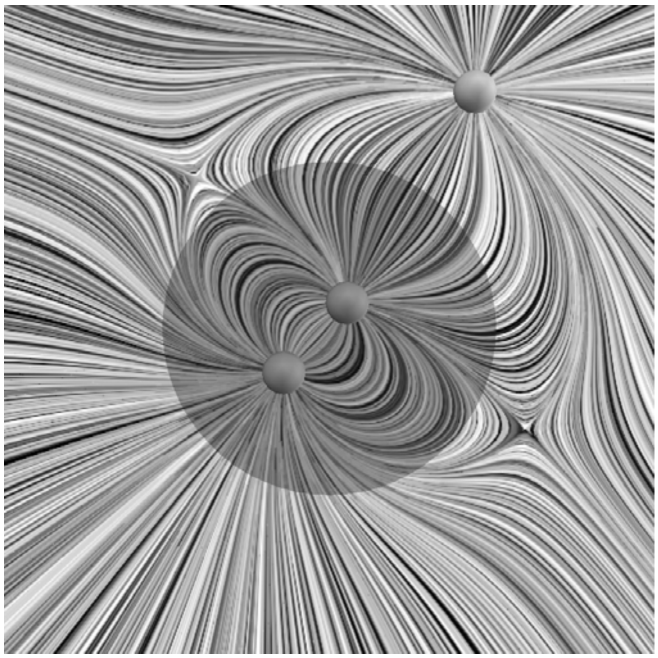
Gauss' Law

Challenge Problem Solutions

Problem 1:

The grass seeds figure below shows the electric field of three charges with charges +1, +1, and -1, The Gaussian surface in the figure is a sphere containing two of the charges.



The total electric flux through the spherical Gaussian surface is

a) Positive

b) Negative

c) Zero

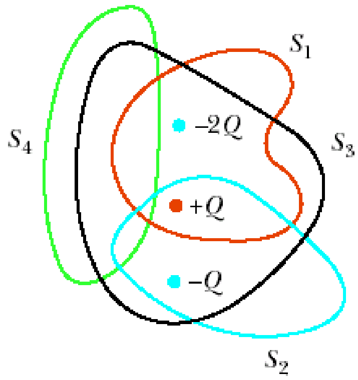
d) Impossible to determine without more information

Problem 1 Solution:

c. Because the field lines connect the two charges within the Gaussian surface they must have opposite sign. Therefore the charge enclosed in the Gaussian surface is zero. Hence the electric flux through the surface of the Gaussian surface is also zero.

Problem 2:

(a) Four closed surfaces, S_1 through S_4, together with the charges $-2Q$, Q, and $-Q$ are sketched in the figure at right. The colored lines are the intersections of the surfaces with the page. Find the electric flux through each surface.



(b) A pyramid has a square base of side a, and four faces which are equilateral triangles. A charge Q is placed at the center of the base of the pyramid. What is the net flux of electric field emerging from one of the triangular faces of the pyramid?

Problem 2 Solutions:

(a) By Gauss's Law, the flux through the closed surfaces is equal to the charge enclosed over ε_0. So,

$$\Phi_{S_1} = -Q/\varepsilon_0 \,;\ \Phi_{S_2} = 0;\ \Phi_{S_3} = -2Q/\varepsilon_0 \,;\ \Phi_{S_4} = 0$$

(b) Two pyramids attached at their base form an eight sided regular octahedron with triangular faces. By Gauss's Law, the flux through the entire closed surface is equal to the charge enclosed over ε_0. So, $\Phi_S = Q/\varepsilon_0$. The flux on each of the eight faces is equal, so the net flux of electric field emerging from one of the triangular faces of the pyramid is

$$\Phi_{face} = Q/8\varepsilon_0 \,.$$

Problem 3:

Careful measurements reveal an electric field

$$\vec{\mathbf{E}}(r) = \begin{cases} \dfrac{a}{r^2}\left(1 - \dfrac{r^3}{R^3}\right)\hat{\mathbf{r}}\,; & r \le R \\ \vec{0}\,; & r \ge R \end{cases}$$

where a and R are constants. Which of the following best describes the charge distribution giving rise to this electric field?

a) A negative point charge at the origin with charge $q = 4\pi\varepsilon_0 a$ and a uniformly positive charged spherical shell of radius R with surface charge density $\sigma = -q/4\pi R^2$.

b) A positive point charge at the origin with charge $q = 4\pi\varepsilon_0 a$ and a uniformly negative charged spherical shell of radius R with surface charge density $\sigma = -q/4\pi R^2$.

c) A positive point charge at the origin with charge $q = 4\pi\varepsilon_0 a$ and a uniformly negative charged sphere of radius R with charge density $\rho = -q/(4\pi R^3/3)$.

d) A negative point charge at the origin with charge $-q = -4\pi\varepsilon_0 a$ and a uniformly positive charged sphere of radius R with charge density $\rho = q/(4\pi R^3/3)$.

e) Impossible to determine from the given information.

