Capacitance & Capacitors, Energy Stored in Capacitors
Challenge Problem Solutions

Problem 1:

A parallel-plate capacitor is charged to a potential V_0, charge Q_0 and then disconnected from the battery. The separation of the plates is then halved. What happens to

(a) the charge on the plates?

(b) the electric field?

(c) the energy stored in the electric field?

(d) the potential?

(e) How much work did you do in halving the distance between the plates?

Problem 1 Solutions:
(a) No Change. We aren't attached to a battery, so the charge is fixed.

(b) No Change. The charge is constant so, in the planar geometry, so is the field.

(c) Halves. The volume in which we have field halves, so the energy does too.

(d) Halves. $V = E\,d$, so if d halves, so does V

(e) The work done is the change in energy. Energy, given the charge and potential, is:

$$U = \tfrac{1}{2}CV^2 = \tfrac{1}{2}QV$$

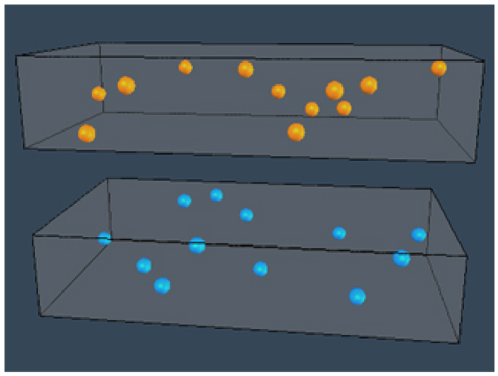
The energy halves, so the change is half the initial energy: $\boxed{W = \Delta U = -\tfrac{1}{4}Q_0V_0}$

Notice the sign – you did negative work bringing the plates together because that is the way they naturally want to move; the field did positive work.

Problem 2:

The simulation at
http://web.mit.edu/viz/EM/visualizations/electrostatics/CapacitorsAndCondcutors/capacitor/capacitor.htm

illustrates the interaction of charged particles inside the two plates of a capacitor. Each plate contains twelve charges interacting via the Coulomb and Pauli forces, where one plate contains positive charges and the other contains negative charges.



(a) Before running the simulation, **PREDICT** will happen to the charges (i.e. how will they arrange themselves). Now run the simulation. Describe what you observe.

(b) Suppose *both* the top and bottom plates now consist of twelve ***negative*** charges. What do you expect to see and why?

(c) Keeping the number of charges on the ***bottom*** array the <u>same (and negative)</u>, what do you suppose would happen if the **top** array had a <u>larger</u> amount of charge (i.e. sixteen positive charges, instead of twelve)? Explain.

(d) Suppose you now have *six positive charges* **AND** *six negative charges* on the <u>top array</u> and further suppose that the <u>bottom array</u> also consists of *six positive charges* **AND** six negative charges. What do you expect will happen and why?

Problem 2 Solutions:

(a) We expect that charges of the same sign will try to get as far away as possible from each other, and as close as possible to charges of opposite sign. This indeed happens—the charges end up on the inner surfaces of the two capacitor plates, spread along the edges.

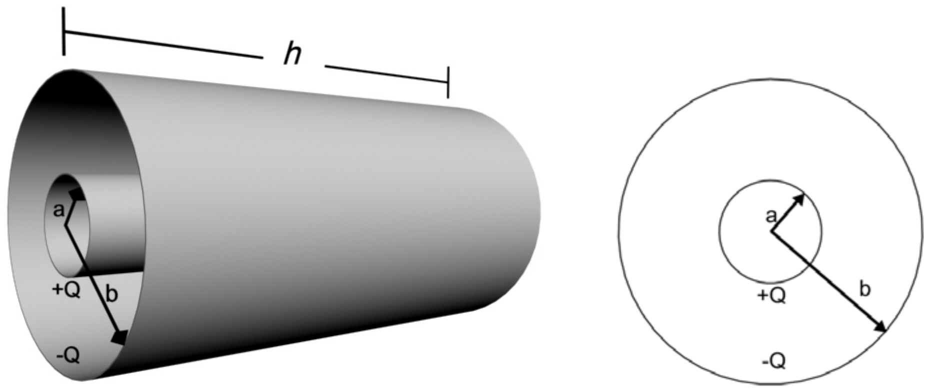
(b)We expect that charges of the same sign will try to get as far away as possible from each other. This indeed happens—the charges end up on the outer surfaces of the two capacitor plates, spread along the edges.

