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Module 15: Sections 7.6 through 7.11[1]

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Direct-Current Circuits

7.1 Introduction

Electrical circuits connect power supplies to *loads* such as resistors, motors, heaters, or lamps. The connection between the supply and the load is made by soldering with wires that are often called *leads*, or with many kinds of connectors and terminals. Energy is delivered from the source to the user on demand at the flick of a switch. Sometimes many circuit elements are connected to the same lead, which is the called a *common lead* for those elements. Various parts of the circuits are called circuit elements, which can be in series or in parallel, as we have already seen in the case of capacitors.

Elements are said to be in *parallel* when they are connected across the same potential difference (see Figure 7.1.1a).

Figure 7.1.1 Elements connected (a) in parallel, and (b) in series.

Generally, loads are connected in parallel across the power supply. On the other hand, when the elements are connected one after another, so that the current passes through each element without any branches, the elements are in *series* (see Figure 7.1.1b).

There are pictorial diagrams that show wires and components roughly as they appear, and schematic diagrams that use conventional symbols, somewhat like road maps. Some frequently used symbols are shown below:

Voltage Source	
Resistor	
Switch	

Often there is a switch in series; when the switch is open the load is disconnected; when the switch is closed, the load is connected.

One can have closed circuits, through which current flows, or open circuits in which there are no currents. Usually by accident, wires may touch, causing a *short circuit*. Most of the current flows through the short, very little will flow through the load. This may burn out a piece of electrical equipment such as a transformer. To prevent damage, a fuse or circuit breaker is put in series. When there is a short the fuse blows, or the breaker opens.

In electrical circuits, a point (or some common lead) is chosen as the *ground*. This point is assigned an arbitrary voltage, usually zero, and the voltage *V* at any point in the circuit is defined as the voltage difference between that point and ground.

7.2 Electromotive Force

In the last Chapter, we have shown that electrical energy must be supplied to maintain a constant current in a closed circuit. The source of energy is commonly referred to as the electromotive force, or emf (symbol ε). Batteries, solar cells and thermocouples are some examples of emf source. They can be thought of as a "charge pump" that moves charges from lower potential to the higher one. Mathematically emf is defined as

$$\varepsilon \equiv \frac{dW}{dq} \tag{7.2.1}$$

which is the work done to move a unit charge in the direction of higher potential. The SI unit for ε is the volt (V).

Consider a simple circuit consisting of a battery as the emf source and a resistor of resistance *R*, as shown in Figure 7.2.1.
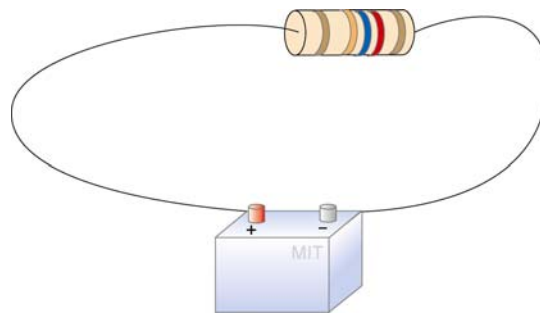

Figure 7.2.1 A simple circuit consisting of a battery and a resistor

Assuming that the battery has no internal resistance, the potential difference ΔV (or terminal voltage) between the positive and the negative terminals of the battery is equal to the emf ε. To drive the current around the circuit, the battery undergoes a discharging process which converts chemical energy to emf (recall that the dimensions of emf are the same as energy per charge). The current *I* can be found by noting that no work is done in

moving a charge q around a closed loop due to the conservative nature of the electrostatic force:

$$W = -q\oint \vec{E} \cdot d\vec{s} = 0 \qquad (7.2.2)$$

Let point a in Figure 7.2.2 be the starting point.

Figure 7.2.2
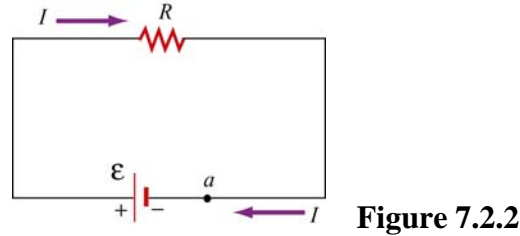
When crossing from the negative to the positive terminal, the potential increases by ε. On the other hand, as we cross the resistor, the potential decreases by an amount IR, and the potential energy is converted into thermal energy in the resistor. Assuming that the connecting wire carries no resistance, upon completing the loop, the net change in potential difference is zero,

$$\varepsilon - IR = 0 \qquad (7.2.3)$$

which implies

$$I = \frac{\varepsilon}{R} \qquad (7.2.4)$$

However, a real battery always carries an internal resistance r (Figure 7.2.3a),
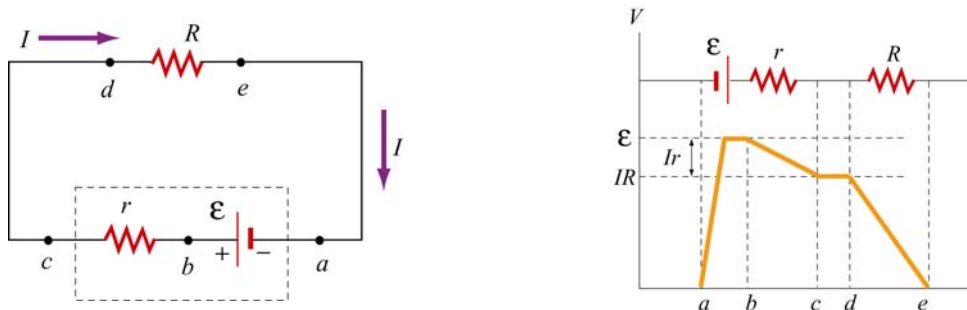

Figure 7.2.3 (a) Circuit with an emf source having an internal resistance r and a resistor of resistance R. (b) Change in electric potential around the circuit.

and the potential difference across the battery terminals becomes

$$\Delta V = \varepsilon - Ir \qquad (7.2.5)$$

Since there is no net change in potential difference around a closed loop, we have

$$\varepsilon - Ir - IR = 0 \tag{7.2.6}$$

or

$$I = \frac{\varepsilon}{R + r} \tag{7.2.7}$$

Figure 7.2.3(b) depicts the change in electric potential as we traverse the circuit clockwise. From the Figure, we see that the highest voltage is immediately after the battery. The voltage drops as each resistor is crossed. Note that the voltage is essentially constant along the wires. This is because the wires have a negligibly small resistance compared to the resistors.

