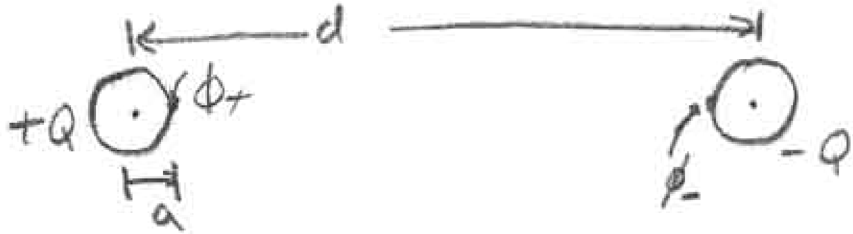
8.022 - FALL 2002 - QUIZ #1
SOLUTIONS

#1



a) $$\phi = \frac{Q}{a}$$

b) $$\phi_+ = \frac{Q}{a} + \frac{-Q}{d-a}$$

$$\phi_- = -\frac{Q}{a} + \frac{Q}{d-a}$$

$$V = \phi_+ - \phi_- = 2Q\left(\frac{1}{a} - \cancelto{0}{\frac{1}{d-a}}\right)$$

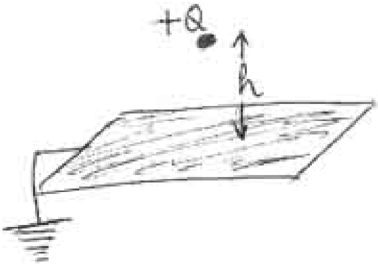
$$= \frac{2Q}{a}$$

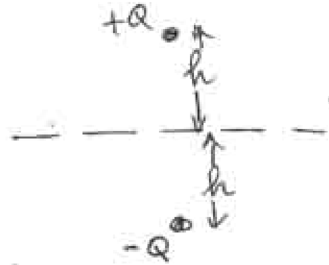
c) $$C = \frac{Q}{V} = \frac{a}{2}$$

d) $$U = \int dV \, \frac{1}{8\pi} E^2 = \frac{1}{2} C V^2$$

$$= \frac{1}{\cancel{2}} \frac{\cancel{a}}{\cancel{2}} \frac{4Q^2}{a^2} = \frac{Q^2}{a}$$

Problem #2 Charges & Conductors: Method of Images.





This is the simplest (probably) demonstration of how (and why) the method of images works.

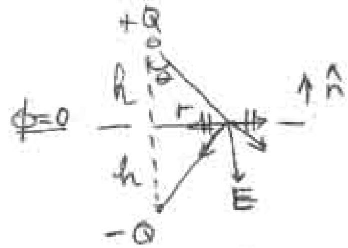
This is a direct result of the UNIQUENESS THEOREM that allows us to reduce an otherwise complex problem to a simpler one by ① honoring Laplace's equation and ② honoring the given boundary conditions.

(a). Remove the conducting plane and replace it by a charge $-Q$ sitting at a distance h "behind" the plane. Every point on the plane is equidistant from $+Q$ and $-Q$, so the potential will be zero.

$$\left[\ \phi = \frac{+Q}{r_1} + \frac{-Q}{r_2} \quad \text{where } r_1 = r_2 \ \forall \text{ point on plane} \Rightarrow \phi = 0\ \right]$$

Then, $+Q$ & $-Q$ should yield the proper solution in the region to the "upper" half of the plane for the point and plane problem.

Then we have:



$$\vec{E}_{\text{on plane}} = \vec{E}_{+Q} + \vec{E}_{-Q}$$

The Components of $\vec{E}_{\pm Q}$, parallel to the plane cancel out.

$$\vec{E} = -\frac{2Q}{r^2+h^2}\cos\theta\,\hat{n} \Rightarrow \vec{E} = -\frac{2Q}{(r^2+h^2)}\cdot\frac{h}{(r^2+h^2)^{1/2}}\,\hat{n} \Rightarrow \boxed{\vec{E} = -\frac{2Qh}{(r^2+h^2)^{3/2}}\,\hat{n}}$$

As expected, the field is perpendicular to the conductor's plane and it is inward due to the negatively induced surface charge density σ.

