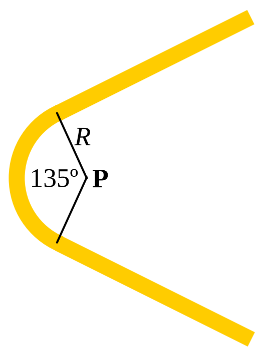
Electric Fields and Continuous Charge Distributions
Challenge Problem Solutions

Problem 1:

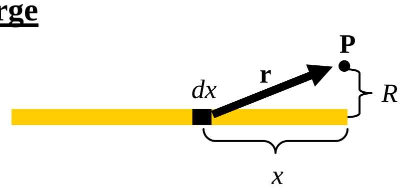
Two thin, semi-infinite rods lie in the same plane. They make an angle of 45º with each other and they are joined by another thin rod bent along an arc of a circle of radius R, with center at P. All the rods carry a uniform charge distribution of λ [C/m]. Find the electric field at point P.



Problem 1 Solution:

At first glance this is an ugly problem because the lines are at an angle so you might think that choosing an integration variable is going to be difficult. But we can easily break this problem into two parts: a semi infinite line of charge and an arc of charge. We'll do these problems individually and then superimpose the results.

<u>Semi-Infinite Line of Charge</u>



The point we want to calculate the field at is a distance *R* above the end of our semi-infinite rod, as pictured above. The field from the small charge segment (length *dx*) is:

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{\left(x^2+R^2\right)^{3/2}}\left(x\hat{\mathbf{i}}+R\hat{\mathbf{j}}\right)$$

There are two different integrals in the x-direction (where we can simply use u-substitution with $u = x^2 + R^2$) and y-directions (where we will need to use trig. substitution $x = R\tan\theta$) so separate the components:

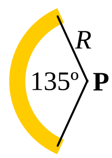
$$E_x = \int dE_x = \int_0^\infty \frac{1}{4\pi\varepsilon_0}\frac{\lambda x dx}{\left(x^2+R^2\right)^{3/2}} = \frac{\lambda}{4\pi\varepsilon_0}\int_{R^2}^\infty \frac{\frac{1}{2}du}{u^{3/2}} = \frac{\lambda}{8\pi\varepsilon_0}\left.\frac{u^{-1/2}}{-1/2}\right|_{R^2}^\infty = \frac{\lambda}{4\pi\varepsilon_0 R}$$

$$E_y = \int dE_y = \int_{x=0}^{\infty} \frac{1}{4\pi\varepsilon_0} \frac{\lambda R dx}{\left(x^2+R^2\right)^{3/2}} = \frac{\lambda R}{4\pi\varepsilon_0} \int_{\theta=0}^{\pi/2} \frac{R\sec^2\theta\, d\theta}{\left(R^2\tan^2\theta + R^2\right)^{3/2}} = \frac{\lambda R^2}{4\pi\varepsilon_0 R^3} \int_{\theta=0}^{\pi/2} \frac{\sec^2\theta\, d\theta}{\left(\tan^2\theta+1\right)^{3/2}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0 R} \int_{\theta=0}^{\pi/2} \frac{\sec^2\theta\, d\theta}{\sec^3\theta} = \frac{\lambda}{4\pi\varepsilon_0 R} \int_{\theta=0}^{\pi/2} \cos\theta\, d\theta = \frac{\lambda}{4\pi\varepsilon_0 R} \sin\theta \Big|_0^{\pi/2} = \frac{\lambda}{4\pi\varepsilon_0 R}$$

Neat, huh? Two very different integrals, same result. So the electric field points at 45º at point P (up and to right) with total amplitude $\sqrt{E_x^2 + E_y^2} = \frac{\sqrt{2}\lambda}{4\pi\varepsilon_0 R}$

Arc of Charge



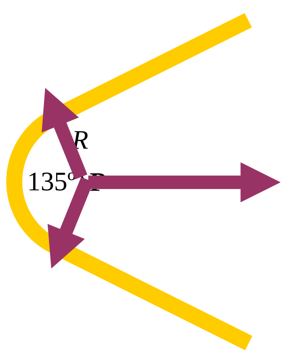
This one is a little easier. By symmetry we know we only have to look at the x-component (the y-components will cancel). Plus we are always a distance R away from point P. So for a little arc-length $Rd\theta$, we have

$$dE_x = \frac{1}{4\pi\varepsilon_0} \frac{dq}{r^3} x = \frac{1}{4\pi\varepsilon_0} \frac{\lambda R d\theta}{R^3} \left(R\cos\theta\right) = \frac{\lambda}{4\pi\varepsilon_0 R} \cos\theta\, d\theta$$

