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Special Relativity, Part II

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Abstract

This note gives the derivation of relativistic momentum and energy, as well as the transformation of momentum and energy between inertial frames. The transformations are then used to derive the transformations of forces between frames, which is necessary for understanding how the electric and magnetic fields are related.

In the last handout, we worked out how to transform events from one inertial frame to another. In electricity and magnetism, we are most interested in the forces (which arise from the fields) and $\vec{F} = d\vec{p}/dt$, so we must also learn how momentum and energy work relativistically.

Momentum

If the maximum velocity attainable by a particle of mass m is c, then, using the $\vec{p} = m\vec{v}$ means there is a maximum momentum $p = mc$ and a maximum kinetic energy $T = mc^2/2$. However, think of the following experiment: a particle of mass m and charge q is accelerated by some means to a velocity close to c and aimed through a set of parallel plates, Fig 1. Just as the

particle reaches plate 1, a potential of strength V is turned on. Traversing the plates gives the particle an additional energy $\Delta T = qV$. After leaving, the potential is turned off. Clearly, it is always possible to add energy to a particle, no matter what its velocity, so we need to come up with a new definition of momentum. In coming up with a new definition, we must be sure we recover the $\vec{p} = m\vec{v}$ and $T = mv^2/2$ when $v << c$.

We are going to now assume that inertial mass is now a function (as yet unknown) of the velocity, $m(v)$. We will work out the function by doing a thought experiment. Suppose two particles of equal mass collide with equal and opposite velocities as shown in Fig. 2a. We consider the problem in frame S_A, Fig. 2b, where particle A moves along the $\hat{y}$ axis. Momentum is conserved, so

$$m(u_o)u_o = m(\sqrt{u^2 + v^2})u \tag{1}$$

where we use the old definition of momentum with the modification that mass is a function of velocity, $\vec{p} = m(v)\vec{v}$.

Frame S_B moves relative to frame S_A with velocity $v\hat{x}$, Fig. 2c, so the $\hat{y}$ component of the velocity of particle A in S_B is gotten from the velocity of A in S_A by

$$u = \frac{u_o}{\gamma}$$

where we have used the rule for finding velocities from the last writeup. Using this with 1, we have

$$m(u_o)u_o = m(\sqrt{u^2 + v^2})\frac{u_o}{\gamma}$$
$$\rightarrow m(\sqrt{u^2 + v^2}) = \gamma m(u_o).$$

since u_o is arbitrary, we take the limit $u_o \rightarrow 0$, which gives $m(v) = \gamma m(0) = \gamma m_o$.