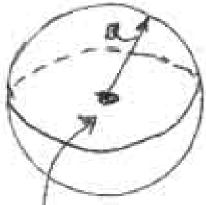
b). In order to find the work, we need to find the force acting on $+Q$ which is nothing but the <u>electrostatic attraction</u> between the charge $+Q$ and the grounded plate. This is obviously equal to the force between $+Q$ and its "image" $-Q$.

$$\vec{F}_{+Q} = -\frac{Q^2}{(2h)^2}\,\hat{n} \qquad W^{E.A.}_{h\to\infty} = \int_h^{\infty} -\vec{F}_{+Q}\cdot dh\,\hat{n} \Rightarrow$$

$$W^{E.A.}_{h\to\infty} = \int_h^{\infty}\frac{Q^2}{4h'^2}\,dh' \Rightarrow W^{E.A.}_{h\to\infty} = \frac{Q^2}{4}\left[-\frac{1}{h}\right]_h^{\infty} \Rightarrow \boxed{W^{E.A.}_{h\to\infty} = \frac{Q^2}{4h}}$$

E.A. = External Agent.

Problem #3: Field & Potential of Symmetric Charge Dist.



(a) by definition: $\rho=\frac{dq}{dV} \Rightarrow dq=\rho\, dV \Rightarrow$

$$Q=\int dq=\int_V \rho\, dV=\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a}\rho_0\left(1-\frac{r^2}{a^2}\right)r^2\sin\theta\, d\theta\, d\phi\, dr.$$

V of sphere.

$$=4\pi\rho_0\int_{r=0}^{a}\left(1-\frac{r^2}{a^2}\right)r^2 dr=4\pi\rho_0\left[\frac{r^3}{3}\Big|_0^a-\frac{r^5}{5a^2}\Big|_0^a\right]$$

$$=4\pi\rho_0\left(\frac{a^3}{3}-\frac{a^3}{5}\right)=\frac{8\pi\rho_0}{15}a^3$$

$\frac{[esu]}{[cm]^3}[cm]^3 \Rightarrow [esu]$ ✓

(b) $\vec{E}$ outside: Use Gaussian-sphere at radius $r>a$

Symmetry imposes $\vec{E}=E(r)\hat{r}$

Surface vector on sphere $d\vec{a}=r\sin\theta d\theta\, r d\phi\, \hat{r}$

Gauss Law: $\oint_{S} \vec{E}\cdot d\vec{a}=4\pi Q \Rightarrow E(r)\cdot 4\pi r^2=4\pi Q \Rightarrow \boxed{E(r)=\frac{Q}{r^2}}$

fixed r

or $\boxed{\vec{E}=\frac{Q}{r^2}\hat{r}}$ where $\boxed{Q=\frac{8\pi\rho_0}{15}a^3}$

$\hookrightarrow$ as expected: outside just as if a point charge

ϕ outside:

by definition $\phi(\vec{r})-\phi(\text{ref})=-\int_{\text{ref}}^{\vec{r}}\vec{E}\,d\vec{r}$

finite extend of charges $\Rightarrow$ $\phi(\vec{r}\to\infty)\to 0$ $\Bigg\}\Rightarrow$

$$\phi(\vec{r})=-\int_{\infty}^{\vec{r}}\vec{E}\cdot d\vec{r}=\int_{\vec{r}}^{\infty}\vec{E}\cdot d\vec{r}=\int_{r}^{\infty}\frac{Q}{r^2}\hat{r}\,dr\,\hat{r}=$$

$$=\left[-\frac{Q}{r}\right]_r^{\infty}=\frac{Q}{r} \quad \text{i.e.} \quad \boxed{\phi(r)=\frac{Q}{r}} \quad r>a$$

(c). $\vec{E}$ inside: use gaussian sphere of radius $r<a$

Same symmetry & surface vectors; from Gauss Law $\Rightarrow$

$\boxed{\oint\vec{E}\cdot d\vec{a}=4\pi Q'}$ where Q' is the total charge enclosed in volume of radius r $(Q(r))$