Integrating (from θ = 0 º to 67.5º and then doubling):

$$E_x = \int dE_x = 2\int_{\theta=0}^{67.5^\circ} \frac{\lambda}{4\pi\varepsilon_0 R} \cos\theta\, d\theta = \frac{\lambda}{2\pi\varepsilon_0 R} \sin\theta \Big|_0^{67.5^\circ} = \frac{\lambda}{2\pi\varepsilon_0 R} \sin\left(67.5^\circ\right)$$

Putting it Together



Drawing our three E field vectors, it is clear that the vertical components of from the lines will cancel as well, so we only need the horizontal component. I'll figure it out for the bottom line and double it. Since the line is rotated down by 22.5º (half of the 45º between the lines), the E field angle is rotated from 45º to 67.5º. We want the horizontal component of that:

$$E_{Horizontal} = \frac{\sqrt{2}\lambda}{4\pi\varepsilon_0 R} \cos\left(67.5^\circ\right)$$

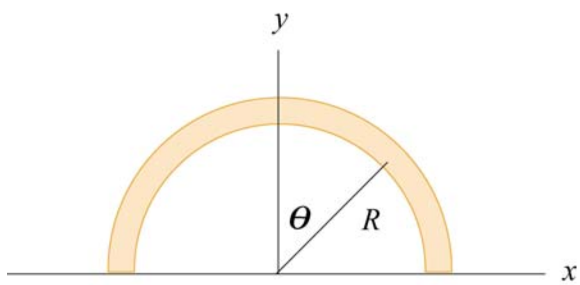
So,

$$\vec{\mathbf{E}}_{Total} = \hat{\mathbf{i}}\left[\frac{\lambda}{2\pi\varepsilon_0 R}\sin\left(67.5^{\circ}\right) - 2\frac{\sqrt{2}\lambda}{4\pi\varepsilon_0 R}\cos\left(67.5^{\circ}\right)\right] = \hat{\mathbf{i}}\frac{\lambda}{2\pi\varepsilon_0 R}\left(\sin\left(67.5^{\circ}\right) - \sqrt{2}\cos\left(67.5^{\circ}\right)\right)$$

$$= 0.061\frac{\lambda}{\varepsilon_0 R}\hat{\mathbf{i}}$$

Problem 2:

A positively charged wire is bent into a semicircle of radius R, as shown in the figure below.



The total charge on the semicircle is Q. However, the charge per unit length along the semicircle is non-uniform and given by $\lambda = \lambda_0 \cos\theta$.

a) What is the relationship between λ_0, R and Q?

b) If a particle with a charge q is placed at the origin, what is the total force on the particle? Show all your work including setting up and integrating any necessary integrals.

Problem 2 Solution:

(a) In order to find a relation between λ_0, R and Q it is necessary to integrate the charge density λ because the charge distribution is non-uniform

$$Q = \int_{wire} \lambda ds = \int_{\theta'=-\pi/2}^{\theta'=\pi/2} \lambda_0 \cos\theta' R d\theta' = R\lambda_0 \sin\theta' \Big|_{\theta'=-\pi/2}^{\theta'=\pi/2} = 2R\lambda_0 .$$

(b) The force on the charged particle at the center P of the semicircle is given by

$$\vec{\mathbf{F}}(P) = q\vec{\mathbf{E}}(P).$$

The electric field at the center P of the semicircle is given by

$$\vec{\mathbf{E}}(P) = \frac{1}{4\pi\varepsilon_0} \int_{wire} \frac{\lambda ds}{r^2} \hat{\mathbf{r}}.$$

The unit vector, $\hat{\mathbf{r}}$, located at the field point, is directed from the source to the field point and in Cartesian coordinates is given by

$$\hat{\mathbf{r}} = -\sin\theta'\,\hat{\mathbf{i}} - \cos\theta'\,\hat{\mathbf{j}}.$$

Therefore the electric field at the center P of the semicircle is given by

$$\vec{\mathbf{E}}(P) = \frac{1}{4\pi\varepsilon_0}\int_{wire}\frac{\lambda ds}{r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\int_{\theta'=-\pi/2}^{\theta'=\pi/2}\frac{\lambda_0\cos\theta' R d\theta'}{R^2}(-\sin\theta'\,\hat{\mathbf{i}} - \cos\theta'\,\hat{\mathbf{j}}).$$