For a source with emf ε, the power or the rate at which energy is delivered is

$$P = I\varepsilon = I(IR + Ir) = I^2 R + I^2 r \tag{7.2.8}$$

That the power of the source emf is equal to the sum of the power dissipated in both the internal and load resistance is required by energy conservation.

7.3 Resistors in Series and in Parallel

The two resistors R_1 and R_2 in Figure 7.3.1 are connected in series to a voltage source ΔV. By current conservation, the same current I is flowing through each resistor.

Figure 7.3.1 (a) Resistors in series. (b) Equivalent circuit.
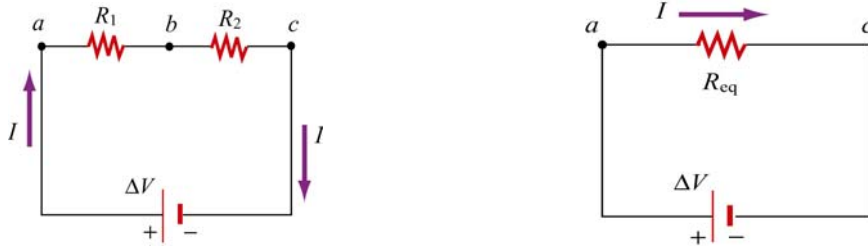
The total voltage drop from a to c across both elements is the sum of the voltage drops across the individual resistors:

$$\Delta V = I R_1 + I R_2 = I \left(R_1 + R_2 \right) \tag{7.3.1}$$

The two resistors in series can be replaced by one equivalent resistor R_{eq} (Figure 7.3.1b) with the identical voltage drop $\Delta V = I R_{eq}$ which implies that

$$R_{eq} = R_1 + R_2 \tag{7.3.2}$$

The above argument can be extended to N resistors placed in series. The equivalent resistance is just the sum of the original resistances,

$$R_{eq} = R_1 + R_2 + \cdots = \sum_{i=1}^{N} R_i \tag{7.3.3}$$

Notice that if one resistance R_1 is much larger than the other resistances R_i, then the equivalent resistance R_{eq} is approximately equal to the largest resistor R_1.

Next let's consider two resistors R_1 and R_2 that are connected in parallel across a voltage source ΔV (Figure 7.3.2a).
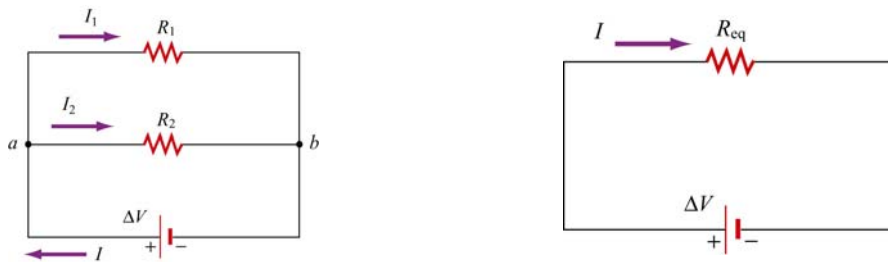

Figure 7.3.2 (a) Two resistors in parallel. (b) Equivalent resistance

By current conservation, the current I that passes through the voltage source must divide into a current I_1 that passes through resistor R_1 and a current I_2 that passes through resistor R_2. Each resistor individually satisfies Ohm's law, $\Delta V_1 = I_1 R_1$ and $\Delta V_2 = I_2 R_2$. However, the potential across the resistors are the same, $\Delta V_1 = \Delta V_2 = \Delta V$. Current conservation then implies

$$I = I_1 + I_2 = \frac{\Delta V}{R_1} + \frac{\Delta V}{R_2} = \Delta V \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \tag{7.3.4}$$

The two resistors in parallel can be replaced by one equivalent resistor R_{eq} with $\Delta V = I R_{eq}$ (Figure 7.3.2b). Comparing these results, the equivalent resistance for two resistors that are connected in parallel is given by

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{7.3.5}$$

This result easily generalizes to N resistors connected in parallel

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots = \sum_{i=1}^{N} \frac{1}{R_i} \qquad (7.3.6)$$

When one resistance R_1 is much smaller than the other resistances R_i, then the equivalent resistance R_{eq} is approximately equal to the smallest resistor R_1. In the case of two resistors,

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2} \approx \frac{R_1 R_2}{R_2} = R_1$$

This means that almost all of the current that enters the node point will pass through the branch containing the smallest resistance. So, when a short develops across a circuit, all of the current passes through this path of nearly zero resistance.

7.4 Kirchhoff's Circuit Rules

In analyzing circuits, there are two fundamental (Kirchhoff's) rules:

1. Junction Rule

At any point where there is a junction between various current carrying branches, by current conservation the sum of the currents into the node must equal the sum of the currents out of the node (otherwise charge would build up at the junction);

$$\sum I_{in} = \sum I_{out} \qquad (7.4.1)$$

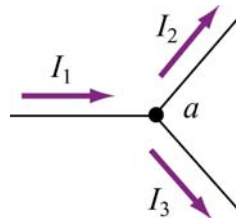As an example, consider Figure 7.4.1 below:

Figure 7.4.1 Kirchhoff's junction rule.

According to the junction rule, the three currents are related by

$$I_1 = I_2 + I_3$$

2. Loop Rule

The sum of the voltage drops ΔV, across any circuit elements that form a closed circuit is zero:

$$\sum_{closed\ loop} \Delta V = 0 \tag{7.4.2}$$

The rules for determining ΔV across a resistor and a battery with a designated travel direction are shown below:
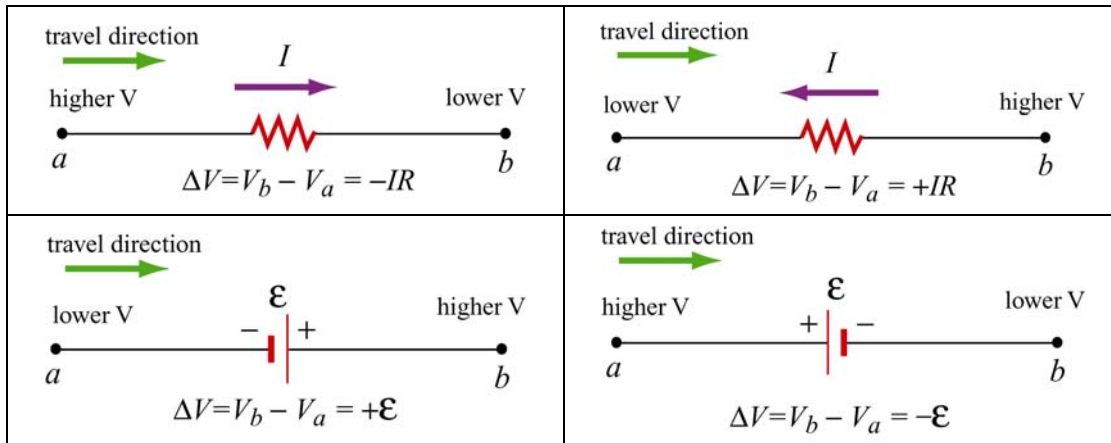

Figure 7.4.2 Convention for determining ΔV.