Find $Q'(r)$: $Q'=\int_{r=0}^{r}\rho\,dv=4\pi\rho_0\int_0^r\left(1-\frac{r^2}{a^2}\right)r^2dr=$

$$=4\pi\rho_0\left[\frac{r^3}{3}-\frac{r^5}{5a^2}\right]$$

Apply Gauss:

$$\cancel{4\pi}r^2\,E(r)=\cancel{4\pi}\cdot 4\pi\rho_0 r^3\left(\frac{1}{3}-\frac{r^2}{5a^2}\right)\Rightarrow\boxed{E(r)=\frac{4\pi}{3}\rho_0 r\left(1-\frac{3r^2}{5a^2}\right)}$$

$\boxed{\begin{array}{l}\underline{\text{Notice}}\ @\ r=a\ \ E^-(r)=\frac{4\pi}{3}\rho_0 a\frac{2}{5}=E^+(r)\\ \quad E \text{ is continuous} \Leftrightarrow \text{no surface charges}\end{array}}$

(compare this with the field in a constant density ball ... $E(r)=\frac{4\pi}{3}\rho_0 r$ ...

ϕ inside: First appreciate that ϕ has to be continuous at $\boxed{r=a}$ implying that $\boxed{\vec{E}\text{ remains finite}}$ (derivative exists)

Thus from definition $\phi(r)-\phi(a)=-\int_a^r\vec{E}\cdot d\vec{r}\Rightarrow$

$$\phi(r)=\phi(a)+\int_r^a\frac{4\pi\rho_0 r'}{3}\left(1-\frac{3r'^2}{5a^2}\right)dr'\Rightarrow\phi(r)=\frac{Q}{a}+\frac{4\pi\rho_0}{3}\left[\frac{r'^2}{2}-\frac{3r'^4}{20a^2}\right]_r^a\Rightarrow$$

$$\phi(r) = \frac{8\pi\rho_0}{15}a^2 + \frac{4\pi\rho_0}{3}\left[\frac{a^2-r^2}{2} - \frac{3(a^4-r^4)}{20a^2}\right] \Rightarrow$$

$$\phi(r) = \frac{8\pi\rho_0}{15}a^2 + \frac{2\pi\rho_0}{3}a^2 - \frac{\cancel{4}^{1}\pi\rho_0}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{20}_{5}}a^2 - \frac{4\pi\rho_0}{6}r^2 + \frac{4\pi\rho_0}{3}\cdot\frac{3}{20}\frac{r^4}{a^2} \Rightarrow$$

$$\phi(r) = \pi\rho_0 a^2\underbrace{\left(\frac{8}{15} + \frac{10}{15} - \frac{3}{15}\right)}_{=1} + \pi\rho_0 r^2\left(\frac{r^2}{5a^2} - \frac{2}{3}\right) \Rightarrow$$

$$\boxed{\phi(r) = \pi\rho_0 a^2 + \pi\rho_0 r^2\left(\frac{r^2}{5a^2} - \frac{2}{3}\right).}$$ Inside

For $\underline{r=0}$ $\phi(0) = \pi\rho_0 a^2$ for $\underline{r=a}$ $\phi(r=a) = \frac{8}{15}\pi\rho_0 a^2$

THE FOLLOWING WAS NOT REQUESTED

Notice that $E(r)$ reaches its maximum value at

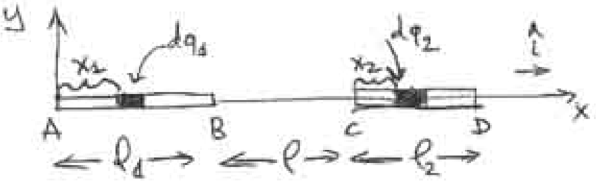
$$\frac{dE}{dr} = 0 \Rightarrow 1 - \frac{9r^2}{5a^2} = 0 \Rightarrow r = \frac{\sqrt{5}}{3}a \Rightarrow \boxed{r = 0.745a}$$

if

$$\frac{d^2E}{dr^2} < 0 \Rightarrow -\frac{18r}{5a^2} < 0 \quad \boxed{\text{TRUE}}$$

Keep this charge distribution handy! You will find it in your chem or nuclear physics course as it describes the charge distribution of light nuclei.

