


$$d\Phi_E = \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

$$\Phi_E - \iint d\Phi_E$$

Concept Question: Flux thru Sphere

The total flux through the below spherical surface is



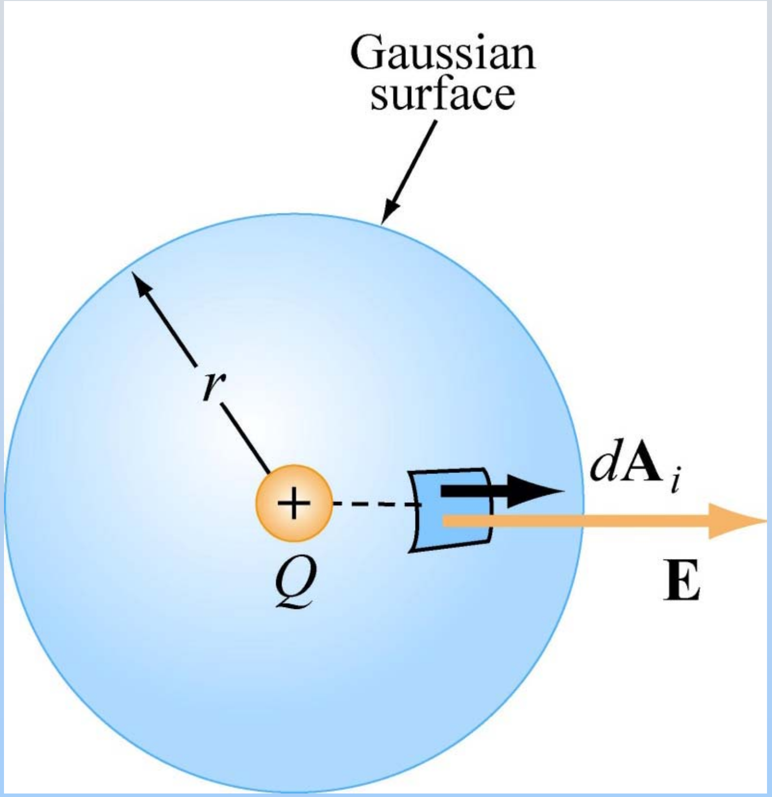
1. positive (net outward flux).
2. negative (net inward flux).
3. zero.
4. I don't know

Electric Flux: Sphere

Point charge ***Q*** at center of sphere, radius *r*

E field at surface:

$$\vec{\mathbf{E}}(r) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$



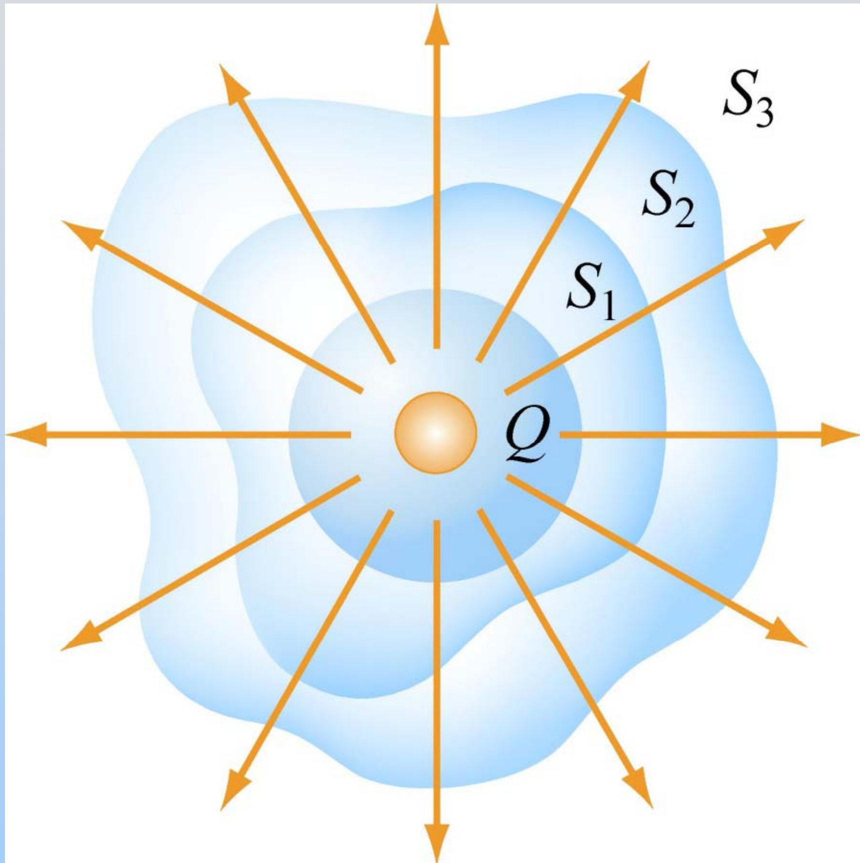
Electric flux through sphere:

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oiint_S \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \cdot dA\hat{\mathbf{r}}$$

$$= \frac{Q}{4\pi\varepsilon_0 r^2}\oiint_S dA = \frac{Q}{4\pi\varepsilon_0 r^2}4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$\boxed{d\vec{\mathbf{A}} = dA\hat{\mathbf{r}}}$$