Problem 3 Solution:
c. As you shall see below the answer should be c. because the problem does not specify the sign of the constant a. However both description c. and d. do seem plausible so we shall accept answers c., d., and e.

Assume $a > 0$. Then the electric field can be thought of as the superposition of two fields, $\vec{\mathbf{E}}_+(r) = \dfrac{a}{r^2}\hat{\mathbf{r}}$ and $\vec{\mathbf{E}}_-(r) = -\dfrac{ar}{R^3}\hat{\mathbf{r}}$. $\vec{\mathbf{E}}_+(r)$ is the electric field of a positive point charge at the origin with $q = 4\pi\varepsilon_0 a$. $\vec{\mathbf{E}}_-(r)$ is the electric field of a uniformly negative charged sphere of radius R. Because the electric field for radius $r > R$ is zero, the sum of the two charges distributions must be zero. Therefore the charge density must satisfy $\rho = -q/(4\pi R^3/3) = -4\pi\varepsilon_0 a/(4\pi R^3/3) = -3\varepsilon_0 a/R^3$.

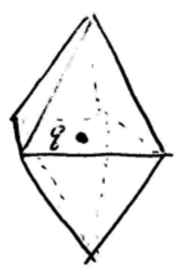
Now assume $a < 0$. Suppose the electric field can now be thought of as the superposition of two fields, $\vec{\mathbf{E}}_{-}(r) = \frac{a}{r^2}\hat{\mathbf{r}}$ and $\vec{\mathbf{E}}_{+}(r) = -\frac{ar}{R^3}\hat{\mathbf{r}}$. $\vec{\mathbf{E}}_{-}(r)$ is the electric field of a negative point charge at the origin with $-q = 4\pi\varepsilon_0 a > 0$, hence $q < 0$. $\vec{\mathbf{E}}_{+}(r)$ is the electric field of a uniformly positively charged sphere of radius R. Because the electric field for radius $r > R$ is zero, the sum of the two charges distributions must be zero. Therefore the charge density must satisfy $\rho = q/(4\pi R^3/3) < 0$. Therefore when $a < 0$ the only possible answer d. cannot be correct.

Problem 4:
A pyramid has a square base of side a, and four faces which are equilateral triangles. A charge Q is placed on the center of the base of the pyramid. What is the net flux of electric field emerging from one of the triangular faces of the pyramid?

1. 0
2. $\dfrac{Q}{8\varepsilon_0}$
3. $\dfrac{Qa^2}{2\varepsilon_0}$
4. $\dfrac{Q}{2\varepsilon_0}$
5. Undetermined: we must know whether Q is infinitesimally above or below the plane?

Problem 4 Solution:

2: Explain your reasoning: Construct an eight faced closed surface consisting of two pyramids with the charge at the center. The total flux by Gauss's law is just Q/ε_0. Since each face is identical, the flux through each face is one eight the total flux or $Q/8\varepsilon_0$.



Problem 5:
A charge distribution generates a radial electric field

$$\vec{\mathbf{E}} = \frac{a}{r^2} e^{-r/b} \hat{\mathbf{r}}$$

where a and b are constants. The total charge giving rise to this electric field is

1. $4\pi\varepsilon_0 a$
2. 0
3. $4\pi\varepsilon_0 b$

Problem 5 Solution:

2: Explain your reasoning: In order to fine the total charge I choose a Gaussian surface that extends over all space. Because the electric field is radially symmetric, I choose my Gaussian surface to be a sphere of radius r and I will take the limit as $r \to \infty$. The flux is given by

$$\lim_{r\to\infty} \oiint_r \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \lim_{r\to\infty} \oiint_r \frac{a}{r^2} e^{-r/b} \hat{\mathbf{r}} \cdot da\hat{\mathbf{r}} = \lim_{r\to\infty} \oiint_r \frac{a}{r^2} e^{-r/b} da = \lim_{r\to\infty} \frac{a}{r^2} e^{-r/b} 4\pi r^2 = 4\pi a \lim_{r\to\infty} e^{-r/b} = 0$$

When I take the limit as $r \to \infty$, the exponential term goes to zero, and so the flux goes to zero. Therefore the charge enclosed is zero.

Problem 6:

The bottom surface of a thundercloud of area A and the earth can be modeled as a pair of infinite parallel plate with equal and opposite surface uniform charge densities. Suppose the vertical electric field at the surface of the earth has a magnitude $\left|\vec{\mathbf{E}}_{atm}\right|$ and points towards the thundercloud.

a) Find an expression for the total charge density σ on the bottom surface of a thundercloud? Is this charge density positive or negative?

b) Suppose that the water in the thundercloud forms water droplets of radius r that carry all the charge of the thundercloud. The drops fall to the ground and make a height h of rainfall directly under the thundercloud. Find an expression for the charge on each droplet of water.

c) For the drops in part b), find an expression for the electric field $\left|\vec{\mathbf{E}}_{drop}\right|$ on the surface of the drop due only to the charge on the drop?

d) If a typical drop has radius $r = 5.0\times 10^{-1}\,mm$ and the rainfall makes a height $h = 2.5\times 10^{-3}\,m$, what is the ratio $f = \left|\vec{\mathbf{E}}_{drop}\right| / \left|\vec{\mathbf{E}}_{atm}\right|$?

