8.022 (E&M) - Lecture 2

Topics:

- Energy stored in a system of charges
- Electric field: concept and problems
- Gauss's law and its applications

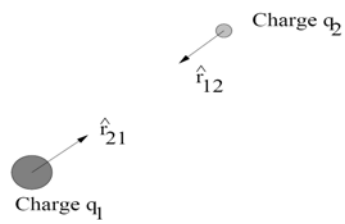
Feedback:

- Thanks for the feedback!
  - Scared by Pset 0? Almost all of the math used in the course is in it…
  - Math review: too fast? Will review new concepts again before using them
  - Pace of lectures: too fast? We have a lot to cover but… please remind me!

Last time…

- Coulomb's law:

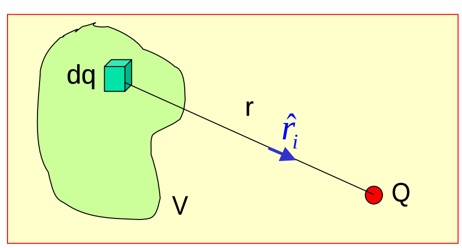
$$\vec{F}_2 = \frac{q_1 q_2}{|r_{21}|^2}\hat{r}_{21}$$



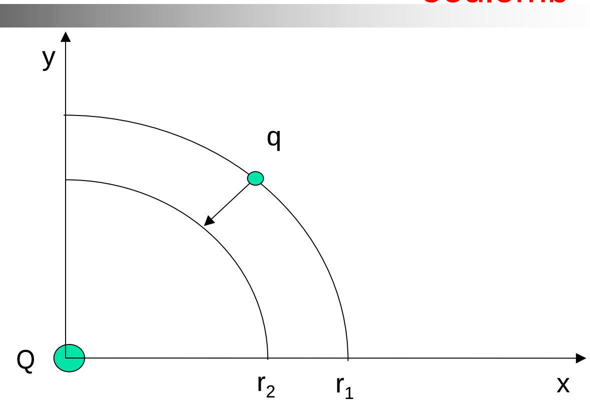
- Superposition principle:

$$\vec{F}_Q = \sum_{i=1}^{i=N} \frac{q_i Q}{|r_i|^2}\hat{r}_i$$

$$\vec{F}_Q = \int_V \frac{\rho \; dV \; Q}{|r|^2} \hat{r}$$



Energy associated with $F_{Coulomb}$



How much work do <u>I</u> have to do to move q from r_1 to r_2?

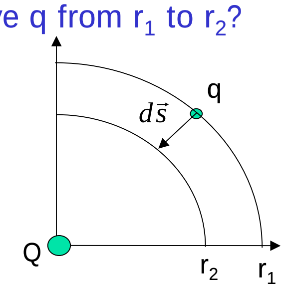
Work done to move charges

- How much work do <u>I</u> have to do to move q from r_1 to r_2?

  $$W = \int \vec{F}_I \bullet d\vec{s} \quad \text{where } \vec{F}_I = -F_{Coulomb} = -\frac{Qq\hat{r}}{r^2}.$$

  - Assuming radial path:

  $$W(r_1 \to r_2) = \int \vec{F}_I \bullet d\vec{s} = -\int_{r1}^{r2} \frac{Qq\hat{r}}{r^2} \bullet d\hat{r} = \frac{Qq}{r_2} - \frac{Qq}{r_1}$$

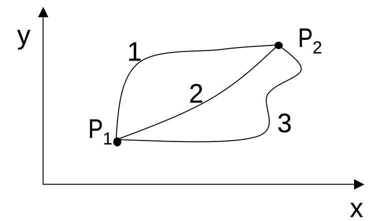


- Does this result depend on the path chosen?
  - No! You can decompose any path in segments // to the radial direction and segments |_ to it. Since the component on the |_ is null the result does not change.