Note that the choice of travel direction is arbitrary. The same equation is obtained whether the closed loop is traversed clockwise or counterclockwise.

As an example, consider a voltage source V_{in} that is connected in series to two resistors, R_1 and R_2
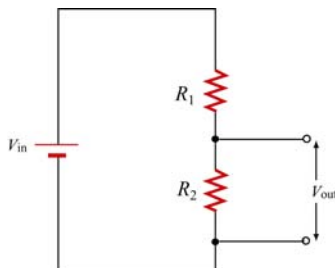

Figure 7.4.3 Voltage divider.

The voltage difference, V_{out}, across resistor R_2 will be less than V_{in}. This circuit is called a voltage divider. From the loop rule,

$$V_{in} - IR_1 - IR_2 = 0 \tag{7.4.3}$$

So the current in the circuit is given by

$$I = \frac{V_{in}}{R_1 + R_2}$$

(7.4.4)

Thus the voltage difference, V_{out}, across resistor R_2 is given by

$$V_{out} = IR_2 = \frac{R_2}{R_1 + R_2} V_{in}$$

(7.4.5)

Note that the ratio of the voltages characterizes the voltage divider and is determined by the resistors:

$$\frac{V_{out}}{V_{in}} = \frac{R_2}{R_1 + R_2}$$

(7.4.6)

7.5 Voltage-Current Measurements

Any instrument that measures voltage or current will disturb the circuit under observation. Some devices, known as ammeters, will indicate the flow of current by a meter movement or a digital display. There will be some voltage drop due to the resistance of the flow of current through the ammeter. An ideal ammeter has zero resistance, but in the case of your multimeter, the resistance is 1Ω on the 250 mDCA range. The drop of 0.25 V may or may not be negligible; knowing the meter resistance allows one to correct for its effect on the circuit.

An ammeter can be converted to a voltmeter by putting a resistor R in series with the coil movement. The voltage across some circuit element can be determined by connecting the coil movement and resistor in parallel with the circuit element. This causes a small amount of current to flow through the coil movement. The voltage across the element can now be determined by measuring I and computing the voltage from $\Delta V = IR$,which is read on a calibrated scale. The larger the resistance R, the smaller the amount of current is diverted through the coil. Thus an ideal voltmeter would have an infinite resistance.

Resistor Value Chart					
0	Black	4	Yellow	8	Gray
1	Brown	5	Green	9	White
2	Red	6	Blue	-1	Gold
3	Orange	7	Violet	-2	Silver

The colored bands on a composition resistor specify numbers according to the chart above (2-7 follow the rainbow spectrum). Starting from the end to which the bands are closest, the first two numbers specify the significant figures of the value of the resistor

and the third number represents a power of ten by which the first two numbers are to be multiplied (gold is 10^{-1}). The fourth specifies the "tolerance," or precision, gold being 5% and silver 10%. As an example, a 43-Ω (43 ohms) resistor with 5% tolerance is represented by yellow, orange, black, gold.

7.6 RC Circuit

7.6.1 Charging a Capacitor

Consider the circuit shown below. The capacitor is connected to a DC voltage source of emf ε. At time $t=0$, the switch S is closed. The capacitor initially is uncharged, $q(t=0)=0$.

Figure 7.6.1 (a) RC circuit diagram for $t < 0$. (b) Circuit diagram for $t > 0$.

In particular for $t < 0$, there is no voltage across the capacitor so the capacitor acts like a short circuit. At $t=0$, the switch is closed and current begins to flow according to

$$I_0 = \frac{\varepsilon}{R} \tag{7.6.1}.$$

At this instant, the potential difference from the battery terminals is the same as that across the resistor. This initiates the charging of the capacitor. As the capacitor starts to charge, the voltage across the capacitor increases in time

$$V_C(t) = \frac{q(t)}{C} \tag{7.6.2}$$

Figure 7.6.2 Kirchhoff's rule for capacitors.

Using Kirchhoff's loop rule shown in Figure 7.6.2 for capacitors and traversing the loop clockwise, we obtain

$$0 = \mathcal{E} - I(t)R - V_C(t)$$
$$= \mathcal{E} - \frac{dq}{dt}R - \frac{q}{C}$$

(7.6.3)

where we have substituted $I = +dq/dt$ for the current. Since I must be the same in all parts of the series circuit, the current across the resistance R is equal to the rate of increase of charge on the capacitor plates. The current flow in the circuit will continue to decrease because the charge already present on the capacitor makes it harder to put more charge on the capacitor. Once the charge on the capacitor plates reaches its maximum value Q, the current in the circuit will drop to zero. This is evident by rewriting the loop law as

$$I(t)R = \mathcal{E} - V_C(t)$$

(7.6.4).

Thus, the charging capacitor satisfies a first order differential equation that relates the rate of change of charge to the charge on the capacitor:

$$\frac{dq}{dt} = \frac{1}{R}\left(\mathcal{E} - \frac{q}{C}\right)$$

(7.6.5)

This equation can be solved by the method of separation of variables. The first step is to separate terms involving charge and time, (this means putting terms involving dq and q on one side of the equality sign and terms involving dt on the other side),

$$\frac{dq}{\left(\mathcal{E} - \dfrac{q}{C}\right)} = \frac{1}{R}dt \quad \Rightarrow \quad \frac{dq}{q - C\mathcal{E}} = -\frac{1}{RC}dt$$

(7.6.6).

Now we can integrate both sides of the above equation,

$$\int_0^q \frac{dq'}{q' - C\mathcal{E}} = -\frac{1}{RC}\int_0^t dt'$$

(7.6.7)

which yields

$$\ln\left(\frac{q - C\mathcal{E}}{-C\mathcal{E}}\right) = -\frac{t}{RC}$$

(7.6.8)

This can now be exponentiated using the fact that $\exp(\ln x) = x$ to yield

$$q(t) = C\varepsilon\left(1 - e^{-t/RC}\right) = Q\left(1 - e^{-t/RC}\right) \qquad (7.6.9)$$

where $Q = C\varepsilon$ is the maximum amount of charge stored on the plates. The time dependence of $q(t)$ is plotted in Figure 7.6.3 below:
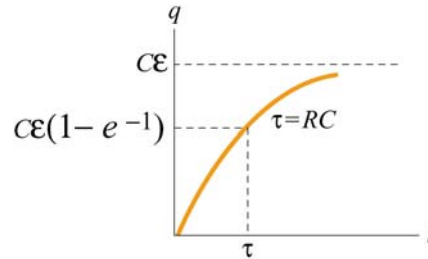

Figure 7.6.3 Charge as a function of time during the charging process.