There are two separate integrals for the x and y components. The x-component of the electric field at the center P of the semicircle is given by

$$E_x(P) = -\frac{1}{4\pi\varepsilon_0}\int_{\theta'=-\pi/2}^{\theta'=\pi/2}\frac{\lambda_0\cos\theta'\sin\theta'\,d\theta'}{R} = \left.\frac{\lambda_0\cos^2\theta'}{8\pi\varepsilon_0 R}\right|_{\theta'=-\pi/2}^{\theta'=\pi/2} = 0.$$

We expected this result by the symmetry of the charge distribution about the y-axis.

The y-component of the electric field at the center P of the semicircle is given by

$$\begin{aligned}
E_y(P) &= -\frac{1}{4\pi\varepsilon_0}\int_{\theta'=-\pi/2}^{\theta'=\pi/2}\frac{\lambda_0\cos^2\theta' d\theta'}{R} = -\frac{1}{4\pi\varepsilon_0}\int_{\theta'=-\pi/2}^{\theta'=\pi/2}\frac{\lambda_0(1+\cos 2\theta')d\theta'}{2R} \\
&= -\frac{\lambda_0}{8\pi\varepsilon_0 R}\theta'\Big|_{\theta'=-\pi/2}^{\theta'=\pi/2} - \frac{\lambda_0}{16\pi\varepsilon_0 R}\sin 2\theta\Big|_{\theta'=-\pi/2}^{\theta'=\pi/2} \\
&= -\frac{\lambda_0}{8\varepsilon_0 R}
\end{aligned}.$$

Therefore the force on the charged particle at the point P is given by

$$\vec{\mathbf{F}}(P) = q\vec{\mathbf{E}}(P) = -\frac{q\lambda_0}{8\varepsilon_0 R}\hat{\mathbf{j}}.$$

Problem 3:
A cylindrical tube of length *L*, radius *R* carries a charge *Q* uniformly distributed over its outer surface. Find the electric field on the axis of the tube at one of its ends.

Problem 3 Solution:

We will find the electric field at one end of the cylinder by dividing the cylinder into small charge elements dq' and then using the fact that the electric field at a field point P due to the small element is given by

$$d\vec{\mathbf{E}}(P) = k\frac{dq'}{r^2}\hat{\mathbf{r}}$$

In the above expression, r is the distance between the charged element and the field point, and $\hat{\mathbf{r}}$ is the unit vector at the field point, pointing from the charged element to the field point.

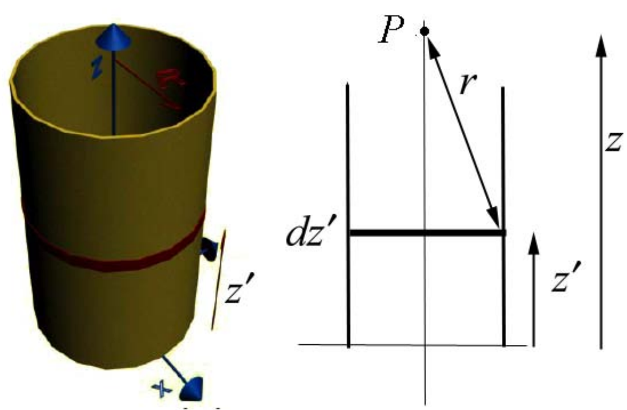
To find the field, we then integrate over the surface of the cylinder.

$$\vec{\mathbf{E}}(P) = \iint_{cylinder} d\vec{\mathbf{E}}(P) = \iint_{cylinder} k\frac{dq'}{r^2}\hat{\mathbf{r}}\,.$$

Methodology: We need to define dq', r, and $\hat{\mathbf{r}}$ with respect to a coordinate system and then set up the integrand. Since the charge is evenly distributed over the surface of the cylinder, the area charge density is given by

$$\sigma = \frac{Q}{A} = \frac{Q}{2\pi RL}$$

We first start by using cylindrical coordinates (R, φ', z') to describe the cylinder. We choose the z-axis to run along the symmetry axis of the cylinder and we locate the top and bottom of the cylinder at the points $z' = 0$ and $z' = L$.