Problem 6 Solutions:

(a) The electric field points from the earth upward to the thundercloud. Therefore the bottom of the thundercloud must carry a negative charge. Therefore

$$\sigma = -\varepsilon_o \left|\vec{\mathbf{E}}_{atm}\right| \tag{17}$$

(b) The raindrops fall to earth and make a layer of water of height h over the area A of the cloud. Therefore the total volume of water is hA. If each droplet had a volume of $\frac{4}{3}\pi r^3$ before it hit the ground and merged into the layer, then the layer is made up of a total number of drops given by $hA/\frac{4}{3}\pi r^3$. Suppose each drop carried a charge q.Then the total charge carried by the layer of water is

$$Q_{layer} = qhA/\tfrac{4}{3}\pi r^3 \tag{18}$$

and therefore the surface charge on the layer is

$$\sigma_{layer} = Q_{layer}/A = qh/\tfrac{4}{3}\pi r^3 \tag{19}$$

If we equate the surface charge density in (19) with that given in (17), we have for q

$$q = \frac{\frac{4}{3}\pi r^3}{h}\varepsilon_o \left|\vec{\mathbf{E}}_{atm}\right| \tag{20}$$

So the electric field at the surface of a drop of radius r with charge q is

$$\left|\vec{\mathbf{E}}_{drop}\right| = \frac{q}{4\pi\varepsilon_o r^2} = \frac{r}{3h}\left|\vec{\mathbf{E}}_{atm}\right| \tag{21}$$

(d) From above,

$$\left|\vec{\mathbf{E}}_{drop}\right| / \left|\vec{\mathbf{E}}_{atm}\right| = \frac{r}{3h} = \frac{5\times 10^{-4}}{(3)(2.5\times 10^{-3})} = 0.067 \tag{22}$$

Problem 7:

A sphere of radius R has a charge density $\rho = \rho_0 (r/R)$ where ρ_0 is a constant and r is the distance from the center of the sphere.

a) What is the total charge inside the sphere?

b) Find the electric field everywhere (both inside and outside the sphere).

Problem 7 Solution:

(a)The total charge inside the sphere is the integral

$$Q = \int_{r'=0}^{r=R} \rho 4\pi r^2 dr = \int_{r'=0}^{r=R} \rho_0 (r/R) 4\pi r^2 dr = \frac{\rho_0 4\pi}{R} \int_{r=0}^{r=R} r^3 dr = \frac{\rho_0 4\pi}{R} \frac{R^4}{4} = \rho_0 \pi R^3$$

(b)There are two regions of space: region I: $r < R$, and region II: $r > R$ so we apply Gauss' Law to each region to find the electric field.

For region I: $r < R$, we choose a sphere of radius r as our Gaussian surface. Then, the electric flux through this closed surface is

$$\oiint \vec{\mathbf{E}}_{\mathbf{I}} \cdot d\vec{\mathbf{A}} = E_I \cdot 4\pi r^2 .$$

Since the charge distribution is non-uniform, we will need to integrate the charge density to find the charge enclosed in our Gaussian surface. In the integral below we use the integration variable r' in order to distinguish it from the radius r of the Gaussian sphere.

$$\frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_{r'=0}^{r'=r} \rho 4\pi r'^2 dr' = \frac{1}{\varepsilon_0} \int_{r'=0}^{r'=r} \rho_0 (r'/R) 4\pi r'^2 dr' = \frac{\rho_0 4\pi}{R\varepsilon_0} \int_{r'=0}^{r'=r} r'^3 dr' = \frac{\rho_0 4\pi r^4}{4R\varepsilon_0} = \frac{\rho_0 \pi r^4}{R\varepsilon_0} .$$

Notice that the integration is primed and the radius of the Gaussian sphere appears as a limit of the integral.

Recall that Gauss's Law equates electric flux to charge enclosed:

$$\oiint \vec{\mathbf{E}}_{\mathbf{I}} \cdot d\vec{\mathbf{A}} = \frac{Q_{enc}}{\varepsilon_0}.$$

So we substitute the two calculations above into Gauss's Law to arrive at:

$$E_I \cdot 4\pi r^2 = \frac{\rho_0 \pi r^4}{R\varepsilon_0}.$$

We can solve this equation for the electric field

$$\vec{\mathbf{E}}_{\mathbf{I}} = E_I \hat{\mathbf{r}} = \frac{\rho_0 r^2}{4R\varepsilon_0}\hat{\mathbf{r}}, \ \ 0 < r < R.$$

The electric field points radially outward and has magnitude $\left|\vec{\mathbf{E}}_{\mathbf{I}}\right| = \frac{\rho_0 r^2}{4\varepsilon_0}$, $0 < r < R$.