(c)Though most of the charges are still expected to be paired up with the opposite charges and stay on the edge of the inner surface as in the part (a), there will be some excess charges that sit on the outer surface of the capacitor plate with the most charge.

(d)The charges in each conductor will form bound systems with overall net zero charge. Depending on the orientation of the bound systems we may get a small positive or negative potential difference between the plates, but it will be random.

Problem 3:

Consider two nested cylindrical conductors of height h and radii a & b respectively. A charge $+Q$ is evenly distributed on the outer surface of the pail (the inner cylinder), $-Q$ on the inner surface of the shield (the outer cylinder).



(a) Calculate the electric field between the two cylinders ($a < r < b$).

(b) Calculate the potential difference between the two cylinders:

(c) Calculate the capacitance of this system, $C = Q/\Delta V$

(d) Numerically evaluate the capacitance for your experimental setup, given:
$h \cong 15$ cm, $a \cong 4.75$ cm and $b \cong 7.25$ cm

e) Find the electric field energy density at any point between the conducting cylinders. How much energy resides in a cylindrical shell between the conductors of radius r (with $a < r < b$), height h**,** thickness dr, and volume $2\pi rh\,dr$? Integrate your expression to find the total energy stored in the capacitor and compare your result with that obtained using $U_E = (1/2)C(\Delta V)^2$.

Problem 3 Solutions:

(a)For this we use Gauss's Law, with a Gaussian cylinder of radius r, height l

$$\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 2\pi r l E = \frac{Q_{inside}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\frac{Q}{h} l \quad \Rightarrow \quad E(r)_{a<r<b} = \frac{Q}{2\pi r \varepsilon_0 h}$$

(b)The potential difference between the outer shell and the inner cylinder is

$$\Delta V = V(a) - V(b) = -\int_b^a \frac{Q}{2\pi r' \varepsilon_0 h} dr' = -\frac{Q}{2\pi \varepsilon_0 h} \ln r' \Big|_b^a = \frac{Q}{2\pi \varepsilon_0 h} \ln\left(\frac{b}{a}\right)$$

(c)

$$C = \frac{|Q|}{|\Delta V|} = \frac{|Q|}{\frac{|Q|}{2\pi\varepsilon_0 h}\ln\left(\frac{b}{a}\right)} = \frac{2\pi\varepsilon_o h}{\ln\left(\frac{b}{a}\right)}$$

(d)

$$C = \frac{2\pi\varepsilon_o h}{\ln\left(\frac{b}{a}\right)} = \frac{1}{2\cdot 9\times 10^9 \text{ m F}^{-1}} \frac{15 \text{ cm}}{\ln\left(\frac{7.25 \text{ cm}}{4.75 \text{ cm}}\right)} \cong 20 \text{ pF}$$

(e) The total energy stored in the capacitor is

$$u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0 \left(\frac{Q}{2\pi r \varepsilon_0 h}\right)^2$$

Then

$$dU = u_E dV = \frac{1}{2}\varepsilon_0 \left(\frac{Q}{2\pi r \varepsilon_0 h}\right)^2 2\pi r h \, dr = \frac{Q^2}{4\pi\varepsilon_0 h}\frac{dr}{r}$$

Integrating we find that

$$U = \int_a^b dU = \int_a^b \frac{Q^2}{4\pi\varepsilon_0 h}\frac{dr}{r} = \frac{Q^2}{4\pi\varepsilon_0 h}\ln(b/a)\,.$$

From part d) $C = 2\pi\varepsilon_o h / \ln(b/a)$, therefore

$$U = \int_a^b dU = \int_a^b \frac{Q^2}{4\pi\varepsilon_0 h}\frac{dr}{r} = \frac{Q^2}{4\pi\varepsilon_0 h}\ln(b/a) = \frac{Q^2}{2C} = \frac{1}{2}C\Delta V^2$$

which agrees with that obtained above.

Problem 4:

A parallel-plate capacitor is charged to a potential V_0, charge Q_0 and then disconnected from the battery. The separation of the plates is then halved. What happens to

(a) the charge on the plates?