Once we know the charge on the capacitor we also can determine the voltage across the capacitor,

$$V_C(t) = \frac{q(t)}{C} = \varepsilon(1 - e^{-t/RC}) \qquad (7.6.10)$$

The graph of voltage as a function of time has the same form as Figure 7.6.3. From the figure, we see that after a sufficiently long time the charge on the capacitor approaches the value

$$q(t = \infty) = C\varepsilon = Q \qquad (7.6.11).$$

At that time, the voltage across the capacitor is equal to the applied voltage source and the charging process effectively ends,

$$V_C = \frac{q(t = \infty)}{C} = \frac{Q}{C} = \varepsilon \qquad (7.6.12).$$

The current that flows in the circuit is equal to the derivative in time of the charge,

$$I(t) = \frac{dq}{dt} = \left(\frac{\varepsilon}{R}\right)e^{-t/RC} = I_0\, e^{-t/RC} \qquad (7.6.13).$$

The coefficient in front of the exponential is equal to the initial current that flows in the circuit when the switch was closed at $t = 0$. The graph of current as a function of time is shown in Figure 7.6.4 below:
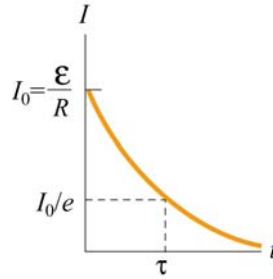
Figure 7.6.4 Current as a function of time during the charging process

The current in the charging circuit decreases exponentially in time, $I(t) = I_0 e^{-t/RC}$. This function is often written as $I(t) = I_0 e^{-t/\tau}$ where $\tau = RC$ is called the *time constant*. The SI units of τ are seconds, as can be seen from the dimensional analysis:

$$[\Omega][F] = ([V]/[A])([C]/[V]) = [C]/[A] = [C]/([C]/[s]) = [s]$$

The time constant τ is a measure of the decay time for the exponential function. This decay rate satisfies the following property:

$$I(t+\tau) = I(t)e^{-1} \tag{7.6.14}$$

which shows that after one time constant τ has elapsed, the current falls off by a factor of $e^{-1} = 0.368$, as indicated in Figure 7.6.4 above. Similarly, the voltage across the capacitor (Figure 7.6.5 below) can also be expressed in terms of the time constant τ:

$$V_C(t) = \mathcal{E}(1 - e^{-t/\tau}) \tag{7.6.15}$$
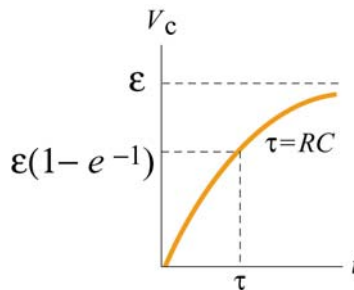
Figure 7.6.5 Voltage across capacitor as a function of time during the charging process.

Notice that initially at time $t = 0$, $V_C(t=0) = 0$. After one time constant τ has elapsed, the potential difference across the capacitor plates has increased by a factor $(1 - e^{-1}) = 0.632$ of its final value:

$$V_C(\tau) = \mathcal{E}(1 - e^{-1}) = 0.632\,\mathcal{E} \tag{7.6.16}$$

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7.6.2 Discharging a Capacitor

Suppose initially the capacitor has been charged to some value Q. For $t < 0$, the switch is open and the potential difference across the capacitor is given by $V_C = Q/C$. On the other hand, the potential difference across the resistor is zero because there is no current flow, that is, $I = 0$. Now suppose at $t = 0$ the switch is closed (Figure 7.6.6). The capacitor will begin to discharge.
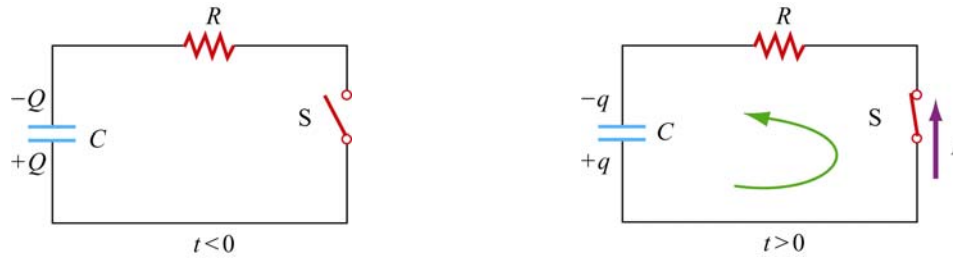

Figure 7.6.6 Discharging the RC circuit

The charged capacitor is now acting like a voltage source to drive current around the circuit. When the capacitor discharges (electrons flow from the negative plate through the wire to the positive plate), the voltage across the capacitor decreases. The capacitor is losing strength as a voltage source. Applying the Kirchhoff's loop rule by traversing the loop counterclockwise, the equation that describes the discharging process is given by

$$\frac{q}{C} - IR = 0 \tag{7.6.17}$$

The current that flows away from the positive plate is proportional to the charge on the plate,

$$I = -\frac{dq}{dt} \tag{7.6.18}$$

The negative sign in the equation is an indication that the rate of change of the charge is proportional to the *negative* of the charge on the capacitor. This is due to the fact that the charge on the positive plate is decreasing as more positive charges leave the positive plate. Thus, charge satisfies a first order differential equation:

$$\frac{q}{C} + R\frac{dq}{dt} = 0 \tag{7.6.19}.$$

This equation can also be integrated by the method of separation of variables

$$\frac{dq}{q} = -\frac{1}{RC}dt \tag{7.6.20}$$

which yields

$$\int_Q^q \frac{dq'}{q'} = -\frac{1}{RC}\int_0^t dt' \quad \Rightarrow \quad \ln\left(\frac{q}{Q}\right) = -\frac{t}{RC} \tag{7.6.21}$$

or

$$\boxed{q(t) = Q\,e^{-t/RC}} \tag{7.6.22}$$

The voltage across the capacitor is then

$$V_C(t) = \frac{q(t)}{C} = \left(\frac{Q}{C}\right)e^{-t/RC} \tag{7.6.23}$$

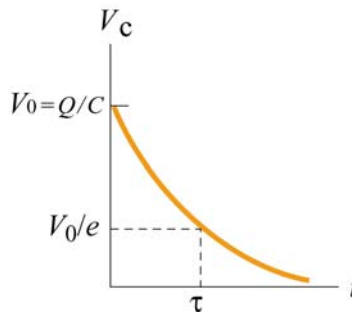A graph of voltage across the capacitor vs. time for the discharging capacitor is shown in Figure 7.6.7.