For region II: $r > R$: we choose the same spherical Gaussian surface of radius $r > R$, and the electric flux has the same form

$$\oiint \vec{\mathbf{E}}_{\mathbf{II}} \cdot d\vec{\mathbf{A}} = E_{II} \cdot 4\pi r^2.$$

All the charge is now enclosed, $Q_{enc} = Q = \rho_0 \pi R^3$, so the right hand side of Gauss's Law becomes

$$\frac{Q_{enc}}{\varepsilon_0} = \frac{Q}{\varepsilon_0} = \frac{\rho_0 \pi R^3}{\varepsilon_0}.$$

Then Gauss's Law becomes

$$E_{II} \cdot 4\pi r^2 = \frac{\rho_0 \pi R^3}{\varepsilon_0}.$$

We can solve this equation for the electric field

$$\vec{\mathbf{E}}_{\mathbf{II}} = E_{II} \hat{\mathbf{r}} = \frac{\rho_0 R^3}{4\varepsilon_0 r^2}\hat{\mathbf{r}}, \ \ r > R.$$

In this region of space, the electric field points radially outward and has magnitude $\left|\vec{\mathbf{E}}_{\mathbf{II}}\right| = \dfrac{\rho_0 R^3}{4\varepsilon_0 r^2}$, $r > R$, so it falls off as $1/r^2$ as we expect since outside the charge distribution, the sphere acts as if it all the charge were concentrated at the origin.

Problem 8:

When two slabs of N-type and P-type semiconductors are put in contact, the relative affinities of the materials cause electrons to migrate out of the N-type material across the junction to the P-type material. This leaves behind a volume in the N-type material that is positively charged and creates a negatively charged volume in the P-type material.

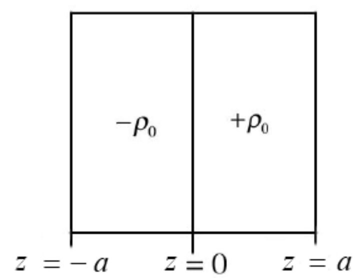
Let us model this as two infinite slabs of charge, both of thickness a with the junction lying on the plane $z = 0$. The N-type material lies in the range $0 < z < a$ and has uniform charge density $+\rho_0$. The adjacent P-type material lies in the range $-a < z < 0$ and has uniform charge density $-\rho_0$. Thus:

$$\rho(x,y,z) = \rho(z) = \begin{cases} +\rho_0 & 0 < z < a \\ -\rho_0 & -a < z < 0 \\ 0 & |z| > a \end{cases}$$

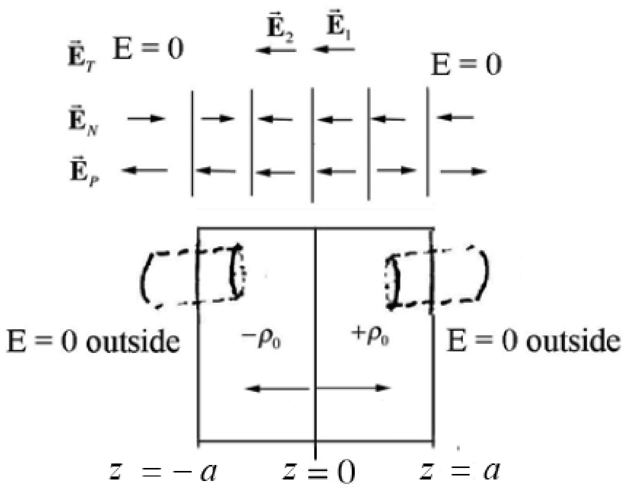
Find the electric field everywhere.

Problem 8 Solution:

In this problem, the electric field is a superposition of two slabs of opposite charge density.



Outside both slabs, the field of a positive slab $\vec{\mathbf{E}}_P$ (due to the P-type semi-conductor) is constant and points away and the field of a negative slab $\vec{\mathbf{E}}_N$ (due to the N-type semi-conductor)is also constant and points towards the slab, so when we add both contributions we find that the electric field is zero outside the slabs. The fields $\vec{\mathbf{E}}_P$ are shown on the figure below. The superposition of these fields $\vec{\mathbf{E}}_T$ is shown on the top line in the figure.