m_o is the rest mass of the particle and is THE mass of the particle, as, in order to satisfy Mr. Einstein, we now require mass be a function of energy. We also have a definition of momentum, $\vec{p} = \gamma m_o \vec{v}$. From now one, we will write m for the rest mass, dropping the subscript.
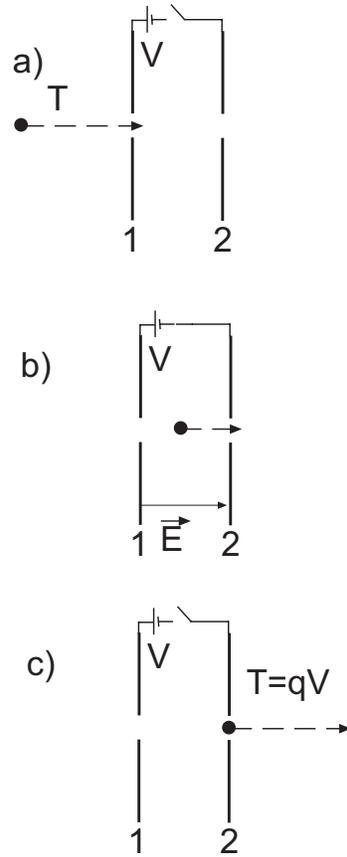
Figure 1: Acceleration of a particle. a) A particle with velocity close to c approaches a set of parallel plates. b) When the particle passes plate 1, a potential V is turned on, creating a field E which exerts a force $F = qE$ on the particle. c) When the particle leaves the plates, the potential is turned off. The particle now has qV more kinetic energy than before, regardless of its initial velocity.
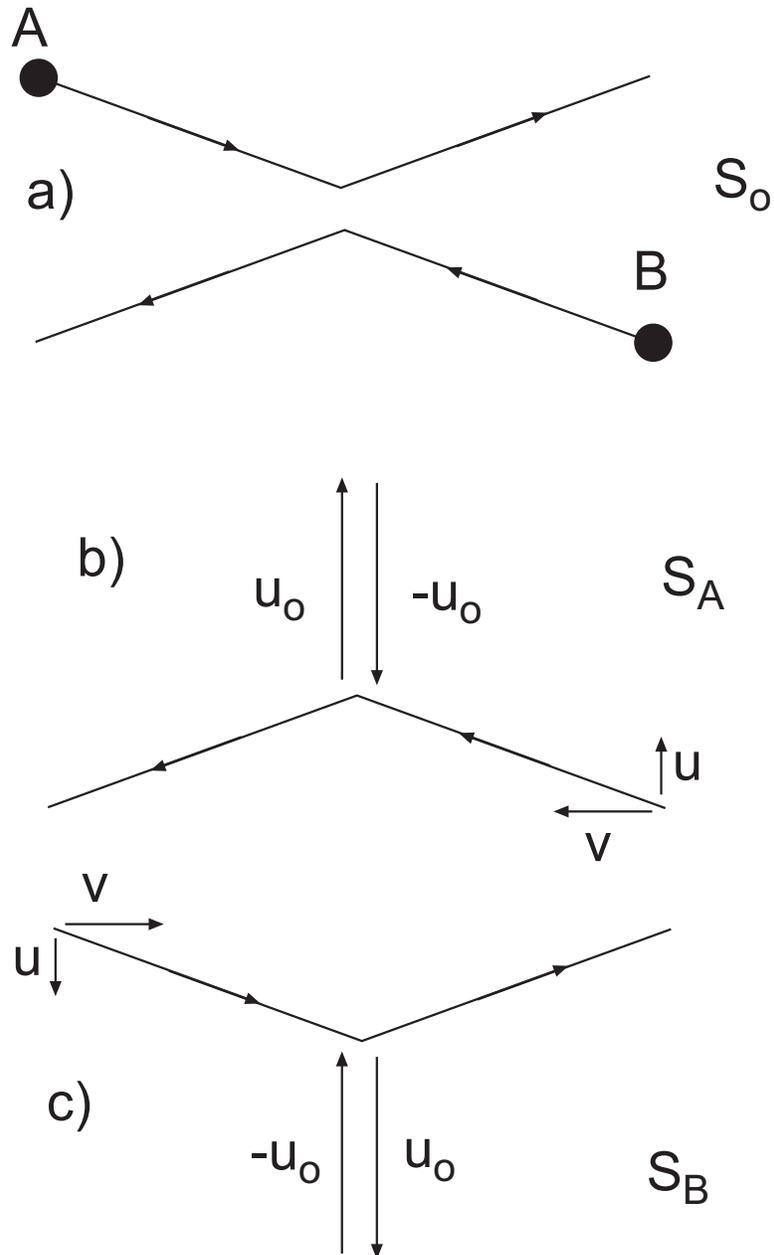
Figure 2: Scattering experiment. a. In frame S_o, the particles A and B collide with equal and opposite momenta. b. In S_A, particle A moves only along the $\hat{y}$ axis. c. In S_B, particle B moves only along the $\hat{y}$ axis. Frame S_B moves in the $-\hat{x}$ direction with velocity v with respect to frame S_A.

Energy

Now we want to know the energy of a particle moving with velocity v. We work this out finding the work necessary to accelerate a particle from rest to velocity v over some path. Then

$$
\begin{aligned}
E &= \int \vec{F} \cdot d\vec{s} = \int \frac{d\vec{p}}{dt} \cdot d\vec{s} \\
&= \int \frac{d\vec{p}}{dt} \cdot \frac{d\vec{s}}{dt} dt = \int \frac{d\vec{p}}{dt} \cdot \vec{v} dt
\end{aligned}
$$

where we have used the definition of velocity, $\vec{v} = d\vec{s}/dt$. The second line results from multiplying by $1 = dt/dt$. Next, use the relation