Figure 7.6.7 Voltage across the capacitor as a function of time for discharging capacitor.

The current also exponentially decays in the circuit as can be seen by differentiating the charge on the capacitor

$$I = -\frac{dq}{dt} = \left(\frac{Q}{RC}\right)e^{-t/RC} \tag{7.6.24}$$

A graph of the current flowing in the circuit as a function of time also has the same form as the voltage graph depicted in Figure 7.6.8.
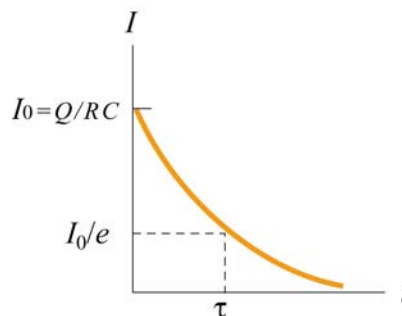
Figure 7.6.8 Current as a function of time for discharging capacitor.

7.7 Summary

- The equivalent resistance of a set of resistors connected in series:

$$R_{eq} = R_1 + R_2 + R_3 + \cdots = \sum_{i=1}^{N} R_i$$

- The quivalent resistance of a set of resistors connected in parallel:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots = \sum_{i=1}^{N} \frac{1}{R_i}$$

- **Kirchhoff's rules**:

(1) The sum of the currents flowing into a junction is equal to the sum of the currents flowing out of the junction:

$$\sum I_{in} = \sum I_{out}$$

(2) The algebraic sum of the changes in electric potential in a closed-circuit loop is zero.

$$\sum_{closed\ loop} \Delta V = 0$$

- In a charging capacitor, the charges and the current as a function of time are

$$q(t) = Q\left(1 - e^{-\frac{t}{RC}}\right), \qquad I(t) = \left(\frac{\varepsilon}{R}\right) e^{-t/RC}$$

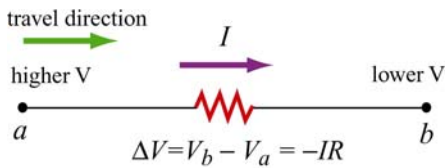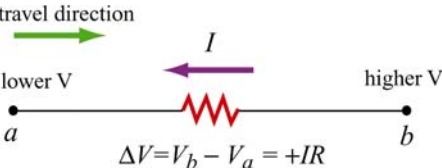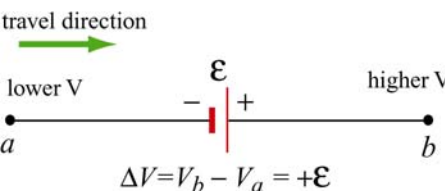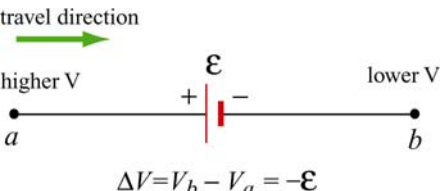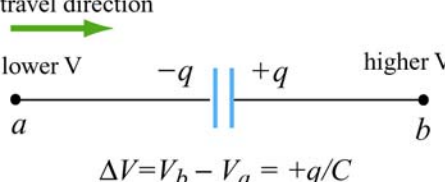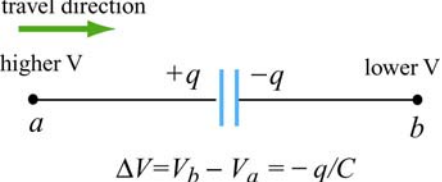- In a discharging capacitor, the charges and the current as a function of time are

$$q(t) = Q\,e^{-t/RC}, \qquad I(t) = \left(\frac{Q}{RC}\right) e^{-t/RC}$$

7.8 Problem-Solving Strategy: Applying Kirchhoff's Rules

In this chapter we have seen how Kirchhoff's rules can be used to analyze multiloop circuits. The steps are summarized below:

(1) Draw a circuit diagram, and label all the quantities, both known and unknown. The number of unknown quantities is equal to the number of linearly independent equations we must look for.

(2) Assign a direction to the current in each branch of the circuit. (If the actual direction is opposite to what you have assumed, your result at the end will be a negative number.)

(3) Apply the junction rule to all but one of the junctions. (Applying the junction rule to the last junction will not yield any independent relationship among the currents.)

(4) Apply the loop rule to the loops until the number of independent equations obtained is the same as the number of unknowns. For example, if there are three unknowns, then we must write down three linearly independent equations in order to have a unique solution.

Traverse the loops using the convention below for ΔV :

resistor	travel direction higher V ———ⱲⱲ——— lower V a I b $\Delta V = V_b - V_a = -IR$	travel direction lower V ———ⱲⱲ——— higher V a I b $\Delta V = V_b - V_a = +IR$
emf source	travel direction ε lower V —— $-\|+$ —— higher V a b $\Delta V = V_b - V_a = +\varepsilon$	travel direction ε higher V —— $+\|-$ —— lower V a b $\Delta V = V_b - V_a = -\varepsilon$
capacitor	travel direction lower V —— $-q\|\|+q$ —— higher V a b $\Delta V = V_b - V_a = +q/C$	travel direction higher V —— $+q\|\|-q$ —— lower V a b $\Delta V = V_b - V_a = -q/C$

The same equation is obtained whether the closed loop is traversed clockwise or counterclockwise. (The expressions actually differ by an overall negative sign. However, using the loop rule, we are led to $0 = -0$, and hence the same equation.)

(5) Solve the simultaneous equations to obtain the solutions for the unknowns.

As an example of illustrating how the above procedures are executed, let's analyze the circuit shown in Figure 7.8.1.
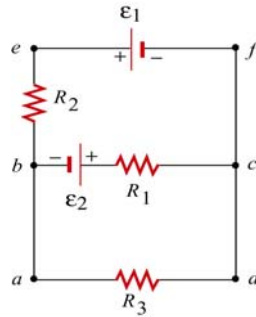
Figure 7.8.1 A multiloop circuit.

Suppose the emf sources ε_1 and ε_2, and the resistances R_1, R_2 and R_3 are all given, and we would like to find the currents through each resistor, using the methodology outlined above.