(b) the electric field?
(c) the energy stored in the electric field?

(d) the potential?

(e) How much work did you do in halving the distance between the plates?

Problem 4 Solutions:

(a)No Change. We aren't attached to a battery, so the charge is fixed.

(b)No Change. The charge is constant so, in the planar geometry, so is the field.

(c)Halves. The volume in which we have field halves, so the energy does too.

(d)Halves. $V = E\,d$, so if d halves, so does V

(e)The work done is the change in energy. Energy, given the charge and potential, is:

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The energy halves, so the change is half the initial energy: $\boxed{W = \Delta U = -\tfrac{1}{4}Q_0V_0}$

Notice the sign – you did negative work bringing the plates together because that is the way they naturally want to move; the field did positive work.

Problem 5:

What, approximately, is the capacitance of a typical MIT student? Check out the exhibit in Strobe Alley (4th floor of building 4) for a hint or just to check your answer.

Problem 5 Solution:

There are lots of ways to do this. The note in strobe alley tells us to use a cylinder of dimensions such that when filled with water it would be your mass. Personally I feel more like a sphere, of which we have already calculated the capacitance in class. All I need to know is my radius, *a*. As a first approximation, probably it's a meter (I'm certainly less than 10 m and more than 10 cm). So my capacitance should be about:

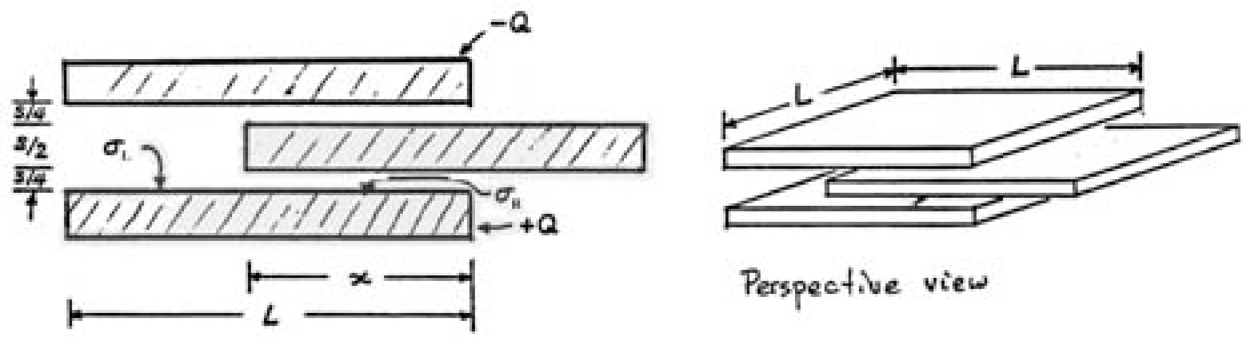
$$C \approx 4\pi\varepsilon_0 a \approx 1\text{m} \big/ 9\times10^9 \text{ F}^{-1}\text{ m} \approx 100 \text{ pF}$$

Not a bad approximation – according to the measurement I'm really ~170 pF.

Note that for simplicity I used the value for k_E rather than ε_0. Always look for ways to recombine constants into things that you know.

Problem 6:

Two flat, square metal plates have sides of length L, and thickness $s/2$, are arranged parallel to each other with a separation of s, where $s << L$ so you may ignore fringing fields. A charge Q is moved from the upper plate to the lower plate. Now a force is applied to a third uncharged conducting plate of the same thickness $s/2$ so that it lies between the other two plates to a depth x, maintaining the same spacing $s/4$ between its surface and the surfaces of the other two. You may neglect edge effects.



a) Using the fact that the metals are equipotential surfaces, what are the surface charge densities σ_L on the lower plate adjacent to the wide gap and σ_R on the lower plate adjacent to the narrow gap?

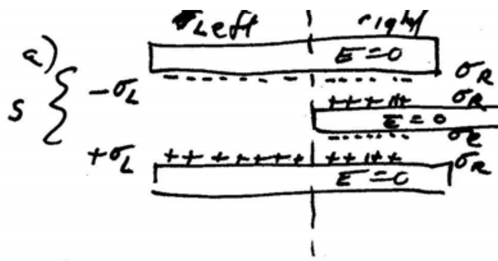
b) What is the electric field in the wide and narrow gaps? Express your answer in terms of L, x, and s.