$$
\frac{d}{dt}(\vec{p} \cdot \vec{v}) = \vec{v} \cdot \frac{d\vec{p}}{dt} + \vec{p} \frac{d\vec{v}}{dt}
$$

to obtain

$$
\begin{aligned}
E &= \int_o^{v_o} \left(\frac{d}{dt}(\vec{p} \cdot \vec{v}) - \vec{p} \cdot \frac{d\vec{v}}{dt} \right) dt \\
&= m\gamma_o v_o^2 - \int_o^{v_o} \vec{p} \cdot d\vec{v} = m\gamma_o v_o^2 - \int_o^{v_o} \frac{m\vec{v} \cdot d\vec{v}}{\sqrt{1 - v^2/c^2}}
\end{aligned}
$$

where we take the final velocity of the particle to be v_o. Since $\vec{v} \cdot d\vec{v} = v dv = d(v^2)/2$, the second term is

$$
\int_o^{v_o} \frac{mv}{\sqrt{1 - v^2/c^2}} dv = -\frac{mc^2}{2} \int_1^{1 - v_o^2/c^2} \frac{du}{\sqrt{u}} = -mc^2(\sqrt{1 - v_o^2/c^2} - 1)
$$

where we have used the substitution $u = 1 - v^2/c^2$. Then,

$$
\begin{aligned}
E &= m\gamma_o v_o + mc^2 \left(\frac{1}{\gamma_o} - 1 \right) \\
&= mc^2 \left(\gamma_o \left(1 - \frac{1}{\gamma_o^2} \right) + \frac{1}{\gamma_o} - 1 \right) \\
&= \gamma_o mc^2 - mc^2.
\end{aligned}
$$

From the last line, a reasonable interpretation is that $E = \gamma_o mc^2$, so when a particle is at rest, $\gamma_o = 1$ and we get the famous $E = mc^2$.[1] We can now also prove other relations:

$$
\begin{aligned}
E^2 &= p^2 c^2 + m^2 c^4 \\
\beta &= \frac{pc}{E}.
\end{aligned}
$$

Given any two of $m, \vec{p}, \vec{v} = \vec{\beta}c$ and E, we can find the other two quantities. The relations are summarized in Fig. 3. You should be sure you know how to show all the relations in the triangle.

Now we have to make sure that we get the old expressions for kinetic energy and momentum when $v \ll c$. Using the Taylor expansion for γ gives $\gamma = 1 + v^2/2c^2$, so we have

$$
\begin{aligned}
p &= \gamma mv \sim mv \\
E &= \gamma mc^2 \sim mc^2 + mv^2/2
\end{aligned}
$$

for a particle moving with velocity v. Notice that the total energy of a particle E is the sum of the kinetic energy and mc^2, which is called the rest energy.

Transformation of energy and momentum between inertial frames

Suppose particle of mass m has momentum $\vec{p}$ nd energy E in frame S. What are the energy and momentum in S'? If we know $\vec{\beta'} = \vec{v'}/c$, then we can easily find $\vec{p'}$ and E', so we just need to work out $\vec{\beta'}$, we will know everything. From the last handout, we use the addition of velocities. To save writing, we will use $\beta = u/v$, where u is the relative velocities of S and S', $\beta_x = v_x/c$ and $\beta_y = v_y/c$, for the components of $\vec{v}$. Then,