Corollaries

- The work performed to move a charge between P_1 and P_2 is the same independently of the path chosen

$$W_{12} = \int_{Path1} \vec{F} \bullet d\vec{s} = \int_{Path2} \vec{F} \bullet d\vec{s} = \int_{Path3} \vec{F} \bullet d\vec{s}$$



- The work to move a charge on a close path is zero:

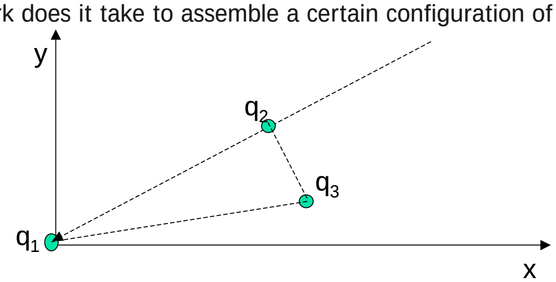
$$W_{11} = \oint_{Any} \vec{F} \bullet d\vec{s} = 0$$

In other words: the electrostatic force is conservative!

This will allow us to introduce the concept of potential (next week)

Energy of a system of charges

How much work does it take to assemble a certain configuration of charges?



$$W(Q) = \int_{\infty}^{P_1} \vec{F} \bullet d\vec{s} = 0 \quad \text{no other charges: F=0}$$

$$W_{1+2} = \int \vec{F}_I \bullet d\vec{s} = \frac{q_1 q_2}{r_{12}}$$

$$W_{1+2+3} = W_{1+2} + W_{1+3} + W_{2+3} = \frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}$$

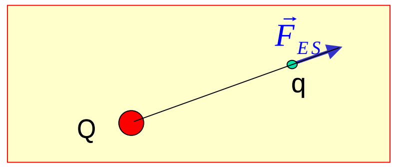
Energy stored by N charges:

$$U = \frac{1}{2} \sum_{i=1}^{i=N} \sum_{\substack{j=1 \\ j \neq i}}^{j=N} \frac{q_i q_j}{r_{ij}}$$

The electric field

Q: what is the best way of describing the effect of charges?

- 1 charge in the Universe
- 2 charges in the Universe



$$\vec{F}_q = \frac{qQ}{|r|^2}\hat{r}$$

But: the force F depends on the test charge q... ☹

→ define a quantity that describes the effect of the charge Q on the surroundings: Electric Field

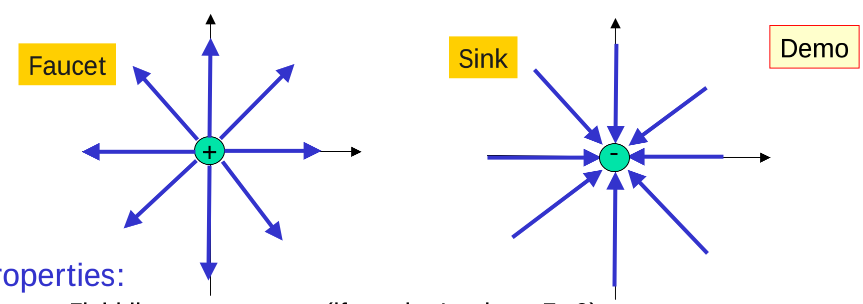
$$\vec{E} = \frac{\vec{F}_q}{q} = \frac{Q}{|r|^2}\hat{r}$$

Units: dynes/e.s.u

Electric field lines

Visualize the direction and strength of the Electric Field:

- Direction: // to E, pointing towards – and away from +
- Magnitude: the denser the lines, the stronger the field.



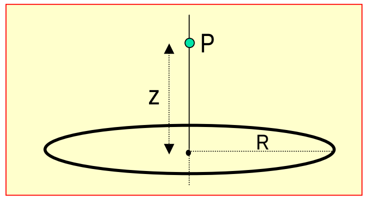
Properties:

- Field lines never cross (if so, that's where E=0)
- They are orthogonal to equipotential surfaces (will see this later).

Electric field of a ring of charge

Problem: Calculate the electric field created by a uniformly charged ring on its axis

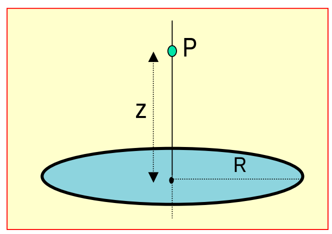
- Special case: center of the ring
- General case: any point P on the axis



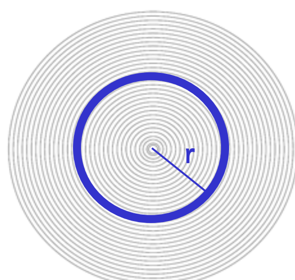
Answers:
- Center of the ring: E=0 by symmetry
- General case: $\vec{E} = \frac{Q z}{(R^2 + z^2)^{\frac{3}{2}}} \hat{z}$

Electric field of disk of charge

Problem:
Find the electric field created by a disk of charges on the axis of the disk



Trick:
a disk is the sum of an infinite number of infinitely thin concentric rings.
And we know E_{ring}…



(creative recycling is fair game in physics)

E of disk of charge (cont.)