Problem #4. Coulomb Force Between Charges



Clearly all forces act on the x axis

$d\vec{F} = dF\,\hat{i}$ → we will work with the signed magnitude of force from now on.

Take dq_2 on CD and find force due to AB

$$dF_{dq_2} = \frac{dq_2 \cdot dq_1}{(x_2+\ell+\ell_1-x_1)^2} = \frac{dq_2 \cdot \lambda_1 dx_1}{(x_2+\ell+\ell_1-x_1)^2}.$$

Integrate over dq_1, i.e, Keep x_2 fixed and let x_1 run from $x_1=0$ to $x_1=\ell_1$:

$$F_{dq_2} = dq_2\,\lambda_1 \int_{x_1=0}^{\ell_1} \frac{dx_1}{(x_2+\ell+\ell_1-x_1)^2} \Rightarrow$$

$$F_{dq_2} = \lambda_1\, dq_2 \int_{x_1=0}^{\ell_1} \frac{d(x_1 \overbrace{-x_2-\ell-\ell_1}^{\text{constants}})}{(x_1-x_2-\ell-\ell_1)^2} \Rightarrow$$

$$F_{dq_2} = \lambda_1\, dq_2 \left[-\frac{1}{(x_1-x_2-\ell-\ell_1)} \right]_{x_1=0}^{x_1=\ell_1} \Rightarrow$$

$$F_{dq_2} = \lambda_1\, dq_2 \left[-\frac{1}{(-x_2-\ell)} + \frac{1}{(-x_2-\ell-\ell_1)} \right] \Rightarrow$$

$$F_{dq_2} = \lambda_1 \cdot dq_2 \left[\frac{1}{\ell+x_2} - \frac{1}{\ell+x_2+\ell_1} \right].$$

This is the force on the little dq_2 due to the entire straight charge $\underline{\underline{AB}}$.

To find the force on the entire charge distribution $\underline{\underline{CD}}$ we have to integrate over all $\underline{\underline{its}}$ pieces, i.e.

$$F_{CD} = \int F_{dq_2} = \lambda_1 \lambda_2 \int_{x_2=0}^{x_2=\ell_2} dx_2 \left[\frac{1}{\ell + x_2} - \frac{1}{\ell + x_2 + \ell_1}\right] \Rightarrow$$

$$F_{CD} = \lambda_1 \lambda_2 \ \ln(\ell + x_2)\Big|_{x_2=0}^{x_2=\ell_2} - \ln(\ell + x_2 + \ell_1)\Big|_{x_2=0}^{x_2=\ell_2} \Rightarrow$$

$$F_{CD} = \lambda_1 \lambda_2 \left[\ln \frac{\ell + \ell_2}{\ell} - \ln \frac{\ell + \ell_1 + \ell_2}{\ell + \ell_1}\right] \Rightarrow$$

$$\boxed{F_{CD} = \underbrace{\lambda_1}_{\frac{[esu]}{[cm]}} \underbrace{\lambda_2}_{\frac{[esu]}{[cm]}} \ \ln \underbrace{\frac{(\ell + \ell_1)(\ell + \ell_2)}{\ell(\ell + \ell_1 + \ell_2)}}_{\text{pure\#}}}$$

$\frac{[esu]^2}{[cm]^2} = [dynes]$ ✓

if $\lambda_1 \cdot \lambda_2 > 0$ $\vec{F}_{CD} \uparrow\uparrow \hat{i}$ (repulsive)

if $\lambda_1 \cdot \lambda_2 < 0$ $\vec{F}_{CD} \uparrow\downarrow \hat{i}$ (attractive)

$\boxed{\vec{F}_{CD} = F_{CD}\,\hat{i}}$ also $\boxed{\vec{F}_{AB} = -\vec{F}_{CD}}$

obviously

if $\ell_1 \ll \ell$, $\ell_2 \ll \ell$, we get:

$$F_{CD} = \lambda_1 \lambda_2 \ \ln \frac{\ell^2 + \ell(\ell_1 + \ell_2) + \ell_1 \ell_2}{\ell^2 + \ell(\ell_1 + \ell_2)} \Rightarrow$$