$$
\beta_x' = \frac{\beta_x - \beta}{1 - \beta_x \beta}
$$

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[1]This is note a rigorous proof, but a heuristic argument.
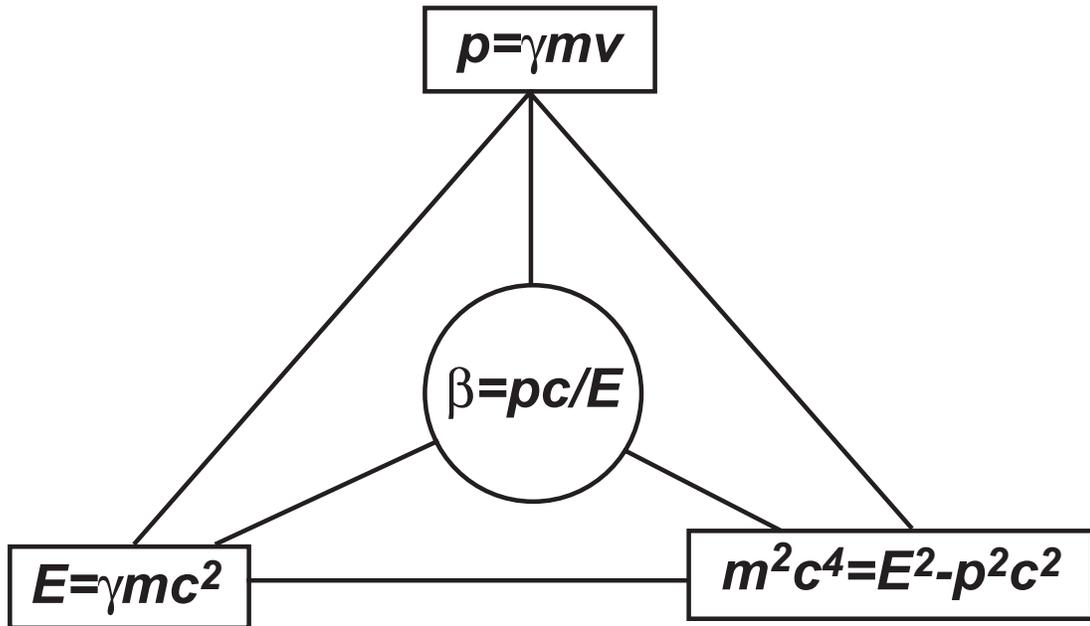
$$\beta = v/x$$
$$\gamma = 1/(1-\beta^2)^{1/2}$$

Figure 3: The Relativity Triangle: knowing any two quantities, you can find the other two.

$$\beta'_y = \frac{\beta_y}{\gamma(1 - \beta_x\beta)}$$

and

$$\gamma'_x = \frac{1}{\sqrt{1 - \beta_x^2}} = \frac{1}{\sqrt{1 - \left(\frac{\beta_x - \beta}{1 - \beta_x\beta}\right)^2}}$$

$$= \frac{1 - \beta_x\beta}{\sqrt{(1 - \beta_x\beta)^2 - (\beta_x - \beta)^2}} = \frac{1 - \beta_x\beta}{1 - 2\beta_x\beta + \beta_x^2\beta^2 - \beta_x^2 + 2\beta_x\beta - \beta^2}$$

$$= \frac{1 - \beta_x\beta}{\sqrt{(1 - \beta_x^2)(1 - \beta^2)}} = (1 - \beta_x\beta)\gamma_x\gamma.$$

Now we just put in to find the momentum and energy in the S' frame:

$$p'_x = m\gamma'\beta'_x c = m\gamma_x\gamma(1 - \beta_x\beta)c \tag{2}$$

$$= \gamma p_x - \frac{\beta\gamma E}{c} \tag{3}$$

$$p'_y = m\gamma'\beta'_y c = m\gamma_x\gamma(1 - \beta_x\beta)\frac{\beta_y}{\gamma_x(1 - \beta_x\beta)}c = p_y \tag{4}$$

$$E = \gamma'mc^2 = (1 - \beta_x\beta)\gamma_x\gamma mc^2 \tag{5}$$

$$= \gamma E - \gamma\beta p_x c. \tag{6}$$

Notice energy and momentum transform in a way similar to time and the spatial coordinates.

Transformation of forces

The electric and magnetic fields create forces and transforming the forces is key to transforming the fields. As usual, we start with a force $\vec{F}$ in S which moves relative to S' with velocity $\vec{u} = \vec{\beta}c = u\hat{x}$ and find $\vec{F'}$. Starting first with F'_x

$$F'_x = \frac{dp'_x}{dt'}\frac{\gamma dp_x\beta\gamma dE/c}{\gamma dt - \frac{\beta\gamma}{c}dx} = \frac{F_x - \beta(dE/dt)/c}{1 - \frac{\beta v_x}{c}}.$$