(1) The unknown quantities are the three currents I_1, I_2 and I_3, associated with the three resistors. Therefore, to solve the system, we must look for three independent equations.

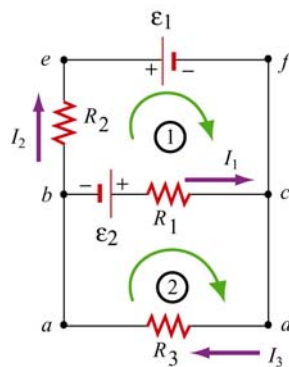(2) The directions for the three currents are arbitrarily assigned, as indicated in Figure 7.8.2.

Figure 7.8.2

(3) Applying Kirchhoff's current rule to junction b yields

$$I_1 + I_2 = I_3$$

since I_1 and I_2 are leaving the junction while I_3 is entering the junction. The same equation is obtained if we consider junction c.

(4) The other two equations can be obtained by using the loop (voltage) rule, which states that the net potential difference across all elements in a closed circuit loop is zero. Traversing the first loop *befcb* in the clockwise direction yields

$$-I_2 R_2 - \varepsilon_1 + I_1 R_1 - \varepsilon_2 = 0$$

Similarly, traversing the second loop *abcda* clockwise gives

$$\varepsilon_2 - I_1 R_1 - I_3 R_3 = 0$$

Note however, that one may also consider the big loop *abefcda*. This leads to

$$-I_2 R_2 - \varepsilon_1 - I_3 R_3 = 0$$

However, the equation is not linearly independent of the other two loop equations since it is simply the sum of those equations.

(5) The solutions to the above three equations are given by, after tedious but straightforward algebra,

$$I_1 = \frac{\varepsilon_1 R_3 + \varepsilon_2 R_3 + \varepsilon_2 R_2}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

$$I_2 = -\frac{\varepsilon_1 R_1 + \varepsilon_1 R_3 + \varepsilon_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

$$I_3 = \frac{\varepsilon_2 R_2 - \varepsilon_1 R_1}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

Note that I_2 is a negative quantity. This simply indicates that the direction of I_2 is opposite of what we have initially assumed.

7.9 Solved Problems

7.9.1 Equivalent Resistance

Consider the circuit shown in Figure 7.9.1. For a given resistance R_0, what must be the value of R_1 so that the equivalent resistance between the terminals is equal to R_0?
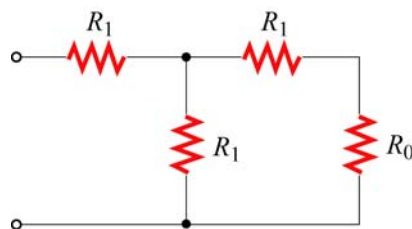

Figure 7.9.1

Solution:

The equivalent resistance, R', due to the three resistors on the right is

$$\frac{1}{R'} = \frac{1}{R_1} + \frac{1}{R_0 + R_1} = \frac{R_0 + 2R_1}{R_1(R_0 + R_1)}$$

or

$$R' = \frac{R_1(R_0 + R_1)}{R_0 + 2R_1}$$

Since R' is in series with the fourth resistor R_1, the equivalent resistance of the entire configuration becomes

$$R_{eq} = R_1 + \frac{R_1(R_0 + R_1)}{R_0 + 2R_1} = \frac{3R_1^2 + 2R_1 R_0}{R_0 + 2R_1}$$

If $R_{eq} = R_0$, then

$$R_0(R_0 + 2R_1) = 3R_1^2 + 2R_1 R_0 \implies R_0^2 = 3R_1^2$$

or

$$R_1 = \frac{R_0}{\sqrt{3}}$$

7.9.2 Variable Resistance

Show that, if a battery of fixed emf ε and internal resistance r is connected to a variable external resistance R, the maximum power is delivered to the external resistor when $R = r$.

Solution:

Using Kirchhoff's rule,

$$\varepsilon = I(R + r)$$

which implies

$$I = \frac{\varepsilon}{R + r}$$

The power dissipated is equal to

$$P = I^2 R = \frac{\varepsilon^2}{(R + r)^2} R$$

To find the value of R which gives out the maximum power, we differentiate P with respect to R and set the derivative equal to 0:

$$\frac{dP}{dR} = \varepsilon^2 \left[\frac{1}{(R+r)^2} - \frac{2R}{(R+r)^2} \right] = \varepsilon^2 \frac{r-R}{(R+r)^3} = 0$$

which implies

$$R = r$$

This is an example of "impedance matching," in which the variable resistance R is adjusted so that the power delivered to it is maximized. The behavior of P as a function of R is depicted in Figure 7.9.2 below.
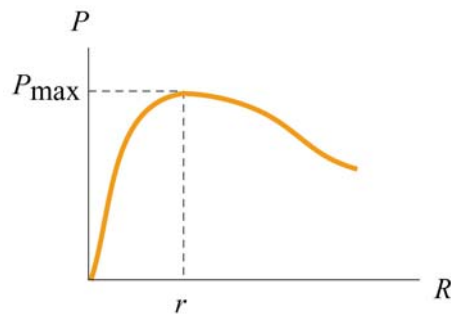

Figure 7.9.2

7.9.3 RC Circuit

In the circuit in figure 7.9.3, suppose the switch has been open for a very long time. At time $t = 0$, it is suddenly closed.
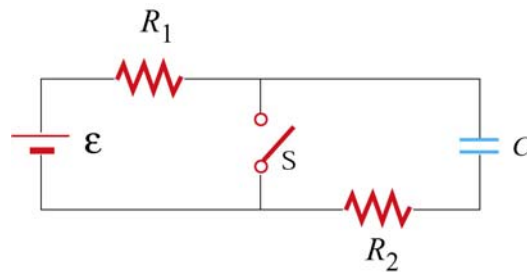

Figure 7.9.3

(a) What is the time constant before the switch is closed?

(b) What is the time constant after the switch is closed?

(c) Find the current through the switch as a function of time after the switch is closed.