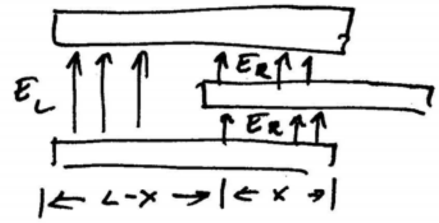
c) What is the potential difference between the lower plate and the upper plate?

d) What is the capacitance of this system?

e) How much energy is stored in the electric field?

Problem 6 Solutions:





a) $\Delta V_L = \Delta V_R$ since upper and lower plates are held at same potential difference

$$\Delta V_L = E_L s = \frac{\sigma_L}{\varepsilon_0} s$$

$$\Delta V_R = E_R \frac{s}{4} + E_R \frac{s}{4} = E_R \frac{s}{2} = \frac{\sigma_R}{\varepsilon_0} \frac{s}{2}$$

$$\Delta V_L = \Delta V_R \Rightarrow \frac{\sigma_L s}{\varepsilon_0} = \frac{\sigma_R}{2} \frac{s}{\varepsilon_0} \Rightarrow \sigma_L = \frac{\sigma_R}{2}$$

$$Q^T = \sigma_L (L-x) L + \sigma_R x L = \frac{\sigma_R}{2} (L-x) L + \sigma_R x L$$

$$Q^T = \sigma_R \frac{L^2}{2} + \sigma_R \frac{xL}{2} = \sigma_R \frac{L}{2} (L+x)$$

$$\sigma_R = \frac{Q^T}{\frac{L}{2}(L+x)} \Rightarrow \sigma_L = \frac{\sigma_R}{2} = \frac{Q^T}{L(L+x)}$$

b) $$E_L = \frac{\sigma_L}{\varepsilon_0} = \frac{Q^T}{\varepsilon_0 L (L+x)} \qquad E_R = \frac{2 Q^T}{(L)(L+x)\varepsilon_0}$$

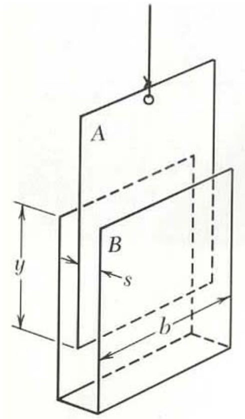
c) $$\Delta V = E_L s = \frac{Q^T}{\varepsilon_0 L (L+x)} s$$

d)= $$C = \frac{Q^T}{\Delta V} = \frac{Q^T}{\frac{Q^T s}{\varepsilon_0 (L(L+x)}} = \frac{\varepsilon_0 (L)(L+x)}{s}$$

e) $$U = \frac{(Q^T)^2}{2C} = \frac{1}{2} \frac{(Q^T)^2 s}{\varepsilon_0 L (L+x)}$$

Problem 7:

A flat conducting sheet A is suspended by an insulating thread between the surfaces formed by the bent conducting sheet B as shown in the figure on the left. The sheets are oppositely charged, the difference in potential, in statvolts, is $\Delta\phi$. This causes a force F, in addition to the weight of A, pulling A downward.



a) What is the capacitance of this arrangement of conductors as a function of y, the distance that plate A is inserted between the sides of plate B?

b) How much energy is needed to increase the inserted distance by Δy?

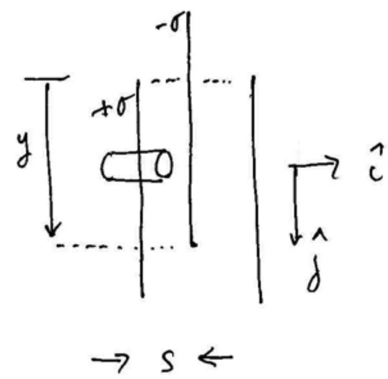
c) Find an expression for the difference in potential $\Delta\phi$ in terms of F and relevant dimensions shown in the figure.

Problem 7 Solutions:

a) We begin by assuming the plates are very large and use Gauss's Law to calculate the electric field between the plates

$$\oint\!\!\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \frac{q_{\text{enc}}}{\varepsilon_0}.$$

Our choice of Gaussian surface is shown in the figure below.