$$F_{CD} = \lambda_1 \lambda_2 \ln\left[1 + \frac{\ell_1 \ell_2}{\ell^2 + \underbrace{\ell(\ell_1 + \ell_2)}_{\text{very small compared to } \ell^2}}\right] \Rightarrow F_{CD} = \lambda_1 \lambda_2 \ln\left(1 + \frac{\ell_1 \ell_2}{\ell^2}\right)$$

$$\Rightarrow F_{CD} = \lambda_1 \lambda_2 \frac{\ell_1 \ell_2}{\ell^2} \Rightarrow F_{CD} = \frac{\overbrace{(\lambda_1 \ell_1)}^{\text{TOTAL CHARGE OF BAR\#1}} \overbrace{(\lambda_2 \ell_2)}^{\text{TOTAL CHARGE OF BAR \#2}}}{\ell^2} \Rightarrow \boxed{F_{CD} = \frac{Q_1 \cdot Q_2}{\ell^2}} \text{ Q.E.D.}$$

#4 An interesting approximation → (NOT REQUESTED)

What if only ONE of the two line charges were $l_2 \gg l_1$, $l_2 \gg l$; $l_2 \gg l + l_1$

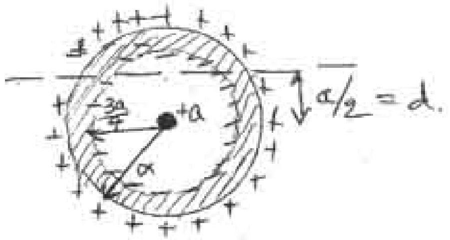
From $\mathbb{F}_{CD} = \lambda_1 \lambda_2 \ln \frac{(l_1+l)(l_2+l)}{l(l_1+l_2+l)} \Rightarrow$

$$\mathbb{F}_{CD} = \lambda_1 \lambda_2 \ln \left[\frac{l_1+l}{l} \; \frac{l_2+l}{l_1+l_2+l} \right] \Rightarrow$$

$$\left.\begin{aligned} &\mathbb{F}_{CD} = \lambda_1 \lambda_2 \ln \left[\frac{l_1+l}{l} \cdot \frac{1+l/l_2}{1+\frac{l+l_1}{l_2}} \right] \\ &\text{for } l_2 \gg l+l_1 , \; l_2 \gg l \quad \frac{1+\frac{l}{l_2}}{1+\frac{l+l_1}{l_2}} \to 1 \end{aligned}\right\} \Rightarrow$$

$$\boxed{\mathbb{F}_{CD} = \lambda_1 \lambda_2 \ln \frac{l+l_1}{l}}$$

Problem #5 Forces on Conductors



The insertion of the small sphere of charge +Q inside the shell will induce charges −Q and +Q on the shell that will uniformly be distributed on the inner and outer parts of shell respectively.

The fact that the total charge will be +Q, −Q is a direct result of Gauss Law and the fact that it will be uniformly distributed on the involved surfaces is a result of symmetry (and the fact that the outside ball is placed in the center - if not symmetry and uniformity won't apply!)

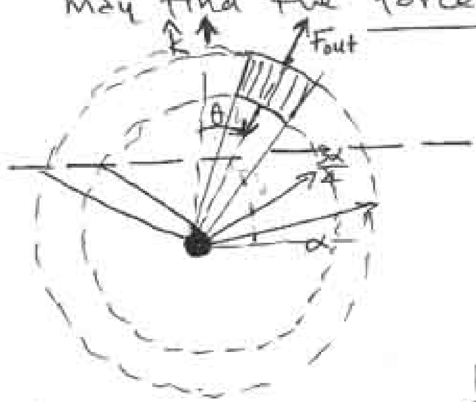
The surface densities of charges on the shell will thus be:

$$\sigma_{out} = \frac{+Q}{4\pi\alpha^2} \qquad \sigma_{IN} = \frac{-Q}{4\pi\left(\frac{3\alpha}{4}\right)^2}. \qquad (1)$$

Clearly they are not equal, but their integrals over their respective surfaces are (absolute values).

There is no net charge densities in the interior of the shell.