Remember $dE = \vec{F} \cdot d\vec{s}$, so $dE/dt = \vec{F} \cdot \vec{v}$ and

$$F'_x = \frac{F_x - \beta\gamma(\vec{F} \cdot \vec{v})/c}{1 - \beta\beta_x}.$$

For a particle at in S, $F'_x = F_x$. For transverse forces (i.e. forces normal to the relative velocity of the frames)

$$F'_y = \frac{dp'_y}{dt'} = \frac{dp_x}{\gamma dt - \frac{\beta\gamma}{c}dx} = \frac{dp_x/dt}{\gamma(1 - \beta v_x)} = \frac{F_y}{\gamma(1 - \beta v_x)}$$

and for a particle at rest in S, $F'_y = F_y/\gamma$.

Some examples

When physicists talk about relativistic kinematics, they usually express energies in electron volts or eV. This is just the energy a particle of one electron of charge has when it sits at a potential of 1 Volt; 1 eV=4.8×10^{-10}esu $\times$ 1Volt/300Volts/statvolt $= 1.6 \times 10^{-12}$ergs. Since $E = mc^2$, we can express masses using units of $eV/c^2 = 1.8 \times 10^{-33}$g. Thus, the electron, which has a mass of 9.1×10^{-28} g would have a mass of $m_e = 511,000\,\text{eV}/c^2$ or $511\,\text{keV}/c^2$. Similarly, momenta my be expressed using units of eV/c.

Pion decay

The lowest mass nuclear particle is the charged pion or π. The π is unstable, with a lifetime of 26 ns and decays to a muon (μ) and a neutrino (ν), $\pi \rightarrow \mu\nu$. The pion has a mass of $m_\pi = 140\,\text{MeV}/c^2$, the muon has a mass $m_\mu = 106\,\text{MeV}/c^2$ and the neutrino is massless.

We start by asking: if a π decays at rest, what are the energies and momenta of the decay products, the μ and the ν? To find the answer, use the fact that energy and momentum are conserved in the decay. Notice that since $m_\nu = 0$, $E_\nu = p_\nu c$. Referring to Fig. 4, $\vec{p}_\nu = -\vec{p}_\mu$, so we take $|\vec{p}_\mu| = |\vec{p}_\nu| = p$. Energy

conservation says $E_\pi = E_\mu + E_\nu$ and, since the pion is at rest, $E_\pi = m_\pi c^2$. Then, using $E_\mu = \sqrt{p^2 c^2 + m_\mu^2 c^4}$,

$$
\begin{aligned}
E_\mu = m_\pi c^2 - pc &= \sqrt{p^2 c^2 + m_\mu^2 c^4} \\
p^2 c^2 - 2 m_\pi p c^3 + m_\pi^2 c^4 &= p^2 c^2 + m_\mu^2 c^4 \\
(m_\pi^2 - m_\mu^2) c^4 &= 2 m_\pi^2 p c^3 \\
\to p &= \frac{(m_\pi^2 - m_\mu^2) c}{2 m_\pi}.
\end{aligned}
$$

Putting the numbers for the masses in gives $p = 30\,\mathrm{MeV/c^2}$ for the muon momentum and $E = \sqrt{p^2 c^2 + m_\mu^2 c^4} = 110 \mathrm{MeV}$ for the muon energy.

Next, we want to think about the decay of a pion moving with velocity v. When it decays, the momentum and energy of the muon and neutrino will depend on the decay angle in the rest frame of the pion, Fig. 4b. We can use relativity to find out the energy and angle in the lab frame in which the pion moves with $\vec{v} = v\hat{x}$. In the pion rest frame, after the decay, the muon has momentum

$$
\begin{aligned}
p_x &= p \cos\theta \\
p_y &= p \sin\theta
\end{aligned}
$$

where $p = 30 \mathrm{MeV/c}$. Using Eq. 3-6 to transform into a the lab frame gives

$$
\begin{aligned}
p'_x &= \gamma p \cos\theta + \beta\gamma E \\
p'_y &= p \sin\theta \\
E' &= \gamma E + \beta\gamma p \cos\theta.
\end{aligned}
$$

The range of possible energies is $\gamma E - \beta\gamma p$ to $\gamma E + \beta\gamma p$. For example, a pion with kinetic energy of 500 MeV would have a total energy of $E_\pi = 640\,\mathrm{MeV}$. Then, $\gamma = E_\pi / m_\pi = 4.6$ and $\beta = \sqrt{1 - 1/\gamma^2} = 0.98$. Then, the minimum muon energy is 371 MeV and maximum is 641 MeV.

