# Current, Current Density, Resistance and Ohm's Law
## Challenge Problem Solutions

**Problem 1:**

A straight cylindrical wire lying along the $x$-axis has a length $L$ and a diameter $d$. It is made of a material described by Ohm's law with a resistivity $\rho$. Assume that a potential $V$ is maintained at $x = 0$, and that $V = 0$ at $x = L$. In terms of $L$, $d$, $V$, $\rho$, and physical constants, determine expressions for

(a) the electric field in the wire.

(b) the resistance of the wire.

(c) the electric current in the wire.

(d) the current density in the wire. Express vectors in vector notation.

(e) Show that $\vec{\mathbf{E}} = \rho\vec{\mathbf{J}}$.

**Problem 1 Solutions:**

(a)

$$\vec{\mathbf{E}} = \frac{V}{L}\hat{\mathbf{x}}$$

**(b)**

$$R = \frac{\rho L}{A} = \frac{\rho L}{\pi (d/2)^2} = \frac{4\rho L}{\pi d^2}$$

**(c)**

$$\vec{I} = \frac{V}{R}\hat{x} = V \Big/ \left(\frac{4\rho L}{\pi d^2}\right)\hat{x} = \frac{\pi d^2 V}{4\rho L}\hat{x}$$

(d)

$$\vec{\mathbf{J}} = \frac{\vec{\mathbf{I}}}{A} = \left(\frac{\pi d^2 V}{4\rho L}\hat{\mathbf{x}}\right) \Big/ \pi (d/2)^2 = \frac{V}{\rho L}\hat{\mathbf{x}}$$

(e)

$$\rho\vec{\mathbf{J}} = \rho\left(\frac{V}{\rho L}\hat{\mathbf{x}}\right) = \frac{V}{L}\hat{\mathbf{x}} = \vec{\mathbf{E}}$$

**Problem 2:**

The first telegraphic messages crossed the Atlantic Ocean in 1858, by a cable 3000 km long laid between Newfoundland and Ireland. The conductor in this cable consisted of seven copper wires, each of diameter 0.73 mm, bundled together and surrounded by an insulating sheath. Calculate the resistance of the conductor. Use $3\times10^{-8}\ \Omega\cdot\text{m}$ for the resistivity of copper, which was of somewhat dubious purity.

**Problem 2 Solution:**
When current flows in the cable, the ends of each of the seven copper wires are held at the same voltage difference, so the wires are in parallel. Recall that when resistors are in parallel, the equivalent resistance adds inversely:

$$\frac{1}{R_{eq}}=\frac{1}{R_1}+\frac{1}{R_2}+\cdots.$$

Since resistance is inversely proportional to area, we have that

$$\frac{1}{R_{eq}}=\frac{1}{R_1}+\frac{1}{R_2}+\cdots=\frac{A_1}{\rho L_1}+\frac{A_2}{\rho L_2}+\cdots.$$

The wires are all the same length and area so for seven wires

$$\frac{1}{R_{eq}}=\frac{7A}{\rho L}.$$

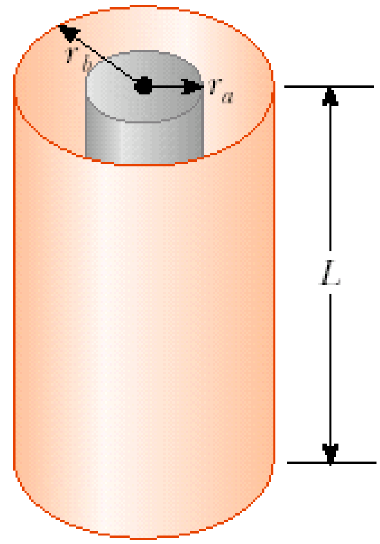
Thus the equivalent resistance is

$$R_{eq}=\frac{\rho L}{7A}=\frac{(3\times10^{-8}\ \Omega\cdot\text{m})(3\times10^{6}\ \text{m})}{(7)(\pi)(7.3\ \times10^{-4}\,\text{m}/2)^2}=3.0\times10^{4}\ \Omega\,.$$

Check: Since resistance is inversely proportional to area, the effective area is seven times the area of one wire.

**Problem 3:**

An oceanographer is studying how the ion concentration in sea water depends on depth. She does this by lowering into the water (until completely submerged) a pair of concentric metallic cylinders (see figure) at the end of a cable and taking data to determine the resistance between these electrodes as a function of depth. The water between the two cylinders forms a cylindrical shell of inner radius $r_a$ , outer radius $r_b$, and length $L$ much larger than $r_b$. The scientist applies a potential difference $\Delta V$ between the inner and outer surfaces, producing an outward radial current $I$. Let $\rho$ represent the resistivity of the water.




(a) Find the resistance of the water between the cylinders in terms of $L$, $\rho$, $r_a$, and $r_b$.

(b) Express the resistivity of the water in terms of the measured quantities $L$, $r_a$, $r_b$, $\Delta V$, and $I$.