Electric field of a ring of radius r:

$$\vec{E}_{ring}(r) = \frac{zQ}{(r^2+z^2)^{\frac{3}{2}}}\hat{z}$$

If charge is uniformly spread:

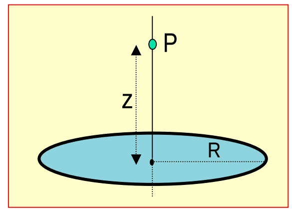
$$dq = \sigma\, da = 2\pi r \sigma\, dr$$

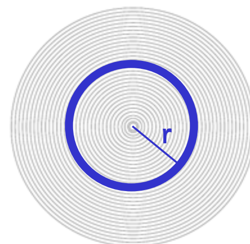
→ Electric field created by the ring is:

$$d\vec{E} = \frac{z\sigma\, 2\pi r\, dr}{(r^2+z^2)^{\frac{3}{2}}}\hat{z}$$

→ Integrating on r: 0→R:

$$\vec{E} = \int_{r=0}^{r=R} d\vec{E} = \int_{r=0}^{r=R} \frac{z\sigma 2\pi dr}{(r^2+z^2)^{\frac{3}{2}}}\hat{z} = 2\pi\sigma z\hat{z}\left(\frac{1}{|z|} - \frac{1}{\sqrt{R^2+z^2}}\right)$$





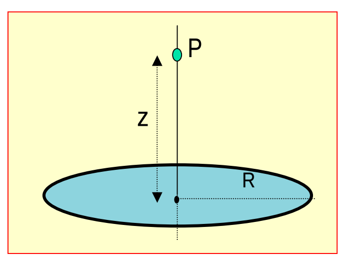
Special case 1: R→infinity

For finite R: $\vec{E} = 2\pi\sigma z\hat{z}\left(\frac{1}{|z|} - \frac{1}{\sqrt{R^2+z^2}}\right)$

What if R→infinity? E.g. what if R>>z?

Since $\lim_{R\to\infty} \frac{1}{\sqrt{R^2+z^2}} = 0$

→ $\vec{E} = 2\pi\sigma\,\hat{z}$

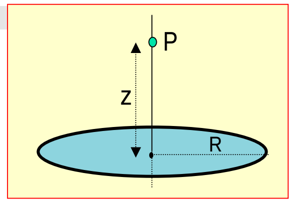


Conclusion:

Electric Field created by an infinite conductive plane:

- Direction: perpendicular to the plane (+/- z)
- Magnitude: $2\pi\sigma$ (constant!)

Special case 2: h>>R



For finite R: $\vec{E} = 2\pi\sigma z\hat{z}\left(\frac{1}{|z|} - \frac{1}{\sqrt{R^2+z^2}}\right)$

What happens when h>>R?

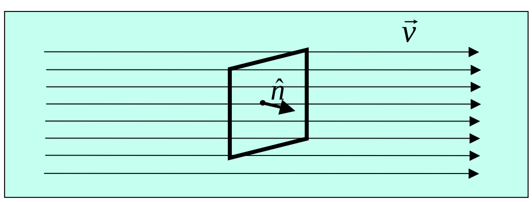
- Physicist's approach:
  - The disk will look like a point charge with $Q=\sigma\pi r^2$

    → $E=Q/z^2$
- Mathematician's approach:
  - Calculate from the previous result for z>>R (Taylor expansion):

$$\vec{E} = 2\pi\sigma z\hat{z}\left(\frac{1}{|z|} - \frac{1}{\sqrt{R^2+z^2}}\right) = 2\pi\sigma z\hat{z}\frac{1}{z}\left(1-\left(1+\left(\frac{R}{z}\right)^2\right)^{-1/2}\right)$$

$$\sim 2\pi\sigma\hat{z}\left(1-(1-\frac{1}{2}\left(\frac{R}{z}\right)^2)\right) = \pi\sigma\hat{z}\left(\frac{R}{z}\right)^2 = \frac{Q}{z^2}\hat{z}$$