The electric field can be described by

$$\vec{\mathbf{E}}_T(z) = \begin{cases} \vec{\mathbf{0}} & z < -a \\ \vec{\mathbf{E}}_2 & -a < z < 0 \\ \vec{\mathbf{E}}_1 & 0 < z < a \\ \vec{\mathbf{0}} & |x| > d \end{cases} .$$

We shall now calculate the electric field in each region using Gauss's Law:

For region $-a < z < 0$: The Gaussian surface is shown on the left hand side of the figure below. Notice that the field is zero outside. Gauss's Law states that

$$\oiint_{\substack{closed\\surface}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \frac{Q_{enclosed}}{\varepsilon_0} .$$

So for our choice of Gaussian surface, on the cap inside the slab the unit normal for the area vector points in the positive z-direction, thus $\hat{\mathbf{n}} = +\hat{\mathbf{k}}$. So the dot product becomes $\vec{\mathbf{E}}_2 \cdot \hat{\mathbf{n}} da = E_{2,z}\, \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} da = E_{2,z}\, da$. Therefore the flux is

$$\oiint_{\substack{closed\\surface}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = E_{2,z} A_{cap}$$

The charge enclosed is

$$\frac{Q_{enclosed}}{\varepsilon_0} = \frac{-\rho_0 A_{cap}(a+z)}{\varepsilon_0}$$

where the length of the Gaussian cylinder is $a+z$ since $z<0$.

Substituting these two results into Gauss's Law yields

$$E_{2,z}A_{cap} = \frac{-\rho_0 A_{cap}(a+z)}{\varepsilon_0}$$

Hence the electric field in the N-type is given by

$$E_{2,x} = \frac{-\rho_0(a+z)}{\varepsilon_0}.$$

The negative sign means that the electric field point in the –z direction so the electric field vector is

$$\vec{\mathbf{E}}_2 = \frac{-\rho_0(a+z)}{\varepsilon_0}\,\hat{\mathbf{k}}\,.$$

Note when $z=-a$ then $\vec{\mathbf{E}}_2 = \vec{\mathbf{0}}$ and when $z=0$, $\vec{\mathbf{E}}_2 = \dfrac{-\rho_0 a}{\varepsilon_0}\hat{\mathbf{k}}$.

We make a similar calculation for the electric field in the P-type noting that the charge density has changed sign and the expression for the length of the Gaussian cylinder is $a-z$ since $z>0$. Also the unit normal now points in the –z-direction. So the dot product becomes

$$\vec{\mathbf{E}}_1 \cdot \hat{\mathbf{n}}da = E_{1,z}(-\hat{\mathbf{k}})\cdot\hat{\mathbf{k}}da = -E_{1,z}\ da$$

Thus Gauss's Law becomes

$$-E_{1,z}A_{cap} = \frac{+\rho_0 A_{cap}(a-z)}{\varepsilon_0}.$$

So the electric field is

$$E_{1,z} = -\frac{\rho_0(a-z)}{\varepsilon_0}.$$

The vector description is then

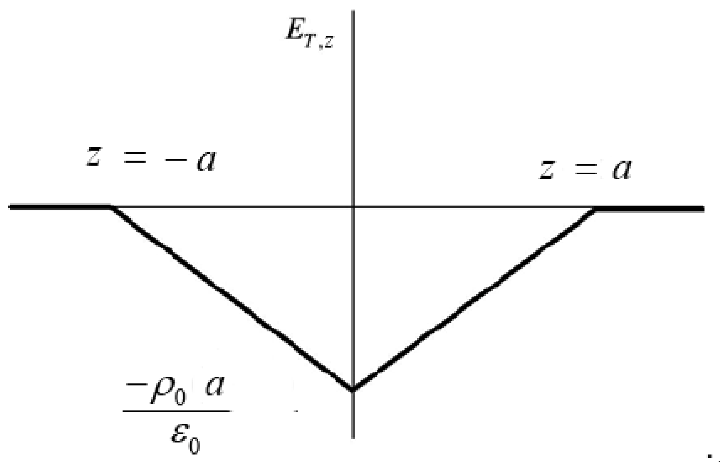
$$\vec{\mathbf{E}}_1 = \frac{-\rho_0(a-a)}{\varepsilon_0}\,\hat{\mathbf{k}}$$

Note when $z = a$ then $\vec{\mathbf{E}}_1 = \vec{\mathbf{0}}$ and when $z = 0$, $\vec{\mathbf{E}}_1 = \dfrac{-\rho_0 a}{\varepsilon_0}\hat{\mathbf{k}}$.

So the resulting field is

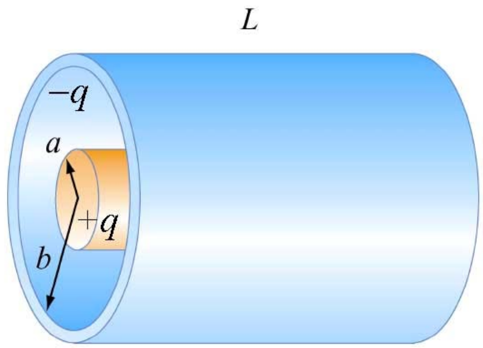
$$\vec{\mathbf{E}}_T(z) = \begin{cases} \vec{\mathbf{0}} & z < -a \\ \vec{\mathbf{E}}_2 = \dfrac{-\rho_0(a+z)}{\varepsilon_0}\hat{\mathbf{k}} & -a < z < 0 \\ \vec{\mathbf{E}}_1 = \dfrac{-\rho_0(a-z)}{\varepsilon_0}\hat{\mathbf{k}} & 0 < z < a \\ \vec{\mathbf{0}} & |z| > a \end{cases} .$$

The graph of the electric field is shown below



Problem 9:

A very long conducting cylinder (length L and radius a) carrying a total charge $+q$ is surrounded by a thin conducting cylindrical shell (length L and radius b) with total charge $-q$, as shown in the figure.