If we know the surface charge densities, we may find the force per unit area $\frac{dF}{da} = 2\pi\sigma^2$



Let us examine the forces acting on a piece of the shell that is described at position $\hat{\theta}$ angle with respect to the vertical:

SUBSTITUTE FROM (A)

$$\left.\begin{aligned} \frac{d\vec{F}_{out}}{da_{out}} &= 2\pi\sigma_{out}^2\,\hat{r} \\ \frac{d\vec{F}_{in}}{da_{in}} &= 2\pi\sigma_{IN}^2\,(-\hat{r}). \end{aligned}\right\} \quad (2) \quad \left|\frac{dF}{da}\right|^{out} = \frac{Q^2}{8\pi a^4} \qquad \left|\frac{dF}{da}\right|^{IN} = \frac{32Q^2}{81\pi a^4}$$

Notice that the two "pressures" have opposite direction.

$$\left.\begin{aligned} da_{out} &= \alpha\sin\theta\, d\theta\, \alpha\, d\phi. = a^2\sin\theta\, d\theta\, d\varphi \\ da_{in} &= \frac{3\alpha}{4}\sin\theta\, d\theta\, \frac{3\alpha}{4}d\phi = \frac{9}{16}a^2\sin\theta\, d\theta\, d\varphi. \end{aligned}\right\} \quad (3)$$

Let us plug in (1) and (3) into (2):

$$d\vec{F}_{out} = 2\pi\frac{Q^2}{(4\pi a^2)^2}\underbrace{a^2\sin\theta\, d\theta\, d\varphi}\,\hat{r} = \frac{Q^2}{8\pi a^2}\sin\theta\, d\theta\, d\varphi\,\hat{r}$$

$$d\vec{F}_{in} = -2\pi\frac{Q^2}{\left[4\pi\left(\frac{3a}{4}\right)^2\right]^2}\cdot\underbrace{\frac{9}{16}a^2\sin\theta\, d\theta\, d\varphi}\,\hat{r} = \frac{2Q^2}{9\pi a^2}\sin\theta\, d\theta\, d\varphi\,(-\hat{r}).$$

As you may appreciate the forces are <u>NOT</u> equal.

FOR PARTS ©, Ⓓ & Ⓔ WE FOCUS ON

PART $\underline{\underline{A}}$ OF THE SHELL:

FORCES $d\vec{F}_{out}$, $d\vec{F}_{IN}$ HAVE X, Y COMPONENTS THAT CANCEL OUT WHEN INTEGRATED FOR $\phi: [0 \to 2\pi]$

THUS THERE IS ONLY A $\underline{\underline{z}}$ COMPONENT THAT SURVIVES IN ALL CASES!

$$\vec{F}_z^{out} = \frac{Q^2}{8\pi a^2}\hat{k} \int_{\theta=0}^{\theta^{OUT}} \int_{\phi=0}^{2\pi} \sin\theta \, d\theta \, d\phi \, \cos\theta$$

projection of dF_z^{out} onto $\hat{k}$

FROM GIVEN GEOMETRY:

$$\cos\theta_{OUT} = \frac{1}{2} \Rightarrow \sin^2\theta_{OUT} = \frac{3}{4}.$$

FOR INNER PART OF THE "A" SHELL: (F_x, F_y CANCEL).

$$\vec{F}_z^{IN} = -\frac{2Q^2}{9\pi a^2}\hat{k} \int_{\theta=0}^{\theta^{IN}} \int_{\phi=0}^{2\pi} \sin\theta \, d\theta \, d\phi \, \cos\theta$$

projection of dF^{IN} onto $\hat{k}$

WARNING! $\theta_{IN} \neq \theta_{OUT}$ $\quad \cos\theta_{IN} = \dfrac{\frac{a}{2}}{\frac{3a}{4}} = \frac{2}{3} \Rightarrow \sin^2\theta_{IN} = \frac{5}{9}$

TOTAL FORCE ON "A" IS THE VECTOR SUM OF $\vec{F}_z^{OUT} + \vec{F}_z^{IN}$ JUST CALCULATED.

IF YOU WORK OUT THE INTEGRALS YOU MAY FIND THAT $\vec{F}_z$ EXISTS (NON ZERO) AND IS ALONG $-\hat{k}$, I.E. TENDS TO HOLD THE TWO PIECES OF SHELL TOGETHER