We locate the area element at the point (R, φ', z'). Note that (φ', z') are the integration variables. (Note that is the reason we prime these coordinates.)
A small area element is given by

$$dq' = \sigma da' = \sigma R d\varphi' dz'$$

where $R d\varphi'$ is a small arc length, and dz' is the small length (infinitesimal) along the vertical z-axis.

For the moment we locate the field point along the z-axis at $(0, 0, z)$. Notice that the variable z is unprimed because it is the field point. If we want to find the field at either end of the cylinder we will substitute $z = 0$ or $z = L$ at the end of our calculation.

The distance r between the charged element and the field point in these coordinates is given by

$$r = (R^2 + (z - z')^2)^{1/2}.$$

By symmetry as we integrate the charged element around the z-axis, only the component along the z-axis will be non-zero. This means that we only need to decompose the vertical component of the unit vector (we denote this by $\hat{\mathbf{r}}_z$). Let θ denote the angle $\hat{\mathbf{r}}$ makes with a horizontal line, then the vertical component is given by

$$\hat{\mathbf{r}}_z = \sin\theta\,\hat{\mathbf{k}} = \frac{(z - z')}{r}\,\hat{\mathbf{k}} = \frac{(z - z')}{(R^2 + (z - z')^2)^{1/2}}\,\hat{\mathbf{k}}$$

Then

$$d\vec{\mathbf{E}}_z(P) = k\frac{dq'}{r^2}\hat{\mathbf{r}}_z = k\frac{\sigma R d\varphi' dz'}{(R^2 + (z - z')^2)}\frac{(z - z')}{(R^2 + (z - z')^2)^{1/2}}\,\hat{\mathbf{k}} = k\frac{(z - z')\sigma R d\varphi' dz'}{(R^2 + (z - z')^2)^{3/2}}\,\hat{\mathbf{k}}.$$

We can now integrate this expression:

$$\vec{\mathbf{E}}_z(P) = \iint_{cylinder} d\vec{\mathbf{E}}_z(P) = \int_{z'=0}^{z'=L}\int_{\varphi'=0}^{\varphi'=2\pi} k\frac{(z - z')\sigma R d\varphi' dz'}{(R^2 + (z - z')^2)^{3/2}}\,\hat{\mathbf{k}}.$$

The first integral with respect to $d\varphi'$ is easy because the integrand is independent of φ', and so the integral is just 2π.

$$\vec{\mathbf{E}}_z(P) = \iint_{cylinder} d\vec{\mathbf{E}}_z(P) = k\int_{z'=0}^{z'=L}\frac{(z - z')2\pi\sigma R dz'}{(R^2 + (z - z')^2)^{3/2}}\hat{\mathbf{k}}.$$

Notice that $2\pi\sigma R dz'$ is the charge on a ring of radius R and “width” dz' along the z-axis.

So the integrand is really the contribution that a ring located at z' makes to z-component of the electric field at the point z along the z-axis.

$$\vec{\mathbf{E}}_z(P) = k \int_{z'=0}^{z'=L} \frac{(z-z')2\pi\sigma R dz'}{(R^2+(z-z')^2)^{3/2}} \hat{\mathbf{k}} = \int_{z'=0}^{z'=L} (d\vec{\mathbf{E}}_z(z))_{ring} \ .$$

We now perform the z'-integral. We can make a change of variables $u = (R^2+(z-z')^2)$. Then
$du = -2(z-z')$. The end points are now integrand $u = R^2 + z^2$ and $u = R^2 + (z-L)^2$.
The integral is then

$$\vec{\mathbf{E}}_z(P) = -\pi\sigma Rk \int_{u=R^2+z^2}^{u=R^2+(z-L)^2} \frac{du}{u^{3/2}} = 2\pi\sigma Rk \left.\frac{1}{u^{1/2}}\right|_{u=R^2+z^2}^{u=R^2+(z-L)^2} \hat{\mathbf{k}} \ .$$

Substituting in the endpoints we arrive at

$$\vec{\mathbf{E}}_z(P) = 2\pi\sigma Rk \left(\frac{1}{(R^2+(z-L)^2)^{1/2}} - \frac{1}{(R^2+z^2)^{1/2}} \right) \hat{\mathbf{k}} \ .$$

The problem asked to find the electric field at the ends of the cylinder, $z = 0$ or $z = L$. At the end $z = 0$, we have that using $k = 1/4\pi\varepsilon_0$ and $2\pi\sigma R = \dfrac{Q}{L}$

$$\vec{\mathbf{E}}_z(z=0) = 2\pi\sigma Rk \left(\frac{1}{(R^2+L^2)^{1/2}} - \frac{1}{R} \right) \hat{\mathbf{k}} = \frac{Q}{4\pi\varepsilon_0 L} \left(\frac{1}{(R^2+L^2)^{1/2}} - \frac{1}{R} \right) \hat{\mathbf{k}} \ .$$

Notice that the second term in the parenthesis is greater than the first term so the field point in the $-\hat{\mathbf{k}}$ direction.