Arbitrary Gaussian Surfaces

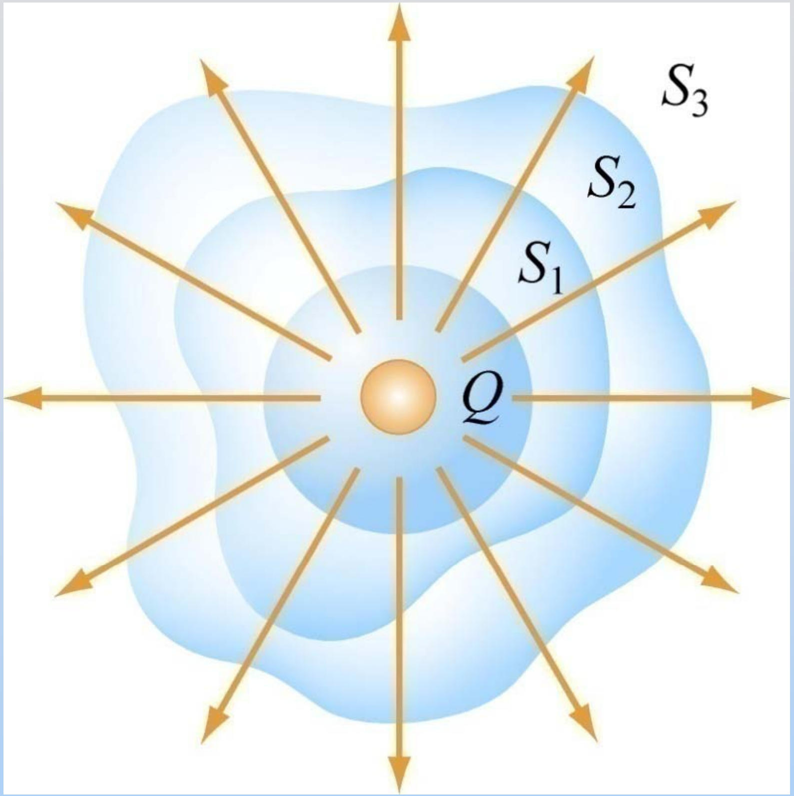


$$\Phi_E = \underset{\text{closed surface S}}{\oint\!\!\oint} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q}{\varepsilon_0}$$

True for all surfaces such as S_1, S_2 or S_3

Why? As A gets bigger E gets smaller

Choosing Gaussian Surface



$$\Phi_E = \oiint_{\text{closed surface } S} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q}{\varepsilon_0}$$

True for ALL surfaces

Useful (to calculate E) for SOME surfaces

Desired **E**: Perpendicular to surface and constant on surface.

Flux is EA or -EA.

Other **E:** Parallel to surface.

Flux is zero

Symmetry & Gaussian Surfaces

Desired **E**: perpendicular to surface and constant on surface. So Gauss's Law useful to calculate E field from **highly symmetric sources**

Source Symmetry	**Gaussian Surface**
Spherical	Concentric Sphere
Cylindrical	Coaxial Cylinder
Planar	Gaussian "Pillbox"

Applying Gauss's Law

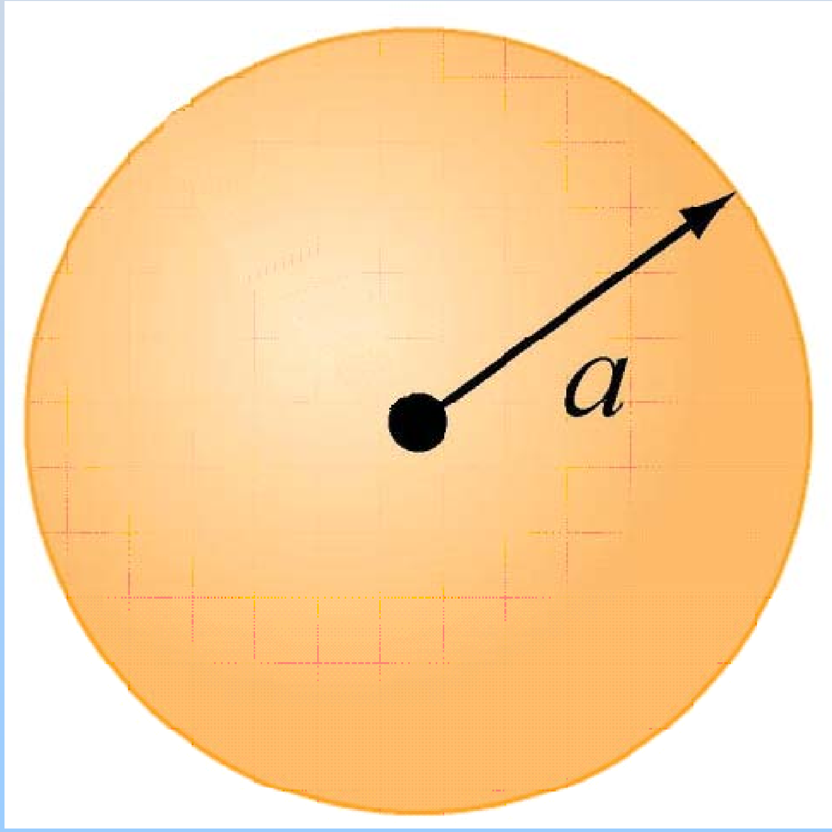
1. Based on the source, identify regions in which to calculate E field.
2. Choose Gaussian surfaces S: Symmetry
3. Calculate $\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$
4. Calculate q_{in}, charge enclosed by surface S
5. Apply Gauss's Law to calculate E:

$$\Phi_E = \oiint_{\text{closed surface}S} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{in}}{\varepsilon_0}$$