(a) Using Gauss's Law, find an expression for the direction and magnitude of the electric field $\vec{\mathbf{E}}$ for the region $r < a$.

(b) Similarly, find an expression for the direction and magnitude of the electric field $\vec{\mathbf{E}}$ for the region $a < r < b$.

Problem 9 Solution:

(a) The electric field is zero inside the inner conducting cylinder.

(b) We use a Gaussian cylinder of length l and radius $a < r < b$. Then, the flux is

$$\iint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E 2\pi r l \,.$$

The charge enclosed is given by

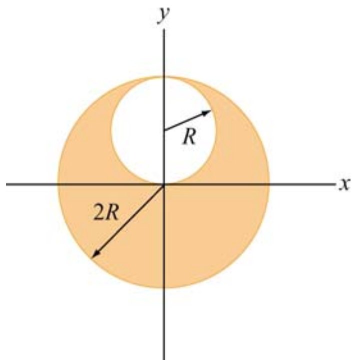
$$Q_{enc} = \lambda l = (q / L) l \,.$$

So Gauss' Law becomes

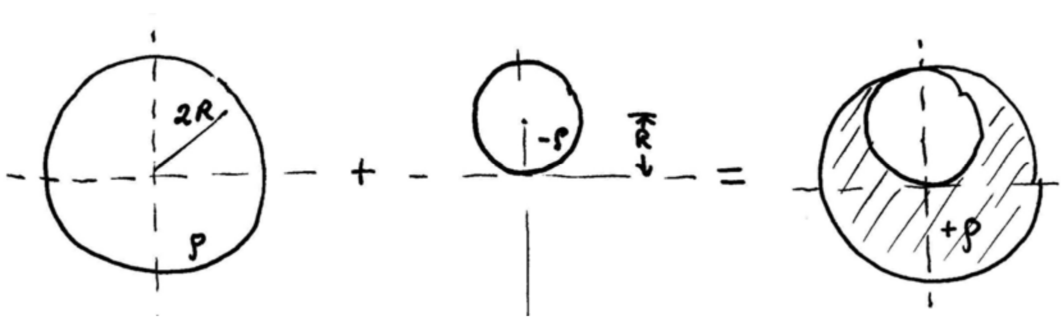
$$\oiint \vec{\mathbf{E}}_I \cdot d\vec{\mathbf{A}} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow E 2\pi r l = \frac{ql}{L\varepsilon_0} \;\; \Rightarrow \vec{\mathbf{E}} = \frac{q}{L 2\pi\varepsilon_0} \frac{1}{r} \hat{\mathbf{r}} ; \; a < r < b$$

Problem 10:

A sphere of radius $2R$ is made of a non-conducting material that has a uniform volume charge density ρ. (Assume that the material does not affect the electric field.) A spherical cavity of radius R is then carved out from the sphere, as shown in the figure below. Find the electric field within the cavity.

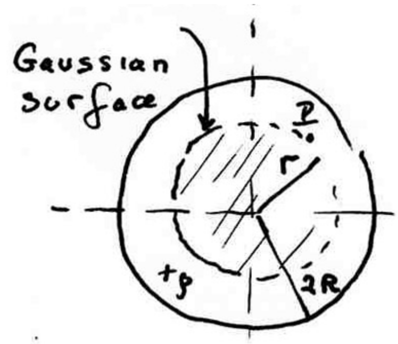


Problem 10 Solution:
At first glance this charge distribution does not seem to have any of the symmetries that enable us to use Gauss's law. However we can consider this charge distribution as the sum of two uniform spherical distributions of charge. The first is a sphere of radius $2R$ centered at the origin with a uniform volume charge density ρ. The second is a sphere of radius R centered at the point along the y-axis a distance R from the origin (the center of the spherical cavity) with a uniform volume charge density $-\rho$.



When we add together these two distributions of charge we obtain the uniform charged sphere with a spherical cavity of radius R as described in the problem. We can then add together the electric fields from these two distributions at any point in the cavity to obtain the electric field of the original distribution at that point inside the cavity (superposition principle). Each of these two distributions are spherically symmetric and therefore we can use Gauss's Law to find the electric field associated with each of them.. We do need to be careful when we add together the electric fields. As you will see that process is somewhat subtle and a good vector diagram will help considerably.

So let's begin by choosing a point P inside the cavity. We will now apply Gauss's Law to our first distribution, the sphere of radius $2R$ centered at the origin with a uniform volume charge density ρ. The point P is a distance $r < 2R$ from the origin. We choose a sphere of radius r as our Gaussian surface with $r < 2R$.