At $z = L$, the field is given by

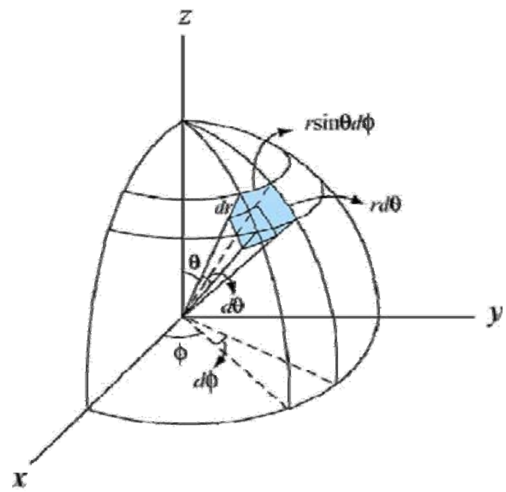
$$\vec{\mathbf{E}}_z(z=L) = 2\pi\sigma Rk \left(\frac{1}{R} - \frac{1}{(R^2+L^2)^{1/2}} \right) \hat{\mathbf{k}} = \frac{Q}{4\pi\varepsilon_0 L} \left(\frac{1}{R} - \frac{1}{(R^2+L^2)^{1/2}} \right) \hat{\mathbf{k}}$$

which has the same magnitude as the field at $z = 0$ but points in the $+\hat{\mathbf{k}}$ direction.

Problem 4:

A hemispherical Plexiglas shell of radius R carries a charge Q uniformly distributed over its surface.

(a) Find the electric field at the "center" of the hemisphere (that is, the center of the sphere from which the hemisphere was cut). HINT: You may be tempted to use the "Ring of Charge" result from class. It's actually much easier to just figure out what dq is, parameterizing your location of the hemisphere with θ and ϕ.



Solution:

Charge is evenly distributed over the surface of the hemisphere, so the charge density:

$$\sigma = \frac{Q}{A} = \frac{Q}{2\pi R^2}$$

Now we need to calculate the electric field. As told in the hint we should just use spherical coordinates and write out dq: $dq = \sigma\left(R\sin\theta\, d\varphi\right)\left(Rd\theta\right) = \sigma R^2 \sin\theta\, d\varphi d\theta$

Why is this easier? Because all dq's are the same distance R away from the origin, which we would lose sight of if we moved to working with rings. We also simplify our task of calculating the E field by realizing that, by symmetry, only the z-component survives:

$$dE_z = -\frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}z = -\frac{1}{4\pi\varepsilon_0}\frac{\sigma R^2\sin\theta\, d\varphi d\theta}{R^3}\left(R\cos\theta\right) = -\frac{\sigma}{4\pi\varepsilon_0}\sin\theta\left(\cos\theta\right)d\varphi d\theta$$

$$E_z = \int_{\theta=0}^{\pi/2}\int_{\varphi=0}^{2\pi} dE_z = \int_{\theta=0}^{\pi/2}\int_{\varphi=0}^{2\pi} -\frac{\sigma}{4\pi\varepsilon_0}\sin\theta\cos\theta\, d\varphi d\theta = -\frac{\sigma}{4\pi\varepsilon_0}\int_{\theta=0}^{\pi/2}\sin\theta\cos\theta\left(2\pi\right)d\theta$$

$$= -\frac{\sigma}{2\varepsilon_0}\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2} = -\frac{\sigma}{2\varepsilon_0}\left[\tfrac{1}{2}\left(1-0\right)\right] = -\frac{\sigma}{4\varepsilon_0}$$

(a) $\boxed{\vec{\mathbf{E}} = -\frac{\sigma}{4\varepsilon_0}\hat{\mathbf{k}}}$, where the direction is using the convention I've chosen above. You were welcome to label your axes as you preferred, but clearly if the hemisphere is positively charged the electric field will point away from it at the origin.