Examples:
Spherical Symmetry
Cylindrical Symmetry
Planar Symmetry

Gauss: Spherical Symmetry

+Q uniformly distributed throughout non-conducting solid sphere of radius a. Find **E** everywhere

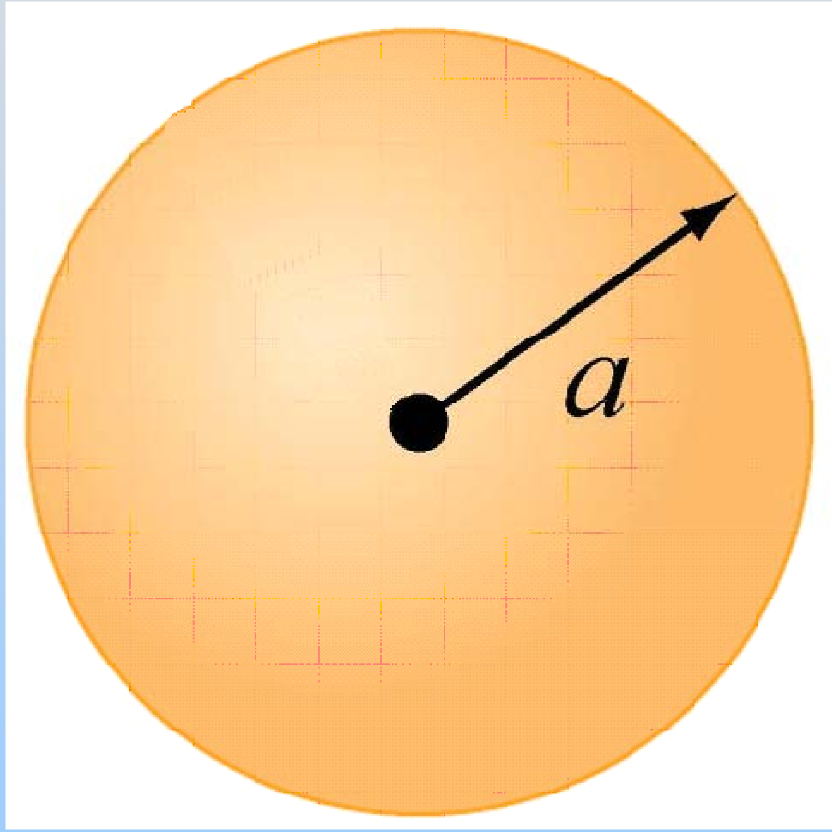


Gauss: Spherical Symmetry

Symmetry is Spherical

$$\vec{\mathbf{E}} = E\,\hat{\mathbf{r}}$$

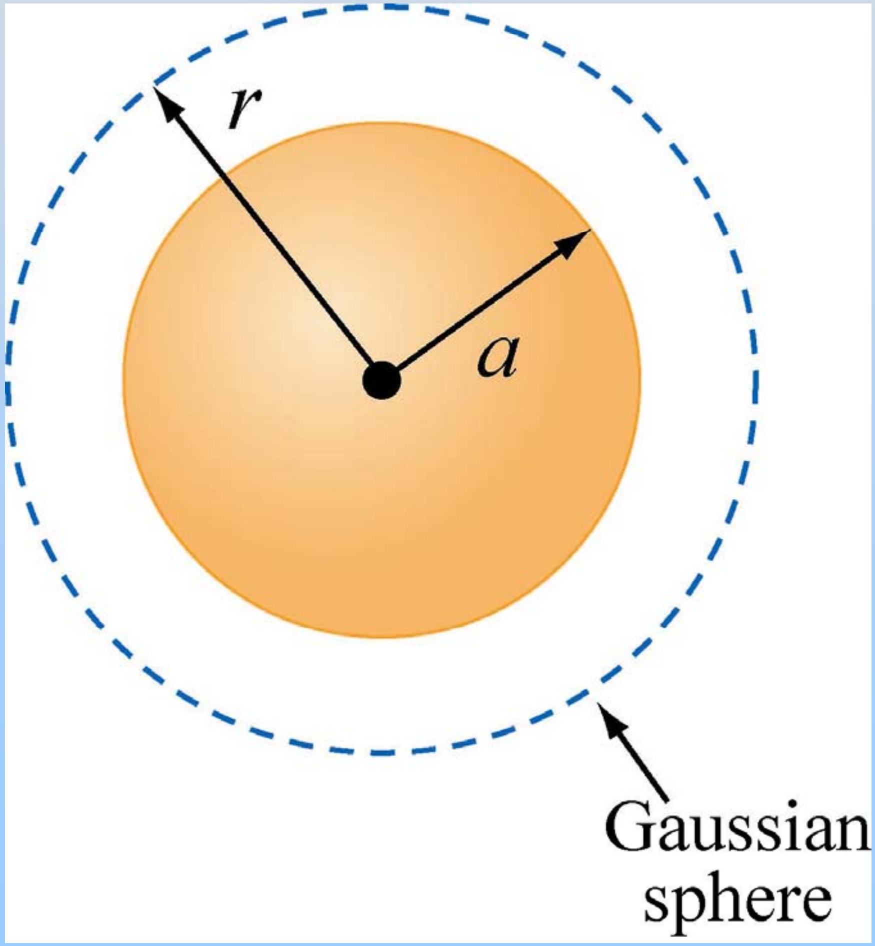
Use Gaussian Spheres



Gauss: Spherical Symmetry