The concept of flux



- Consider the flow of water in a river
- The water velocity is described by

$$\vec{v}(x,y,z) \equiv v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \equiv (v_x, v_y, v_z)$$

- Immerse a squared wire loop of area A in the water (surface S)
- Define the loop area vector as $\vec{A} \equiv A\hat{n}$

Q: how much water will flow through the loop? E.g.:

What is the "flux of the velocity" through the surface S?

What is the flux of the velocity?

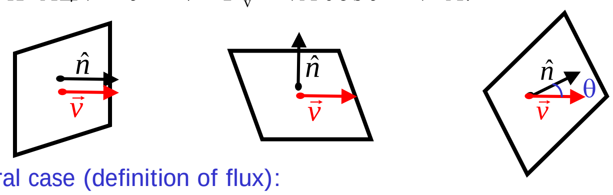
It depends on how the loop is oriented w.r.t. the water…

- Assuming constant velocity and plane loop:

$$1)\ \text{if}\ \vec{A} \perp \vec{v} \quad \rightarrow \quad \Phi_{v} = 0;$$

$$2)\ \text{if}\ \vec{A} \parallel \vec{v} \quad \rightarrow \quad \Phi_{v} = vA;$$

$$3)\ \text{if}\ \vec{A} \measuredangle \vec{v} = \theta \quad \rightarrow \quad \Phi_{v} = vA\cos\theta = \vec{v} \bullet \vec{A}.$$

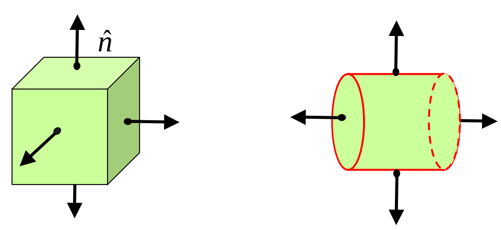


- General case (definition of flux):

$$\Phi_{\vec{v}} = \int_S \vec{v} \bullet d\vec{A}$$

F.A.Q.:
what is the direction of $d\vec{A}$?

- Defined unambiguously only for a 3d surface:
  - At any point in space, dA is perpendicular to the surface
  - It points towards the “outside” of the surface
- Examples:



- Intuitively:
  - “dA is oriented in such a way that if we have a hose inside the surface the flux through the surface will be positive”

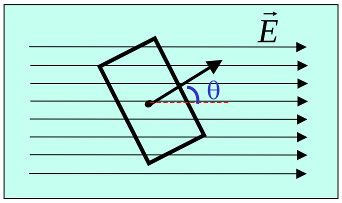
Flux of Electric Field

Definition: $\Phi_{\vec{E}} \equiv \Phi = \int_S \vec{E} \bullet d\vec{A}$

Example: uniform electric field + flat surface

- Calculate the flux:

$$\Phi = \int_S \vec{E} \bullet d\vec{A} = \vec{E} \bullet \vec{A} = EA\cos\theta$$



- Interpretation:

Represent E using field lines:

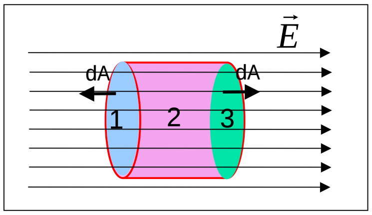
Φ_E is proportional to $N_{\text{field lines}}$ that go through the loop

NB: this interpretation is valid for any electric field and/or surface!

Φ_E through closed (3d) surface

- Consider the total flux of E through a cylinder:

$$\Phi_{tot} = \Phi_1 + \Phi_2 + \Phi_3$$



- Calculate Φ_1, Φ_2, Φ_3
  - Cylinder axis is // to field lines
  - $\Phi_2=0$ because $\vec{E} \perp \hat{n}$
  - $|\Phi_1|=|\Phi_3|$ but opposite sign since

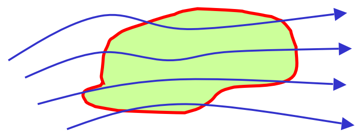
$$\Phi = \int_S \vec{E} \bullet d\vec{A} = EA\cos\theta$$

→ The total flux through the cylinder is zero!