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Cutoff in cosmic rays

There is a very interesting effect in cosmic rays. Very high energy protons travel vast distances without losing much energy. However, above a certain energy, a new loss mechanism becomes important. A proton can absorb a photon and make a Δ particle, the next lightest nuclear particle. The reaction is $p + \gamma \to \Delta$. The Δ is unstable and quickly decays to a proton and a pion: $\Delta \to p + \pi$.

The universe is filled with photons emitted when electrons bound to protons 300,000 years after the Big Bang. At the time, the photons had energies of around 10eV. Since then, the expansion of the universe has caused them to cool to about 200 μeV. These relic photons are hit by very high energy protons to make Δs. If the mass of the Δ is $m_\Delta = 1150 \, \mathrm{MeV/c^2}$, what energy proton is necessary to produce a Δ?

In order to make a Δ, the total energy of the proton and photon must be at least the $m_\Delta c^2$: $E_p + E_\gamma = m_\Delta c^2$. In addition, total momentum is conserved: $E_\gamma/c + p_p = p_\Delta$. Remember, for a massless particle like a photon, $E = pc$. Inthe universe (unprimed) frame, the Δ has and energy E_Δ and momentum p_Δ. Then

$$
\begin{aligned}
m_\Delta c^2 &= E_\Delta^2 - p_\Delta^2 c^2 \\
&= E_\gamma^2 + 2E_\gamma E_p + E_p^2 - p_p^2 c^2 - 2p_p c E_\gamma - E_\gamma^2 \\
\to p_p &= \frac{2E_\gamma E_p - (m_\Delta^2 - m_p^2)c^4}{2E_\gamma}.
\end{aligned}
$$

Now that we have square both sides and solve for E_p. We'll use the notation $\Delta^2 = (m_\Delta^2 - m_p^2)$.

$$
\begin{aligned}
p_p^2 c^2 = E_p^2 - m_p^2 c^4 &= \frac{4E_\gamma^2 E_p^2 - 4E_\gamma E_p \Delta^2 c^4 + \Delta^4 c^8}{4E_\gamma^2} \\
4E_\gamma^2 m_p^2 c^4 + \Delta^4 c^8 &= E_\gamma E_p \Delta^2 c^4 \\
\to E_p &= \frac{4E_\gamma^2 m_p^2 + \Delta^4 c^8}{4E_\gamma \Delta^2 c^4}.
\end{aligned}
\tag{7}
$$

$\Delta^2 = m_\Delta^2 - m_p^2 = 450,000(\,\mathrm{MeV/c^2})^2$. Notice there are two terms in the denominator of Eq. 7; the first is $4E_\gamma^2 m_p^2 = 1.4 \times 10^{11} \, \mathrm{eV^4}$ while the second

is $\Delta^4 = 2 \times 10^{33}\,\text{eV}^4/c^4$, so for all practical purposes

$$E_p = \frac{\Delta^2 c^4}{4E_\gamma} = 6 \times 10^{20}\,\text{eV}.$$

Once produced, the Δ moves with essentially the proton energy and, after a very short time, decays to a proton and a pion, $\Delta \to p + \pi$. The interesting question is than: what is the maximum energy the final proton can have?

In the Δ rest (primed) frame, we have energy conservation: $m_\Delta c^2 = E'_p + E'_\pi$ and momentum conservation: $\vec{p}'_p = -\vec{p}'_\pi$. From momentum conservation

$$
\begin{aligned}
p'_\pi &= p'_p \\
E'^2_p - m^2_p c^4 &= E'^2_\pi - m^2_\pi c^4 \\
&= (m_\Delta c^2 - E'_p)^2 - m^2_\pi c^4 \\
\to E'_p &= \frac{m^2_\Delta c^4 - m^2_\pi c^4 + m^2_p c^4}{2m_\Delta c^2} = 949\,\text{MeV/c}^2
\end{aligned}
$$