Then

$$\oint\!\!\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = EA_{cap}$$

and

$$\frac{1}{\varepsilon_0} q_{\text{enc}} = \frac{1}{\varepsilon_0} \sigma A_{cap} \, .$$

Thus Gauss's Law implies that

$$\vec{\mathbf{E}} = \frac{1}{\varepsilon_0} \sigma \hat{\mathbf{i}}, \text{ between the plates} \, .$$

Note that the potential difference between the positive and negative plates is

$$V(+) - V(-) = -\int_{x=s}^{x=0} E_x dx = \frac{1}{\varepsilon_0} \sigma s \, .$$

So the surface charge density is equal to

$$\sigma = \frac{\varepsilon_0 (V(+) - V(-))}{s}$$

The area between the plates is yb so the capacitance is

$$C(y) = \frac{Q}{V(+) - V(-)} = \frac{2\sigma yb\varepsilon_0}{\sigma s} = \frac{2yb\varepsilon_0}{s} \, .$$

b) How much energy is needed to increase inserted distance by Δy ?

When the inserted distance is increased by

$$\Delta C = \frac{2\Delta y b \varepsilon_0}{s}$$

Because the charge on the plates is fixed, the energy stored in the capacitor is given by

$$U(y) = \frac{Q^2}{2C(y)}$$

When the inserted distance is increased by Δy, the energy stored between the plates decreases by

$$\Delta U = \frac{dU}{dC}\Delta C = -\frac{Q^2}{2C^2}\Delta C = -\frac{(V(+)-V(-))^2}{2}\frac{2\Delta y b \varepsilon_0}{s}$$

c) This decrease in energy is used to pull the hanging plate in between the two positive charged plates. The work done in pulling the hanging plate a distance Δy is given by

$$\Delta W = F_y \Delta y \, .$$

By conservation of energy

$$0 = \Delta U + \Delta W = -\frac{(V(+)-V(-))^2}{2}\frac{2\Delta y b \varepsilon_0}{s} + F_y \Delta y \, .$$

We can solve this equation for the y-component of the force

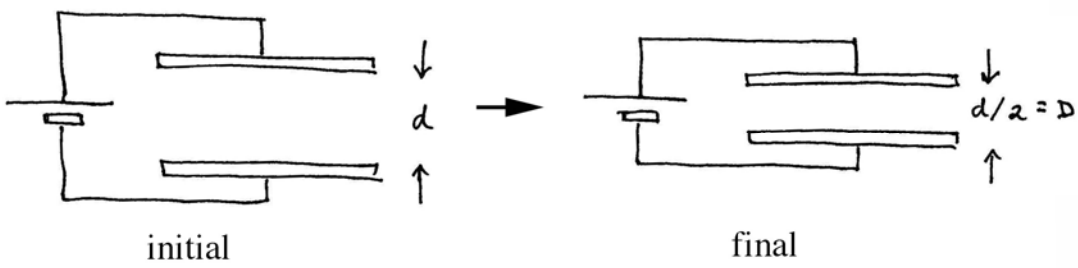
$$F_y = \frac{(V(+)-V(-))^2}{2}\frac{2b\varepsilon_0}{s}$$

or

$$V(+)-V(-) = \sqrt{\frac{sF_y}{\varepsilon_0 b}} \, .$$

Problem 8:

Consider a simple parallel-plate capacitor whose plates are given equal and opposite charges and are separated by a distance d. The capacitor is connected to a battery. Suppose the plates are pushed together until they are separated by a distance D = d/2. How does the final electrostatic energy stored in the capacitor compare to the initial energy?



a) Final is half the initial.

b) Final is one fourth the initial.

c) Final is twice than initial.

d) Final is four times the initial.

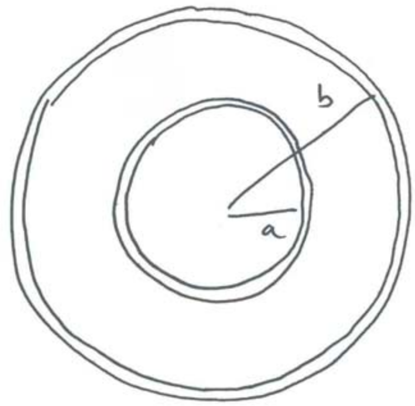
e) They are the same.