Φ_E through closed empty surface

Q1: Is this a coincidence due to shape/orientation of the cylinder?

- Clue:
  - Think about interpretation of Φ_E: proportional # of field lines through the surface…
- Answer:
  - No: all field lines that get into the surface have to come out!



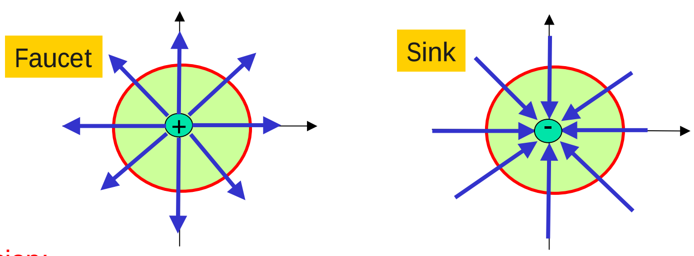
Conclusion:

The electric flux through a closed surface that does not contain charges is zero.

Φ_E through surface containing Q

Q1: What if the surface contains charges?

- Clue:
  - Think about interpretation of Φ_E: the lines will either originate in the surface (positive flux) or terminate inside the surface (negative flux)

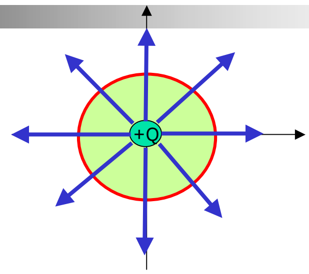


Conclusion:

The electric flux through a closed surface that does contain a net charge is non zero.

Simple example:

Φ_E of charge at center of sphere



Problem:

- Calculate Φ_E for point charge +Q at the center of a sphere of radius R

Solution:

- $\vec{E} \,//\, d\vec{A}$ everywhere on the sphere
- Point charge at distance R: $\vec{E} = \frac{Q}{R^2}\hat{r}$

→ $\Phi = \int_S \vec{E} \bullet d\vec{A} = \int_S \frac{Q}{R^2} dA = \frac{Q}{R^2}\int_S dA = \frac{Q}{R^2} 4\pi R^2 = 4\pi Q$

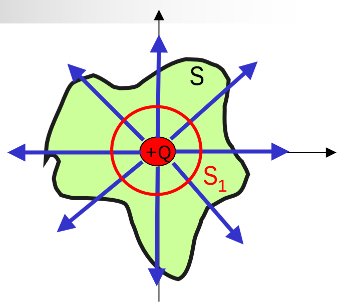
Φ_E through a generic surface

What if the surface is not spherical S?

Impossible integral?

Use intuition and interpretation of flux!

- Version 1:
  - Consider the sphere S_1
  - Field lines are always continuous
  - → $\Phi_{S1} = \Phi_S = 4\pi Q$
- Version 2:
  - Purcell 1.10 or next lecture



Conclusion:

The electric flux Φ through any closed surface S containing a net charge Q is proportional to the charge enclosed:

$$\Phi = \oint_S \vec{E} \bullet d\vec{A} = 4\pi Q_{enc}$$

Gauss's law

Thoughts on Gauss's law

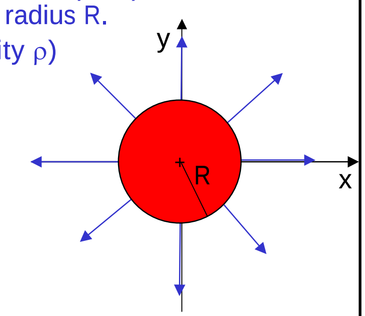
$$\Phi = \oint_S \vec{E} \bullet d\vec{A} = 4\pi Q_{encl}$$ (Gauss's law in integral form)

- Why is Gauss's law so important?
  - Because it relates the electric field E with its sources Q
    - Given Q distribution → find E (integral form)
    - Given E → find Q (differential form, next week)
- Is Gauss's law always true?
  - Yes, no matter what E or what S, the flux is always = $4\pi Q$
- Is Gauss's law always useful?
  - No, it's useful only when the problem has symmetries