which gives a proton momentum of $p'_p = 144\,\text{MeV}/c$. If the Δ has an energy of $6 \times 10^{20}\,\text{eV}$, then $\gamma = 5.2 \times 10^{11}$ (and of course $\beta = 1$ to a very good approximation). Then, the maximum energy the proton can have in the lab frame is

$$E'_p = \gamma E_p + \beta \gamma p_p c = 5.6 \times 10^{20}\,\text{eV}.$$

Thus, the production and decay of the Δ by very high energy protons results in a limit to the proton energy of $6 \times 10^{20}\,\text{eV}$. Protons with higher energies get knocked back down by the Δ production and decay. Interestingly, some experiments claim to have observed cosmic rays above this limit, which either the originate nearby (from as yet unknown sources) or they are not protons or nucleons. Several new experiments are attempting to search for these unusual cosmics rays with higher sensitivity.
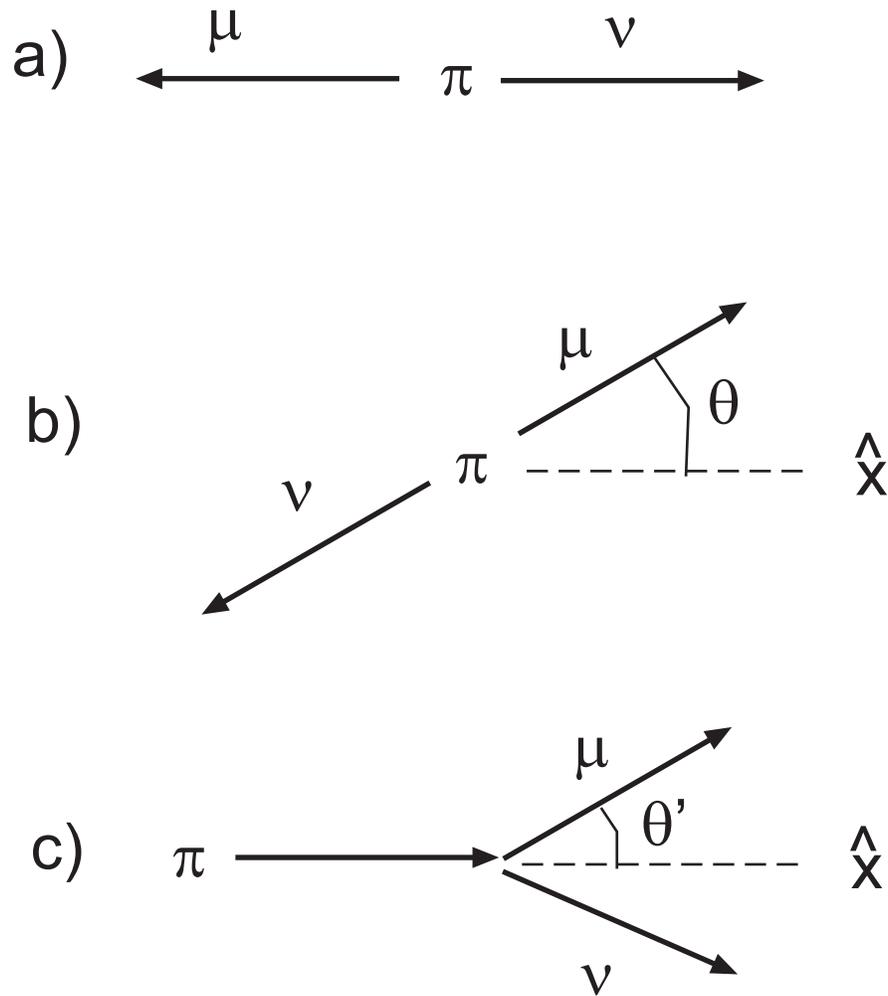
Figure 4: Decay of a pion to a muon and neutrino: a) Decay in pion rest frame. Since the total momentum is zero, the neutrino and muon have equal and opposite momenta. b) Pion decay in pion rest frame. The muon momentum makes an angle θ with the x axis. c) Decay of a pion moving along the x axis. In this frame, the muon momentum makes an angle θ' with the x axis.