Problem 8 Solution:

c. Because the capacitor is connected to the battery, the potential difference across the plates is constant. Therefore the ratio of the final stored energy to the initial stored energy is proportional ratio of the final capacitance to the initial capacitance, $U_f / U_i = \frac{1}{2} C_f \Delta V_C^{\,2} / \frac{1}{2} C_i \Delta V_C^{\,2} = C_f / C_i$. For a parallel plate capacitor, the capacitance is inversely proportional to the distance separating the plates, $C = \dfrac{Q}{\Delta V} = \dfrac{Q}{Ed} = \dfrac{\sigma A}{(\sigma / \varepsilon_0) d} = \dfrac{\varepsilon_0 A}{d}$. Therefore the ratio of the capacitance is $C_f / C_i = d_i / d_f = d / (d/2) = 2$. So $U_f / U_i = 2$.

Problem 9:

Consider a spherical vacuum capacitor consisting of inner and outer thin conducting spherical shells with charge $+Q$ on the inner shell of radius a and charge $-Q$ on the outer shell of radius b. You may neglect the thickness of each shell.



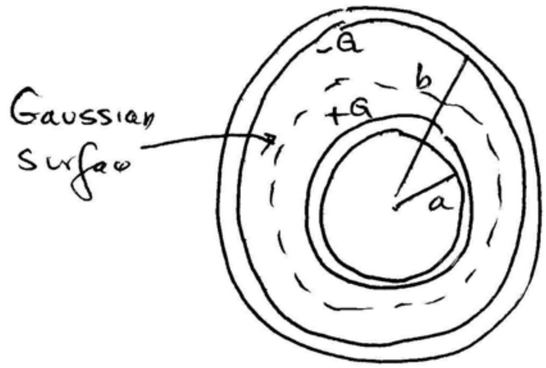
a) What are the magnitude and direction of the electric field everywhere in space as a function of r, the distance from the center of the spherical conductors?

b) What is the capacitance of this capacitor?

c) Now consider the case that the dimension of the outer shell is doubled from b to $2b$. Assuming that the charge on the shells is not changed, how does the stored potential energy change? That is, find an expression for $\Delta U \equiv U_{after} - U_{before}$ in terms of Q a, b, and ε_0 as needed.

Problem 9 Solutions:

(a) There are three regions $r < a$, $a < r < b$, and $b < r$. The electric field is zero for $r < a$ and $b < r$ because the charge enclosed in a Gaussian sphere of radius r is zero for both of those regions.



For the region $a < r < b$, Gauss's Law implies that $E4\pi r^2 = Q/\varepsilon_0$. Hence the magnitude of the electric field is $E = Q/4\pi\varepsilon_0 r^2$ and the direction is radially outward.

$$\vec{E} = \begin{cases} \vec{0} & ; r < a \\ = \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r} & ; a < r < b \\ \vec{0} & ; b < r \end{cases}$$

(b) The capacitance is given by

$$C = \frac{Q}{|\Delta V_C|} = \frac{Q}{-\int\limits_{r=b}^{r=a} \vec{E}\cdot d\vec{s}} = \frac{Q}{-\int\limits_{r=b}^{r=a} \dfrac{Q}{4\pi\varepsilon_0 r^2}\,dr} = \frac{4\pi\varepsilon_0}{\left(\dfrac{1}{a}-\dfrac{1}{b}\right)} = \frac{4\pi\varepsilon_0 ab}{(b-a)}.$$

(c) The energy stored in the capacitor is $U = Q^2/2C$. Therefore the change in stored energy is

$$\Delta U = \frac{Q^2}{2}\left(\frac{(2b-a)}{4\pi\varepsilon_0 a2b} - \frac{(b-a)}{4\pi\varepsilon_0 ab}\right) = \frac{Q^2}{2}\left(\frac{(2b-a)-2(b-a)}{4\pi\varepsilon_0 a2b}\right) = \frac{Q^2}{16\pi\varepsilon_0 b}.$$

Problem 10:

A parallel-plate capacitor has fixed charges $+Q$ and $-Q$. The separation of the plates is then doubled.