Applications of Gauss's law:

Electric field of spherical distribution of charges

Problem: Calculate the electric field (everywhere in space) due to a spherical distribution of positive charges or radius R.
(NB: solid sphere with volume charge density ρ)



Approach #1 (mathematician)

- I know the E due to a point charge dq: $dE=dq/r^2$
- I know how to integrate
- Solve the integral inside and outside the sphere (e.g. r<R and r>R)

$$\int_{r'=0}^{r'=r} dE = \int_{r'=0}^{r'=r} \frac{dq}{r'} = \int_{r'=0}^{r'=r} \frac{\rho dV}{r'^2} = \int d\theta \int d\phi \int_{r'=0}^{r'=r} \frac{\rho r'^2}{r''^2} \sin\theta d\theta d\phi$$

Comment: correct but usually heavy on math!

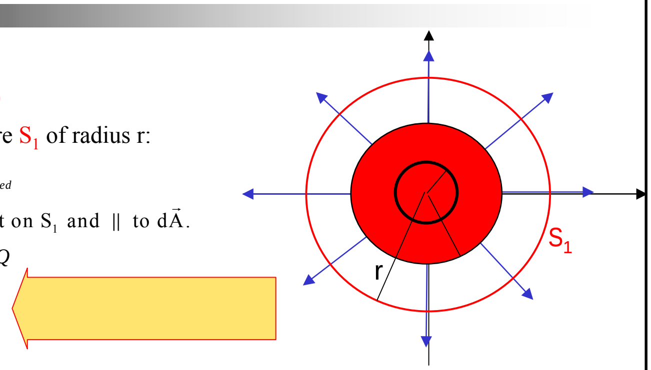
Approach #2 (physicist)

- Why would I ever solve an integral is somebody (Gauss) already did it for me?
- Just use Gauss's theorem…

Comment: correct, much much less time consuming!

Applications of Gauss's law:

Electric field of spherical distribution of charges



Physicist's solution:

1) Outside the sphere (r>R)

Apply Gauss on a sphere S_1 of radius r:

$$\Phi = \oint_{S_1} \vec{E} \bullet d\vec{A} = 4\pi Q_{enclosed}$$

Symmetry: E is constant on S_1 and $\parallel$ to $d\vec{A}$.

$$\oint_{S_1} \vec{E} \bullet d\vec{A} = E 4\pi r^2 = 4\pi Q$$

$$\rightarrow \quad E = \frac{Q}{r^2}$$

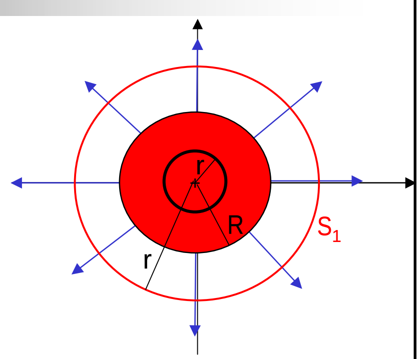
2) Inside the sphere (r<R)

Apply Gauss on a sphere S_2 of radius r:

Again: $\Phi = \oint_{S_2} \vec{E} \bullet d\vec{A} = 4\pi Q_{enclosed}$; symmetry: E is constant on S_2 and $\parallel$ to $d\vec{A}$.

$$\oint_{S_2} \vec{E} \bullet d\vec{A} = E 4\pi r^2; \qquad Q_{enc} = \int \rho dV = \rho \frac{4}{3}\pi r^3 \rightarrow \quad E = \frac{4}{3}\pi \rho r$$

Do I get full credit for this solution?



Did I answer the question completely?

No! I was asked to determine the electric field.
The electric field is a vector
→ magnitude and direction

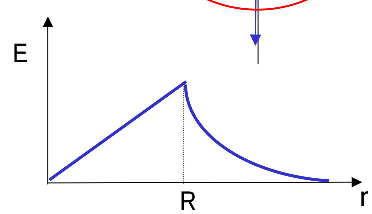
How to get the E direction?