Region 1: $r > a$

Draw Gaussian Sphere in Region 1 ($r > a$)

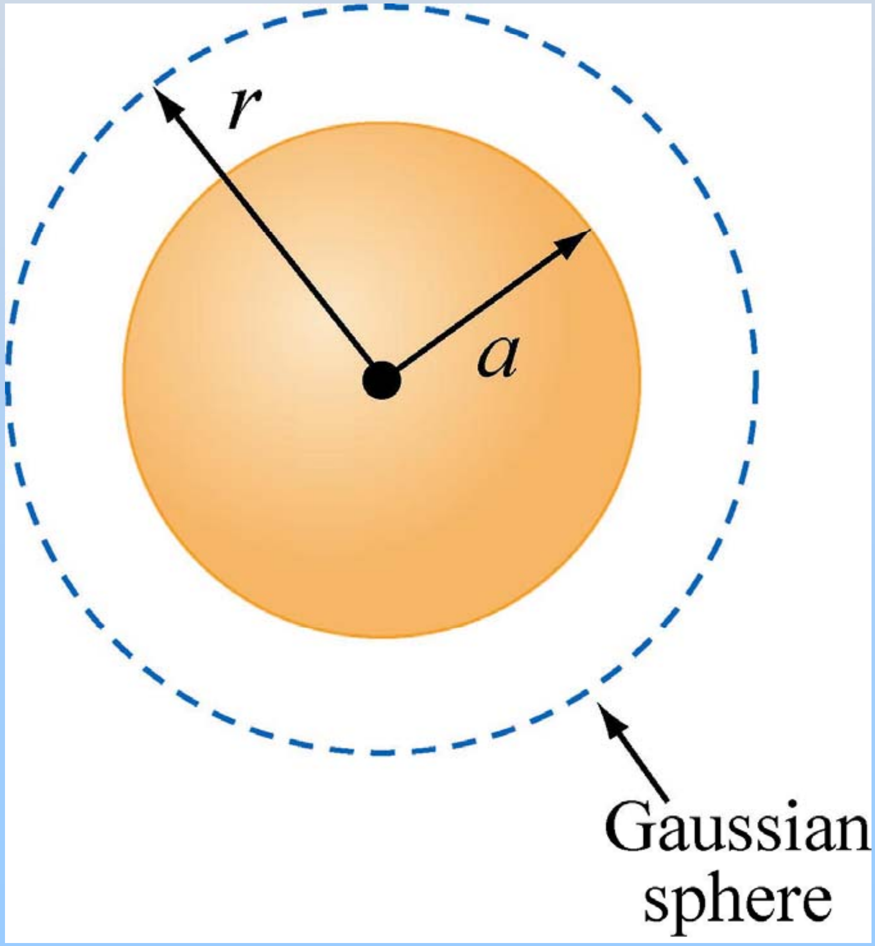


Note: *r* is arbitrary **but** is the radius for which you will calculate the E field!

Problem: Outside Sphere

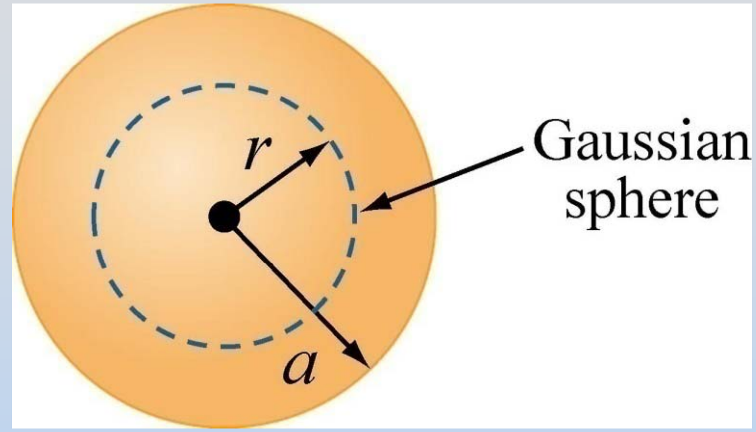
Region 1: $r > a$

Use Gauss's Law in Region 1 ($r > a$)



Again: Remember that *r* is arbitrary **but** is the radius for which you will calculate the E field!