Solutions:

(a) Before the switch is closed, the two resistors R_1 and R_2 are in series with the capacitor. Since the equivalent resistance is $R_{eq} = R_1 + R_2$, the time constant is given by

$$\tau = R_{eq}C = (R_1 + R_2)C$$

The amount of charge stored in the capacitor is

$$q(t) = C\varepsilon(1 - e^{-t/\tau})$$

(b) After the switch is closed, the closed loop on the right becomes a decaying RC circuit with time constant $\tau' = R_2 C$. Charge begins to decay according to

$$q'(t) = C\varepsilon e^{-t/\tau'}$$

(c) The current passing through the switch consists of two sources: the steady current I_1 from the left circuit, and the decaying current I_2 from the RC circuit. The currents are given by

$$I_1 = \frac{\varepsilon}{R_1}$$

$$I'(t) = \frac{dq'}{dt} = -\left(\frac{C\varepsilon}{\tau'}\right)e^{-t/\tau'} = -\left(\frac{\varepsilon}{R_2}\right)e^{-t/R_2 C}$$

The negative sign in $I'(t)$ indicates that the direction of flow is opposite of the charging process. Thus, since both I_1 and I' move downward across the switch, the total current is

$$I(t) = I_1 + I'(t) = \frac{\varepsilon}{R_1} + \left(\frac{\varepsilon}{R_2}\right)e^{-t/R_2 C}$$

7.9.4 Parallel vs. Series Connections

Figure 7.9.4 show two resistors with resistances R_1 and R_2 connected in parallel and in series. The battery has a terminal voltage of ε.
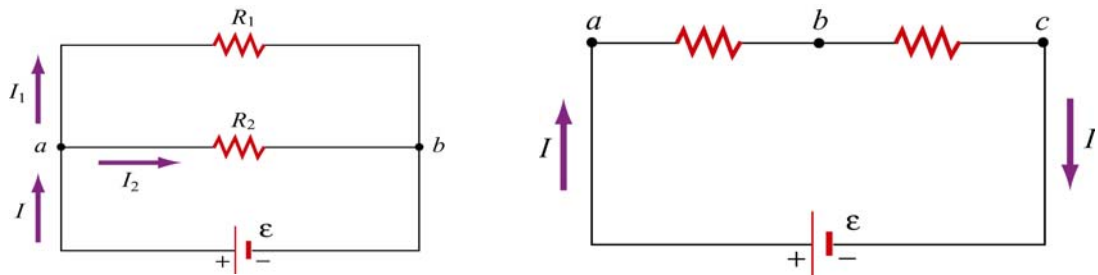
Figure 7.9.4

Suppose R_1 and R_2 are connected in parallel.

(a) Find the power delivered to each resistor.

(b) Show that the sum of the power used by each resistor is equal to the power supplied by the battery.

 Suppose R_1 and R_2 are now connected in series.

(c) Find the power delivered to each resistor.

(d) Show that the sum of the power used by each resistor is equal to the power supplied by the battery.

(e) Which configuration, parallel or series, uses more power?

Solutions:

(a) When two resistors are connected in parallel, the current through each resistor is

$$I_1 = \frac{\varepsilon}{R_1}, \quad I_2 = \frac{\varepsilon}{R_2}$$

and the power delivered to each resistor is given by

$$P_1 = I_1^2 R_1 = \frac{\varepsilon^2}{R_1}, \quad P_2 = I_2^2 R_2 = \frac{\varepsilon^2}{R_2}$$

The results indicate that the smaller the resistance, the greater the amount of power delivered. If the loads are the light bulbs, then the one with smaller resistance will be brighter since more power is delivered to it.

(b) The total power delivered to the two resistors is

$$P_R = P_1 + P_2 = \frac{\varepsilon^2}{R_1} + \frac{\varepsilon^2}{R_2} = \frac{\varepsilon^2}{R_{eq}}$$

where

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \quad \Rightarrow \quad R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

is the equivalent resistance of the circuit. On the other hand, the total power supplied by the battery is $P_\varepsilon = I\varepsilon$, where $I = I_1 + I_2$, as seen from the figure. Thus,

$$P_\varepsilon = I_1\varepsilon + I_2\varepsilon = \left(\frac{\varepsilon}{R_1}\right)\varepsilon + \left(\frac{\varepsilon}{R_2}\right)\varepsilon = \frac{\varepsilon^2}{R_1} + \frac{\varepsilon^2}{R_2} = \frac{\varepsilon^2}{R_{eq}} = P_R$$

as required by energy conservation.

(c) When the two resistors are connected in series, the equivalent resistance becomes

$$R'_{eq} = R_1 + R_2$$

and the currents through the resistors are

$$I_1 = I_2 = I = \frac{\varepsilon}{R_1 + R_2}$$

Therefore, the power delivered to each resistor is

$$P_1 = I_1^2 R_1 = \left(\frac{\varepsilon}{R_1 + R_2}\right)^2 R_1 \, , \quad P_2 = I_2^2 R_2 = \left(\frac{\varepsilon}{R_1 + R_2}\right)^2 R_2$$

Contrary to what we have seen in the parallel case, when connected in series, the greater the resistance, the greater the fraction of the power delivered. Once again, if the loads are light bulbs, the one with greater resistance will be brighter.

(d) The total power delivered to the resistors is

$$P'_R = P_1 + P_2 = \left(\frac{\varepsilon}{R_1 + R_2}\right)^2 R_1 + \left(\frac{\varepsilon}{R_1 + R_2}\right)^2 R_2 = \frac{\varepsilon^2}{R_1 + R_2} = \frac{\varepsilon^2}{R'_{eq}}$$

On the other hand, the power supplied by the battery is

$$P'_\varepsilon = I\varepsilon = \left(\frac{\varepsilon}{R_1 + R_2}\right)\varepsilon = \frac{\varepsilon^2}{R_1 + R_2} = \frac{\varepsilon^2}{R'_{eq}}$$

Again, we see that $P_\varepsilon' = P_R'$, as required by energy conservation.

(e) Comparing the results obtained in (b) and (d), we see that

$$P_\varepsilon = \frac{\varepsilon^2}{R_1} + \frac{\varepsilon^2}{R_2} > \frac{\varepsilon^2}{R_1 + R_2} = P_\varepsilon'$$

which means that the parallel connection uses more power. The equivalent resistance of two resistors connected in parallel is always smaller than that connected in series.

7.9.5 Resistor Network

Consider a cube which has identical resistors with resistance R along each edge, as shown in Figure 7.9.5.
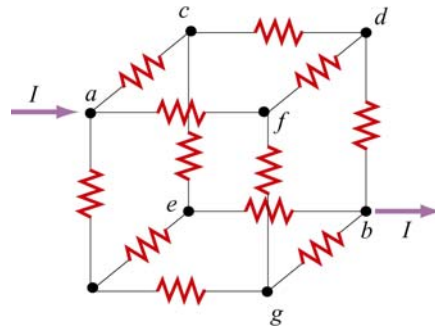
Figure 7.9.5 Resistor network

Show that the equivalent resistance between points a and b is $R_{eq} = 5R/6$.