Look at the symmetry of the problem:
Spherical symmetry → E must point radially

Complete solution:

$$\vec{E} = \frac{Q}{r^2}\hat{r} \text{ for r>R}$$

$$\vec{E} = \frac{4}{3}\pi \rho r \hat{r} \text{ for r<R}$$



Another application of Gauss's law:

Electric field of spherical shell

Problem: Calculate the electric field (everywhere in space) due to a positively charged spherical shell or radius R (surface charge density σ)

Physicist's solution:apply Gauss

1) Outside the sphere (r>R)

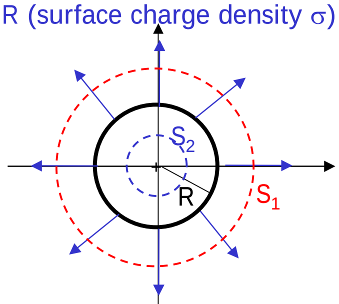
Apply Gauss on a sphere S_1 of radius r:

$$\Phi = \oint_{S_1} \vec{E} \bullet d\vec{A} = 4\pi Q_{enclosed}$$

Symmetry: E is constant on S_1 and $\parallel$ to $d\vec{A}$.

$$\oint_{S_1} \vec{E} \bullet d\vec{A} = E 4\pi r^2 = 4\pi Q_{encl} = 4\pi\sigma(4\pi R^2)$$

$$\rightarrow \quad \vec{E} = \frac{4\pi\sigma R^2}{r^2}\hat{r} = \frac{Q}{r^2}\hat{r} \quad \text{same as point charge!}$$



NB: spherical symmetry
→ E is radial

1) Inside the sphere (r<R)

Apply Gauss on a sphere S_2 of radius r. But sphere is hollow → $Q_{enclosed}=0$ → E=0

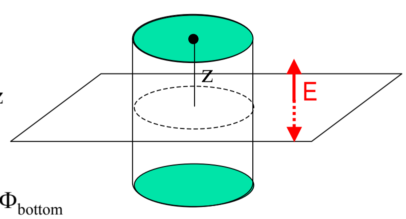
Still another application of Gauss's law:

Electric field of infinite sheet of charge

Problem: Calculate the electric field at a distance z from a positively charged infinite plane of surface charge density σ

Again apply Gauss

- Trick #1: choose the right Gaussian surface!
  - Look at the symmetry of the problem
  - Choose a cylinder of area A and height +/- z
- Trick #2: apply Gauss's theorem
  - $\Phi_{tot} = \Phi_{side} + \Phi_{top} + \Phi_{bottom}$
  - Symmetry: E // z axis → $\Phi_{side}=0$ and $\Phi_{top} = \Phi_{bottom}$



$$\Phi = \oint_{cylinder} \vec{E} \bullet d\vec{A} = 4\pi Q_{enclosed}$$

$$\oint_{cylinder} \vec{E} \bullet d\vec{A} = 2\int_{top} EdA = 2EA = 4\pi(\sigma A) \quad \rightarrow \quad \boxed{\vec{E} = 2\pi\sigma\hat{z}}$$

Checklist for solving 8.022 problems

- Read the problem (I am not joking!)
- Look at the symmetries before choosing the best coordinate system
- Look at the symmetries again and find out what cancels what and the direction of the vectors involved
- Look for a way to avoid all complicated integration
  - Remember physicists are lazy: complicated integral → you screwed up somewhere or there is an easier way out!
- Turn the math crank...
- Write down the complete solution (magnitudes and directions for all the different regions)
- Box the solution: your graders will love you!
- If you encounter expansions:
  - Find your expansion coefficient ($x<<1$) and "massage" the result until you get something that looks like $(1+x)^N$, $(1-x)^N$, or $\ln(1+x)$ or e^x
  - Don't stop the expansion too early: Taylor expansions are more than limits...

Summary and outlook

- What have we learned so far:
  - Energy of a system of charges
  - Concept of electric field E
    - To describe the effect of charges independently from the test charge
  - Gauss's theorem in integral form:
    - $$\Phi = \oint_S \vec{E} \bullet d\vec{A} = 4\pi Q_{encl}$$
    - Useful to derive E from charge distribution with easy calculations
- Next time:
  - Derive Gauss's theorem in a more rigorous way
    - See Purcell 1.10 if you cannot wait...
  - Gauss's law in differential form
    - ... with some more intro to vector calculus... ☺
    - Useful to derive charge distribution given the electric fields
  - Energy associated with an electric field