Gauss: Spherical Symmetry



Region 2: $r < a$

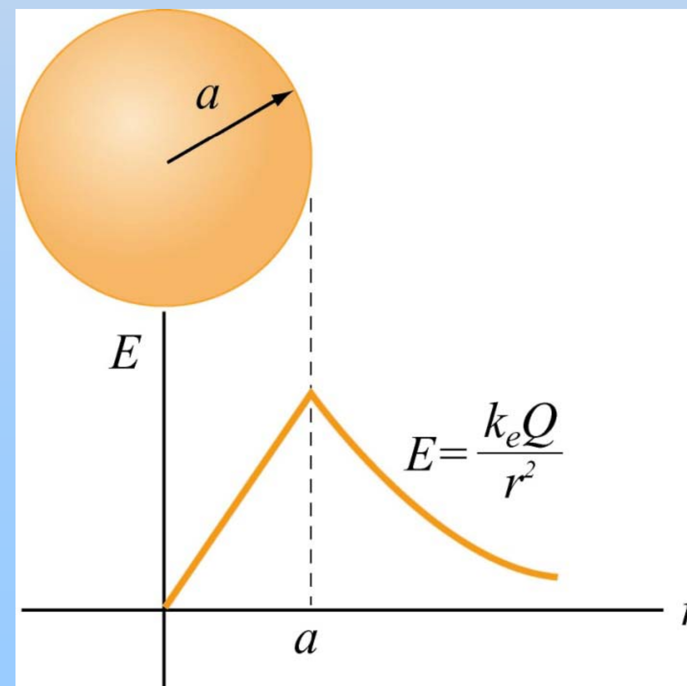
Total charge enclosed:

$$q_{in} = \left(\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi a^3} \right) Q = \left(\frac{r^3}{a^3} \right) Q \quad \text{OR} \quad q_{in} = \rho V$$

Gauss's law:

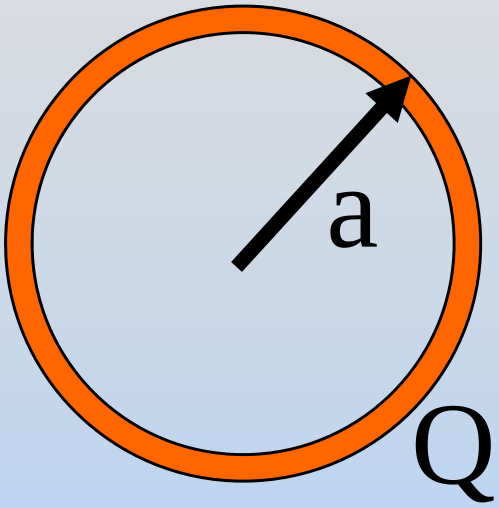
$$\Phi_E = E\left(4\pi r^2\right) = \frac{q_{in}}{\varepsilon_0} = \left(\frac{r^3}{a^3} \right) \frac{Q}{\varepsilon_0}$$

$$E = \frac{Q}{4\pi\varepsilon_0} \frac{r}{a^3} \quad \boxed{\Rightarrow \vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_0} \frac{r}{a^3} \hat{\mathbf{r}}}$$



Concept Question: Spherical Shell

We just saw that in a solid sphere of charge the electric field grows linearly with distance. Inside the charged spherical shell at right (r<a) what does the electric field do?



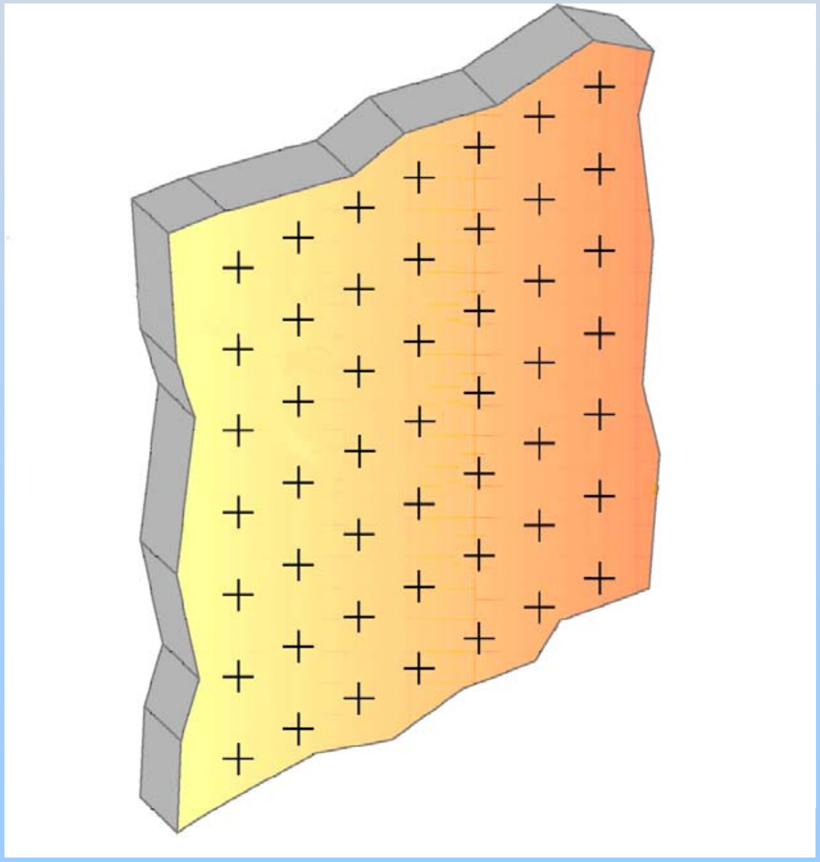
1. Constant and Zero
2. Constant but Non-Zero
3. Still grows linearly
4. Some other functional form (use Gauss' Law)
5. Can't determine with Gauss Law