(a) By what factor does the energy stored in the electric field change?

(b) How much work must be done if the separation of the plates is doubled from d to $2d$? The area of each plate is A.

Consider now a cylindrical capacitor with inner and outer radii a and b, respectively.

(c) Suppose the outer radius b of a cylindrical capacitor is doubled, but the charge is kept constant. By what factor would the stored energy change? Where would the energy come from?

(d) Repeat (c), assuming the voltage remains constant.

Problem 10 Solutions:
(a)Since the capacitor has fixed charges $+Q$ and $-Q$,

$$U = \frac{1}{2}\frac{Q^2}{C}$$

Then the ratio of the energy stored is

$$\frac{U_{\text{after}}}{U_{\text{before}}} = \left(\frac{1}{2}\frac{Q^2}{C_{\text{after}}}\right)\Bigg/\left(\frac{1}{2}\frac{Q^2}{C_{\text{before}}}\right) = \frac{C_{\text{before}}}{C_{\text{after}}} = \left(\frac{\varepsilon_0 A}{d_{\text{before}}}\right)\Bigg/\left(\frac{\varepsilon_0 A}{d_{\text{after}}}\right) = \frac{d_{\text{after}}}{d_{\text{before}}} = 2$$

(b)The electric field on the top plate due to the bottom plate is given by

$$\vec{\mathbf{E}}_{\text{bottom}} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}} = \frac{-Q}{2\varepsilon_0 A}\hat{\mathbf{k}}$$

The force acting on the top plate is

$$\vec{\mathbf{F}}_{\text{top}} = q_{\text{top}}\vec{\mathbf{E}}_{\text{bottom}} = Q\frac{-Q}{2\varepsilon_0 A}\hat{\mathbf{k}} = \frac{-Q^2}{2\varepsilon_0 A}\hat{\mathbf{k}}$$

There fore the work done by an external agent to separate the plate from d to $2d$ is calculated as

$$W = -\int_{z=d}^{2d} \vec{\mathbf{F}}_{\text{top}} \cdot d\,\vec{\mathbf{s}} = -\int_{d}^{2d} \frac{-Q^2}{2\varepsilon_0 A} dz = \frac{Q^2 d}{2\varepsilon_0 A}$$

but this work is just the amount of additional energy that appeared in the electric field when we moved the plate. Therefore this extra energy in the capacitor comes from the fact that we had to do work exactly equal to that amount of extra energy to separate the plates.

(c) From part (a)

$$\frac{U_{\text{after}}}{U_{\text{before}}} = \frac{C_{\text{before}}}{C_{\text{after}}}$$

Thus,

$$\frac{U_{\text{after}}}{U_{\text{before}}} = \left(\frac{2\pi\varepsilon_0 l}{\ln(b/a)}\right) \Big/ \left(\frac{2\pi\varepsilon_0 l}{\ln(2b/a)}\right) = \frac{\ln(2b/a)}{\ln(b/a)}$$

The energy stored increases when the outer radius b is doubled. It is because the force acting between the two cylindrical shells is always attractive and positive work is required to separate them.

(d)Now the voltage remains constant instead of the charge stored. The energy stored is

$$U = \frac{1}{2} C V^2$$

Then the ratio of the energy stored is

$$\frac{U_{\text{after}}}{U_{\text{before}}} = \left(\frac{1}{2} C_{\text{after}} V^2\right) \Big/ \left(\frac{1}{2} C_{\text{before}} V^2\right) = \frac{C_{\text{after}}}{C_{\text{before}}} = \left(\frac{2\pi\varepsilon_0 l}{\ln(2b/a)}\right) \Big/ \left(\frac{2\pi\varepsilon_0 l}{\ln(b/a)}\right) = \frac{\ln(b/a)}{\ln(2b/a)}$$

The energy stored goes down in this case when the outer radius b is doubled. This is because a part of the charge initially stored on the capacitor flows out of the capacitor through the battery opposite the direction charge usually flows through a battery to keep the voltage constant. This charge flow in the "wrong" direction actually charges up the battery, e.g. increases its stored energy. The battery gains both the energy stored in the capacitor and the work we do to separate the shells, which is always positive.