**Problem 3 Solution:**

(a)We will build up the total resistance of the water by considering a number of cylindrical shells of water in series. Consider a thin cylindrical shell of radius $r$, thickness $dr$, and length $L$. Its contribution to the overall resistance of the water is

$$dR = \frac{\rho d\ell}{A} = \frac{\rho dr}{(2\pi r)L} = \left(\frac{\rho}{2\pi L}\right)\frac{dr}{r}$$

The resistance of the whole annulus is the series summation of the contributions of the thin shells:

$$R = \frac{\rho}{2\pi L}\int_{r_a}^{r_b}\frac{dr}{r} = \frac{\rho}{2\pi L}\ln\left(\frac{r_b}{r_a}\right)$$

(b) Since $R = \frac{\rho}{2\pi L}\ln\left(\frac{r_b}{r_a}\right) = \frac{\Delta V}{I}$, the resistivity can be written as

$$\rho = \frac{2\pi L(\Delta V / I)}{\ln\left(r_b / r_a\right)}$$

**Problem 4:**

A cylindrical glass rod is heated with a torch until it conducts enough current to cause a light bulb to glow. The rod has a length L = 2 cm, a diameter d = 0.5 cm, and its ends, plated with material of infinite conductivity, are connected to the rest of the circuit. When red hot, the rod's conductivity varies with position x measured from the center of the rod as $\sigma(x) = \sigma_0 L^4/x^4$ , with $\sigma_0 = 4\times10^{-2}(\Omega - m)^{-1}$.

a) What is the resistance of the glass rod? Express your answer both symbolically and as a value in ohms.

b) When a voltage $\Delta V$ is applied between the two ends, what is the current density $\vec{\mathbf{J}}(x)$ and what is the steady-state electric field $\vec{\mathbf{E}}(x)$?

**Problem 4 Solution:**

(a) We shall consider a small slice of the rod of thickness $dx$ located a distance $x$ measured from the center of the rod . The resistance $dR$ of this small section is given by

$$dR = \frac{\rho_r dx}{A} = \frac{dx}{\sigma_c A} = \frac{4x^4 dx}{\sigma_0 L^4 \pi d^2}.$$

The total resistance of the rod is then the integral over the whole rod since these small sections can be thought of as in series,

$$R = \int_{-L/2}^{L/2} dR = \int_{-L/2}^{L/2} \frac{4x^4 dx}{\sigma_0 L^4 \pi d^2} = \left.\frac{4x^5}{5\sigma_0 L^4 \pi d^2}\right|_{-L/2}^{L/2} = \frac{L}{20\sigma_0 \pi d^2}.$$

$$R = \frac{(2\times10^{-2}\text{ m})}{(20)(4\times10^{-2}(\Omega\cdot\text{m})^{-1})(\pi)(0.5\times10^{-2}\text{ m})^2} = 318\,\Omega$$

(b) The current density is given by

$$\left|\vec{\mathbf{J}}(x)\right| = \frac{I}{A} = \frac{4\Delta V}{R\pi d^2} = \frac{80\sigma_0 \Delta V}{L}$$

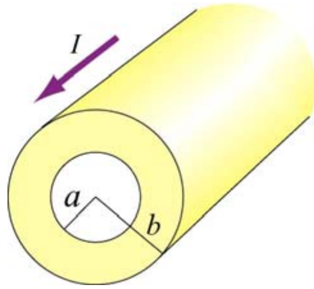
The direction of the current density $\vec{\mathbf{J}}(x)$ is from the high voltage to the low voltage side of the voltage source. The magnitude of the current density is independent of the position in the rod.

The electric field is given by

$$\vec{\mathbf{E}}(x) = \frac{\vec{\mathbf{J}}(x)}{\sigma} = \frac{80\sigma_0 \Delta V / L}{\sigma_0 L^4 / x^4}\,\hat{\mathbf{i}} = \frac{80\Delta V}{L^5} x^4 \hat{\mathbf{i}}\,.$$

**Problem 5:**

Consider a cylindrical conductor with a hollow center and walls of inner and outer radii $a$ and $b$ respectively. The current $I$ is ***non-uniformly*** spread over the cross section of the conductor, with a density that falls exponentially from the outer surface: $J_0 e^{\left(r^2-b^2\right)/\delta^2}$.




Find the constant $J_0$ in order to have total current $I$, given that δ, the "skin depth," is a distance much smaller than the wall thickness (b-a).

**Problem 5 Solution:**

We just need to integrate the current density and set it equal to the current, then solve:

$$I = \iint_{\text{wire}} J \cdot dA = \int_{r=a}^{b} J_0 e^{\left(r^2-b^2\right)/\delta^2} \cdot 2\pi r\, dr; \qquad u-\text{sub: } u = \left(r^2 - b^2\right)/\delta^2; \quad du = 2rdr/\delta^2$$

$$I = J_0 \pi \delta^2 \int_{u=\left(a^2-b^2\right)/\delta^2}^{0} e^u\, du = J_0 \pi \delta^2 \, e^u \Big|_{\left(a^2-b^2\right)/\delta^2}^{0} = J_0 \pi \delta^2 \left( 1 - \underbrace{e^{\left(a^2-b^2\right)/\delta^2}}_{\approx 0 \text{ since } \delta \ll (b-a)} \right) \approx J_0 \pi \delta^2$$

$$\Rightarrow \boxed{J_0 = \frac{I}{\pi \delta^2}}$$