Demonstration
Field Inside Spherical Shell
(Grass Seeds):

Gauss: Planar Symmetry

Infinite slab with uniform charge density σ

Find **E** outside the plane



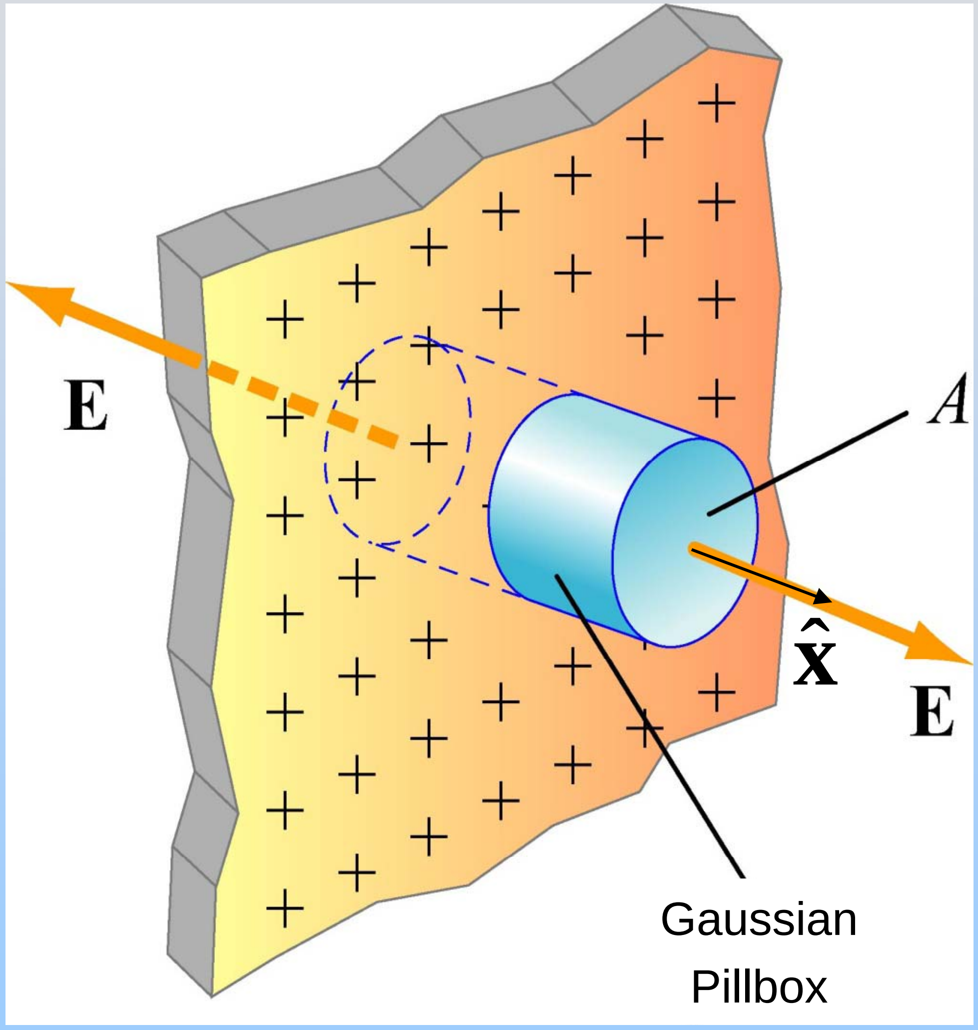
Gauss: Planar Symmetry

Symmetry is Planar

$$\vec{\mathbf{E}} = \pm E\,\hat{\mathbf{x}}$$

Use Gaussian Pillbox

Note: *A* is arbitrary (its size and shape) and should divide out



Gauss: Planar Symmetry

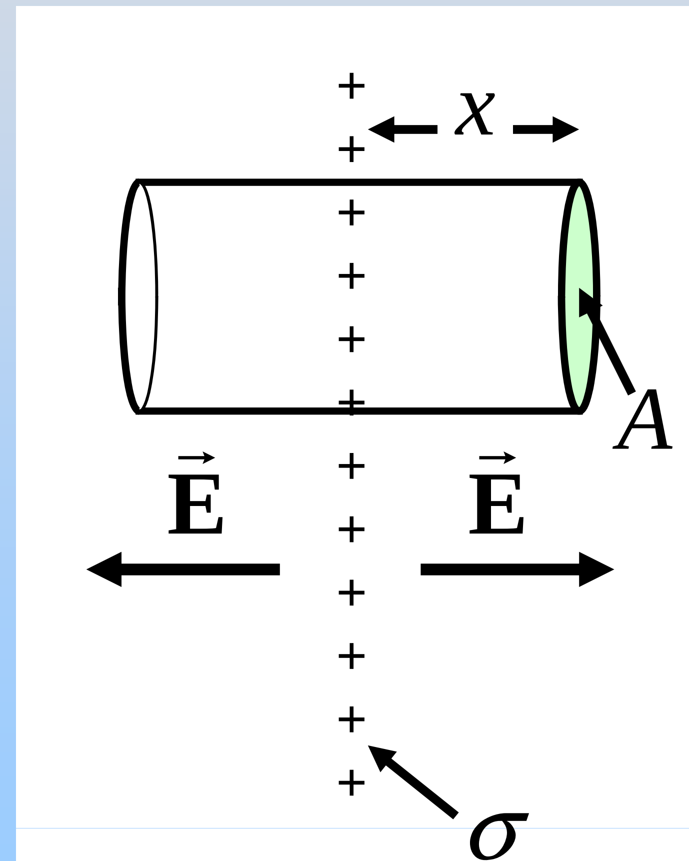
Total charge enclosed: $q_{in} = \sigma A$