Then, the electric flux through this closed surface is

$$\oiint \vec{\mathbf{E}}_{\rho} \cdot d\vec{\mathbf{A}} = E_{\rho} \cdot 4\pi r^2,$$

where E_{ρ} denotes the outward normal component of the electric field at the point P associated to the spherical distribution with uniform volume charge density ρ. Because the charge distribution is uniform, the charge enclosed in the Gaussisan surface is

$$\frac{Q_{enc}}{\varepsilon_0} = \frac{\rho(4\pi r^3 / 3)}{\varepsilon_0}.$$

Recall that Gauss' Law equates electric flux to charge enclosed:

$$\oiint \vec{\mathbf{E}}_{\rho} \cdot d\vec{\mathbf{A}} = \frac{Q_{enc}}{\varepsilon_0}.$$

So we substitute the two calculations above into Gauss' law to arrive at:

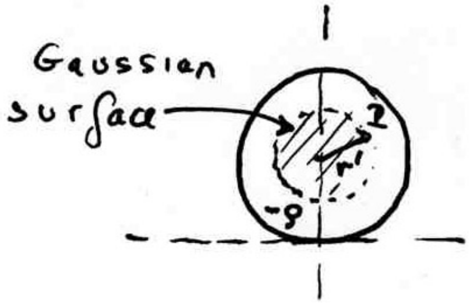
$$E_{\rho} \cdot 4\pi r^2 = \frac{\rho(4\pi r^3 / 3)}{\varepsilon_0}.$$

We can solve this equation for the electric field

$$\vec{\mathbf{E}}_{\rho}(P) = E_{\rho}\hat{\mathbf{r}} = \frac{\rho r}{3\varepsilon_0}\hat{\mathbf{r}} \;.$$

where $\hat{\mathbf{r}}$ is a unit vector at the point P pointing radially away from the origin.

We now apply Gauss's Law to our second distribution, a sphere of radius R centered at the point along the y-axis a distance R from the origin with a uniform volume charge density $-\rho$. The point P is a distance $r' < R$ from the center of the cavity.



We choose a sphere of radius r' as our Gaussian surface with $r' < R$. Then, the electric flux through this closed surface is

$$\oiint \vec{\mathbf{E}}_{-\rho} \cdot d\vec{\mathbf{A}} = E_{-\rho} \cdot 4\pi r'^2 ,$$

where $E_{-\rho}$ denotes the outward normal component of the electric field at the point P associated to the spherical distribution with uniform volume charge density $-\rho$. Because the charge distribution is uniform, the charge enclosed in the Gaussisan surface is

$$\frac{Q_{enc}}{\varepsilon_0} = -\frac{\rho(4\pi r'^3/3)}{\varepsilon_0} .$$

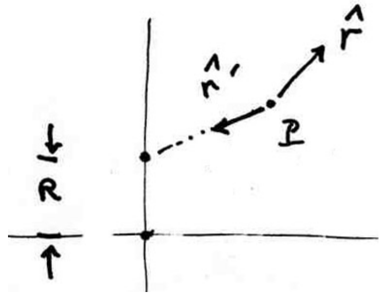
Therefore applying Gauss's Law yields

$$E_{-\rho} \cdot 4\pi r^2 = -\frac{\rho(4\pi r'^3/3)}{\varepsilon_0} .$$

We can solve this equation for the electric field

$$\vec{\mathbf{E}}_{-\rho}(P) = E_{-\rho}\hat{\mathbf{r}}' = -\frac{\rho r'}{3\varepsilon_0}\hat{\mathbf{r}}' .$$

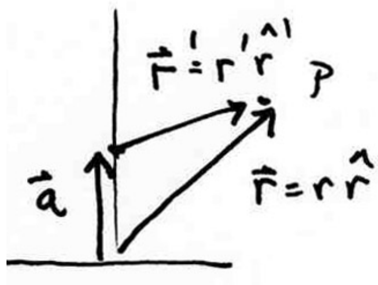
where $\hat{\mathbf{r}}'$ is a unit vector at the point P pointing radially away from the center of the cavity.



The electric field associated with our original distribution is then

$$\vec{\mathbf{E}}(P) = \vec{\mathbf{E}}_{\rho}(P) + \vec{\mathbf{E}}_{-\rho}(P) = E_{\rho}\hat{\mathbf{r}} + E_{-\rho}\hat{\mathbf{r}}' = \frac{\rho r}{3\varepsilon_0}\hat{\mathbf{r}} - \frac{\rho r'}{3\varepsilon_0}\hat{\mathbf{r}}' = \frac{\rho}{3\varepsilon_0}(r\hat{\mathbf{r}} - r'\hat{\mathbf{r}}') = \frac{\rho}{3\varepsilon_0}(\vec{\mathbf{r}} - \vec{\mathbf{r}}')$$

where $\vec{\mathbf{r}}$ is a vector from the origin to the point P and $\vec{\mathbf{r}}'$ is a vector from the center of the cavity to the point P. From our diagram we see that $\vec{\mathbf{a}} = \vec{\mathbf{r}} - \vec{\mathbf{r}}'$.