Solution:

From symmetry arguments, the current which enters a must split evenly, with $I/3$ going to each branch. At the next junction, say c, $I/3$ must further split evenly with $I/6$ going through the two paths ce and cd. The current going through the resistor in db is the sum of the currents from fd and cd : $I/6 + I/6 = I/3$.

Thus, the potential difference between a and b can be obtained as

$$V_{ab} = V_{ac} + V_{cd} + V_{db} = \frac{I}{3}R + \frac{I}{6}R + \frac{I}{3}R = \frac{5}{6}IR$$

which shows that the equivalent resistance is

$$R_{eq} = \frac{5}{6}R$$

7.10 Conceptual Questions

1. Given three resistors of resistances R_1, R_2 and R_3, how should they be connected to (a) maximize (b) minimize the equivalent resistance?

2. Why do the headlights on the car become dim when the car is starting?

3. Does the resistor in an RC circuit affect the maximum amount of charge that can be stored in a capacitor? Explain.

4. Can one construct a circuit such that the potential difference across the terminals of the battery is zero? Explain.

7.11 Additional Problems

7.11.1 Resistive Circuits

Consider two identical batteries of emf ε and internal resistance r. They may be connected in series or in parallel and are used to establish a current in resistance R as shown in Figure 7.11.1.
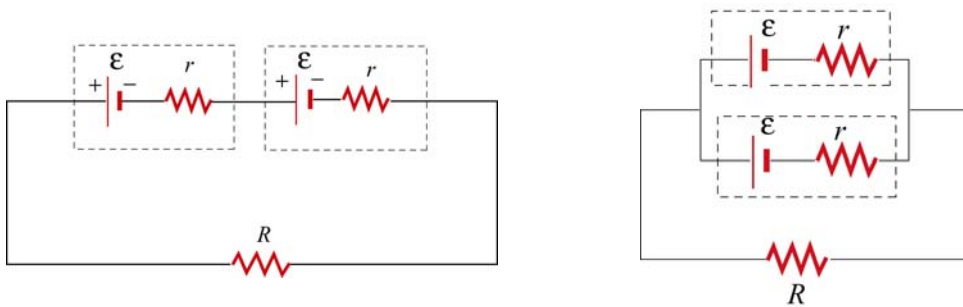

Figure 7.11.1 Two batteries connected in (a) series, and (b) parallel.

(a) Derive an expression for the current in R for the series connection shown in Figure 7.11.1(a). Be sure to indicate the current on the sketch (to establish a sign convention for the direction) and apply Kirchhoff's loop rule.

(b) Find the current for the parallel connection shown in Figure 7.11.1(b).

(c) For what relative values of r and R would the currents in the two configurations be the same?; be larger in Figure 7.11.1(a)?; be larger in 7.11.1(b)?

7.11.2 Multiloop Circuit

Consider the circuit shown in Figure 7.11.2. Neglecting the internal resistance of the batteries, calculate the currents through each of the three resistors.
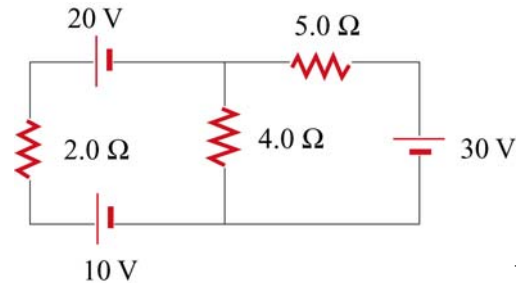
Figure 7.11.2

7.11.3 Power Delivered to the Resistors

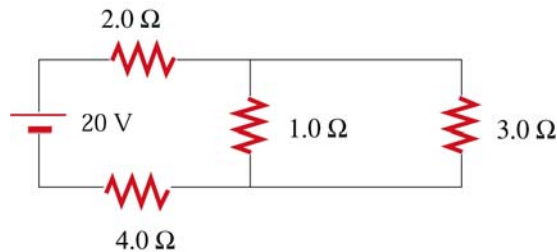Consider the circuit shown in Figure 7.11.3. Find the power delivered to each resistor.

Figure 7.11.3

7.11.4 Resistor Network

Consider an infinite network of resistors of resistances R_0 and R_1 shown in Figure 7.11.4. Show that the equivalent resistance of this network is
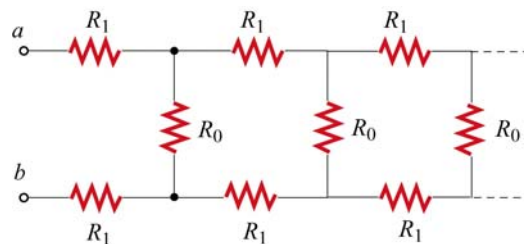
$$R_{eq} = R_1 + \sqrt{R_1^2 + 2R_1 R_0}$$

Figure 7.11.4

7.11.5 RC Circuit

Consider the circuit shown in Figure 7.11.5. Let $\varepsilon = 40$ V, $R_1 = 8.0\ \Omega$, $R_2 = 6.0\ \Omega$, $R_3 = 4.0\ \Omega$ and $C = 4.0\ \mu F$. The capacitor is initially uncharged.
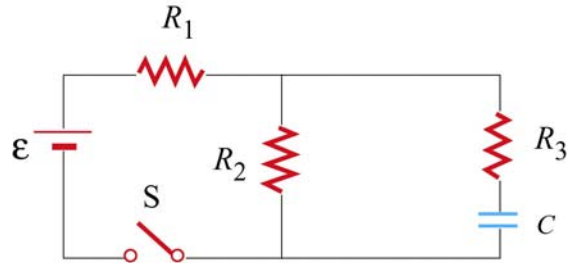

Figure 7.11.5

At $t = 0$, the switch is closed.

(a) Find the current through each resistor immediately after the switch is closed.

(b) Find the final charge on the capacitor.

7.11.6 Resistors in Series and Parallel

A circuit containing five resistors and a 12 V battery is shown in Figure 7.11.6. Find the potential drop across the 5Ω resistor. [Ans: 7.5 V].
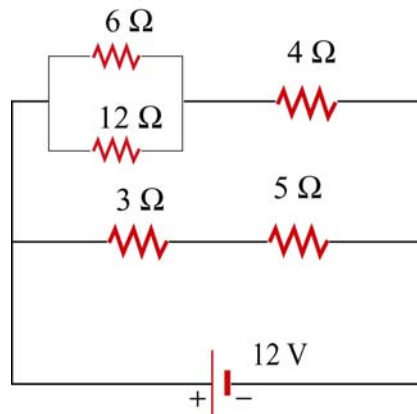

Figure 7.11.6

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Fall 2010