NOTE: No flux through side of cylinder, only endcaps

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E \oiint_S dA = EA_{Endcaps}$$

$$= E(2A) = \frac{q_{in}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

$$E = \frac{\sigma}{2\varepsilon_0} \Rightarrow \vec{\mathbf{E}} = \frac{\sigma}{2\varepsilon_0} \begin{cases} \hat{\mathbf{x}} & \text{to right} \\ -\hat{\mathbf{x}} & \text{to left} \end{cases}$$



E for Plane is Constant????

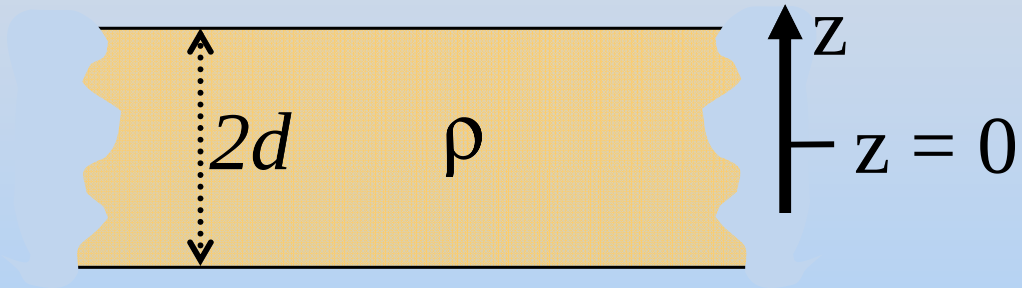
1) Dipole: E falls off like $1/r^3$
2) Point charge: E falls off like $1/r^2$
3) Line of charge: E falls off like $1/r$
4) Plane of charge: E constant

Concept Question: Slab of Charge

Consider positive, semi-infinite (in x & y) flat slab

z-axis is perp. to the sheet, with center at z = 0.

At the plane's center (z = 0), **E**



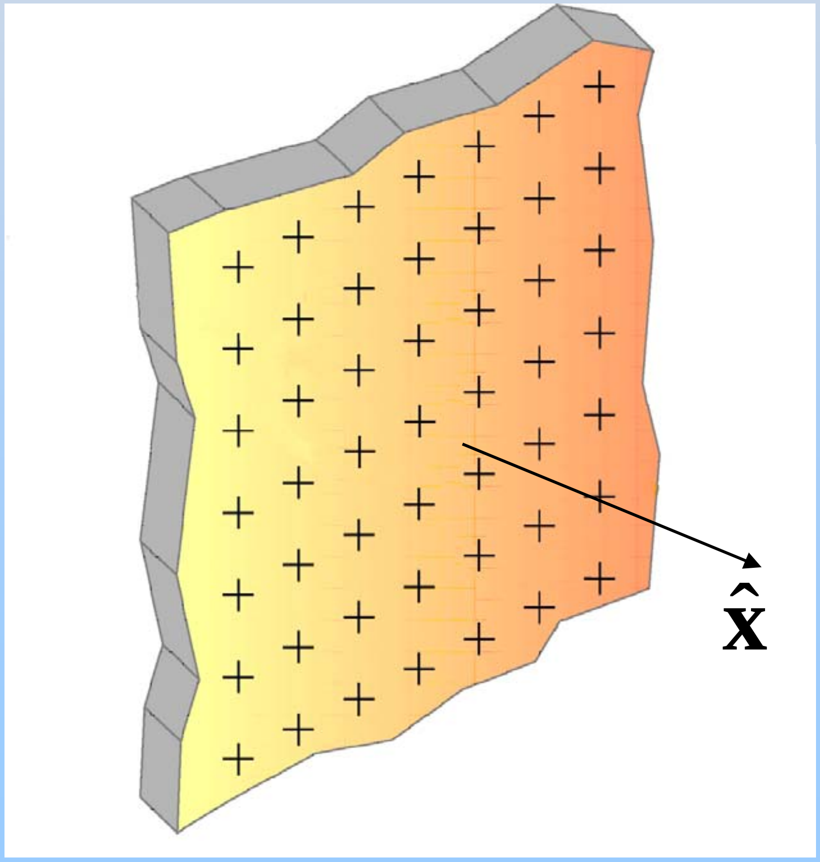
1. points in the positive z-direction.
2. points in the negative z-direction.
3. points in some other (x,y) direction.
4. is zero.
5. I don't know

Problem: Charge Slab

Infinite slab with uniform charge density ρ

Thickness is 2d (from x=-d to x=d).