Therefore the electric field at the point P is given by

$$\vec{\mathbf{E}}(P) = \frac{\rho}{3\varepsilon_0}\vec{\mathbf{a}}.$$

This is a remarkable result. The electric field inside the cavity is uniform. The direction of the electric field points from the center of entire sphere to the center of the cavity. This direction is uniquely specified and is an example of ‘broken symmetry’.

Problem 11:

(a) This problem demonstrates how one can use Gauss's law to draw important conclusions about the electric field associated with charged conductors. The following points will help you answer the questions posed in the problem

(i) In general charge resides *on* the surface of a conductor.

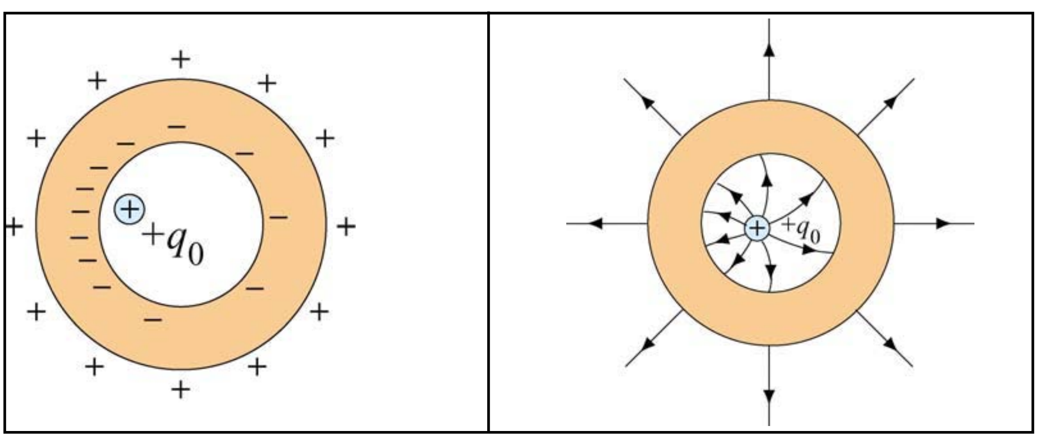
(ii) The electric field is zero inside the conductor. (This must be so in a static situation; otherwise electric currents would be flowing, contrary to the assumption.

(iii) Induced charge on the inner surface is exactly equal to $-q$. (A Gaussian surface, enclosing the $+q$ charge inside the cavity and the $-q$ charge on the inner surface, and staying entirely inside the conductor proves the above statement with the help of Gauss's Law.

(iv) Since the conductor has no net charge, the outer surface must carry $+q$ charge.

(v) The electric field outside any metallic surface is normal to the surface; its magnitude is σ / ε_0 by virtue of Gauss's law. (Recall, any metallic surface is an equipotential surface.)

Although we cannot derive the distribution of the charge $-q$ on the inner surface of the conductor without more sophisticated mathematics, we can nonetheless say that since the fields from all inner charges must add to give a zero field inside the metal, there must be more negative charge near the $+q$ (and less farther away). So the charge distribution must look like this:



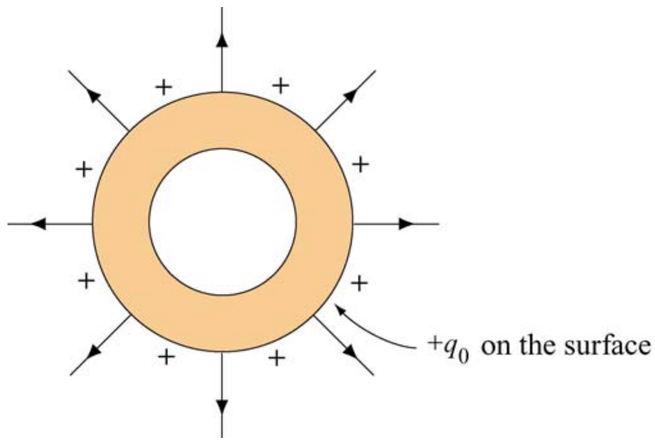
In particular, notice that the charges on the outside of the shell are *uniformly distributed*!

(b) The electric field lines are shown in the sketch above. Notice that the field lines are closer together where the density of negative charges is greatest. Outside the sphere, the field looks like that from a point charge $+q_0$.

(c) No. The negative charge on the inside of the metal does, of course, rearrange itself in order to keep the field zero inside the conductor. The positive charge induced on the outside is totally uninfluenced because of the arguments presented in (a).

(d) When the "source charge" $+q_0$ touches the inner surface, a total neutralization inside the sphere (inner surface plus cavity) takes place; only the induced charge outside remains and is distributed uniformly on the surface.

(e) The behavior of the field just before contact is shown in the figure below:



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