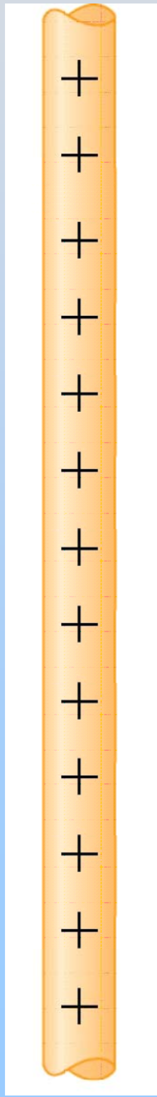
Find **E** for x > 0 (how many regions is that?)



Gauss: Cylindrical Symmetry

Infinitely long rod with uniform charge density λ

Find **E** outside the rod.



Gauss: Cylindrical Symmetry

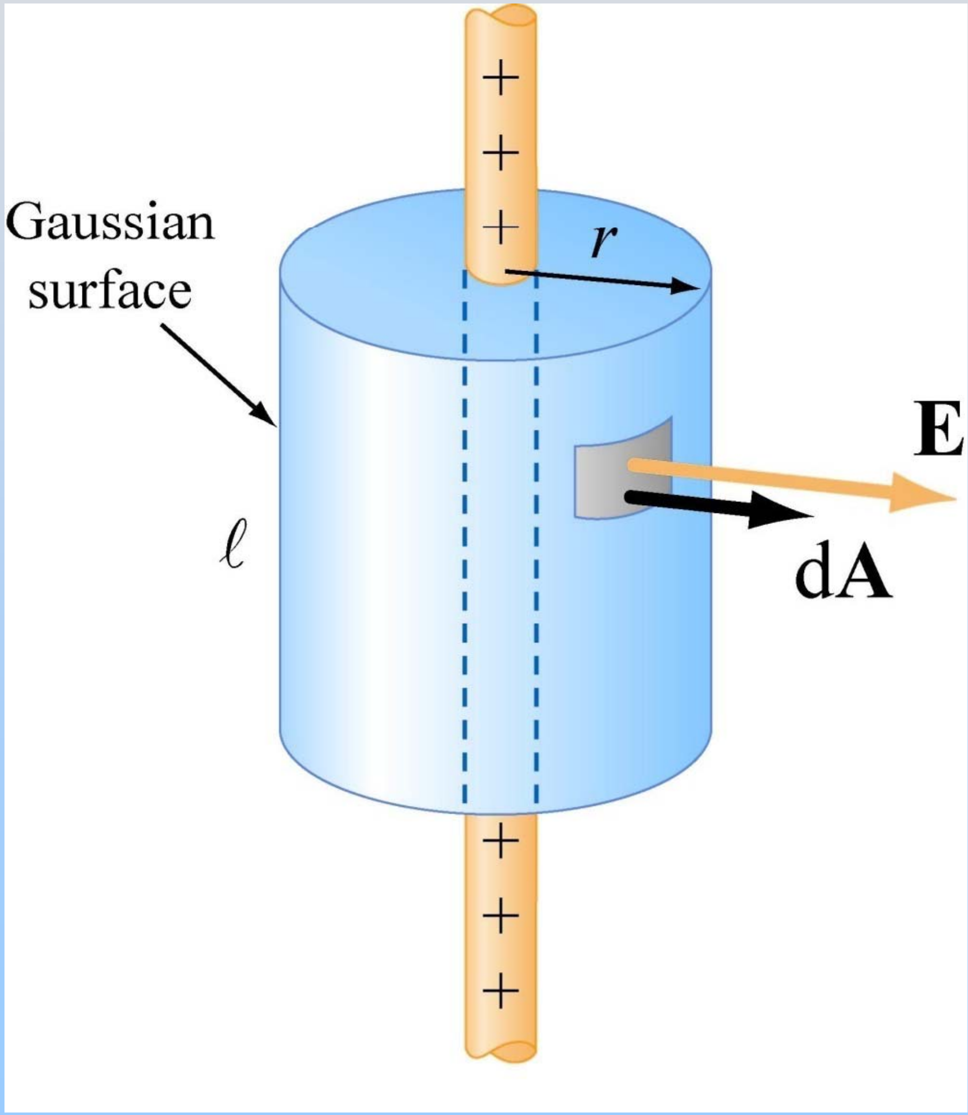
Symmetry is Cylindrical

$$\vec{\mathbf{E}} = E\,\hat{\mathbf{r}}$$

Use Gaussian Cylinder

Note: *r* is arbitrary **but** is the radius for which you will calculate the E field!

ℓ is arbitrary and should divide out



Gauss: Cylindrical Symmetry

Total charge enclosed: $q_{in} = \lambda\ell$

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E \oiint_S dA = EA$$

$$= E\left(2\pi r\ell\right) = \frac{q_{in}}{\varepsilon_0} = \frac{\lambda\ell}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} \Rightarrow \boxed{\vec{\mathbf{E}} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{\mathbf{r}}}$$