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Final Exam Solutions

Problem 1

(a) The force on $+Q$ due to any particular charge $+q$ on the ring is exactly balanced by the force due to the charge $+q$ diametrically opposite. So the net force on $+Q$ is zero.

(b) With the 3:00 charge removed, the 9:00 charge is now unbalanced, and $+Q$ thus experiences a force $\frac{1}{4\pi\epsilon_0}(qQ/R^2)$ to the right. (We may also obtain the same result by considering the effectively equivalent process of placing a charge $-q$ at 3:00 instead of removing the charge $+q$ from there.)

Problem 2

The now-familiar application of Ampère's law gives

$$\oint \mathbf{B} \cdot d\boldsymbol{l} = \mu_0 I_{\substack{\text{piercing} \\ \text{surface}}} \quad \Longrightarrow \quad B = \frac{\mu_0 I_{\text{p.s.}}}{2\pi r} \quad .$$

For definiteness, consider a cross-section where the inner-cylinder current flows into the page and the outer-cylinder current flows out of the page. Take currents out of the page to be positive, so that positive B corresponds to a counterclockwise magnetic field (by right-handed convention) and negative B corresponds to a clockwise field.

$$\text{Region (i):} \qquad I_{\text{p.s.}} = 0 \;\Rightarrow\; B = 0$$

$$\text{Region (ii):} \qquad I_{\text{p.s.}} = -I \;\Rightarrow\; B = -\frac{\mu_0 I}{2\pi r}$$

$$\text{Region (iii):} \qquad I_{\text{p.s.}} = 3I - I = 2I \;\Rightarrow\; B = \frac{\mu_0 I}{\pi r}$$

(One's own choices of sign convention and system orientation may differ from those made here.)

Problem 3

If no current flows through the meter A, then one current (call it I_1) flows through both R_3 and R_x and another current (call it I_2) flows through both R_1 and R_2. Applying Kirchhoff's loop rule to a simple loop through all four resistors gives $I_1(R_3 + R_x) = I_2(R_1 + R_2)$. Taking another loop through R_3, R_1, and A, we get $I_1 R_3 = I_2 R_1$. Dividing the first equation by the second eliminates both I_1 and I_2, and solves to give $R_x = R_3 R_2 / R_1$.

Problem 4 *(See homework problem 1.6.)*
(There are many ways to do the details of this problem.)

By symmetry, the field must be directed normal to the slab and sheet. Take positive E to be up and negative E to be down. By the principle of linear superposition, the electric field at any point will be the sum of the field due to the sheet and the field due to the slab.

First consider the sheet. We apply Gauss's law to the familiar "Gaussian pillbox" enclosing a small patch, and with its two endcaps equidistant from the sheet. This gives us $E_{\text{sheet}} = -\sigma/2\epsilon_0$ above the sheet and $E_{\text{sheet}} = +\sigma/2\epsilon_0$ below, independent of exact distance from the sheet.

Next consider the slab. We apply Gauss's law to a Gaussian pillbox with one endcap outside and above the slab and the other endcap outside and below the slab (at the same distance from the slab as is the first endcap). This gives $E_{\text{slab}} = +\rho D/2\epsilon_0$ above the slab and $E_{\text{slab}} = -\rho D/2\epsilon_0$ below, independent of exact distance from the slab. Inside the slab, since the charge density is constant, E_{slab} changes linearly with position from the "above" value to the "below" value.

(a) At a distance h above the sheet, we are also above the slab, and the total electric field is

$$E = E_{\text{sheet}} + E_{\text{slab}} = -\frac{\sigma}{2\epsilon_0} + \frac{\rho D}{2\epsilon_0} = \frac{(\rho D - \sigma)}{2\epsilon_0} \quad .$$

(b) At a distance $d < D$ below the top surface of the slab, $E_{\text{slab}} = \rho(D - 2d)/2\epsilon_0$: this expression is linear in d, and goes to $+\rho D/2\epsilon_0$ when $d = 0$ and $-\rho D/2\epsilon_0$ when $d = D$. Also, we are below the sheet. Thus,

$$E = E_{\text{sheet}} + E_{\text{slab}} = \frac{\sigma}{2\epsilon_0} + \frac{\rho(D - 2d)}{2\epsilon_0} = \frac{[\sigma + \rho(D - 2d)]}{2\epsilon_0} \quad .$$

(c) At a distance H below the bottom surface of the slab, we are outside the slab and below the sheet as well. The electric field is

$$E = E_{\text{sheet}} + E_{\text{slab}} = \frac{\sigma}{2\epsilon_0} - \frac{\rho D}{2\epsilon_0} = \frac{(\sigma - \rho D)}{2\epsilon_0} \quad .$$

Problem 5
(a) Since there is no current flowing after a long time, the resistor and the self-inductor are irrelevant. The voltage across both capacitors is V, and the electric field between the plates of each is simply $E = V/d$ (directed downward, since the upper plates carry the positive charge).

(b) Since the battery is still connected, the voltage across each capacitor is still V, and we still have $E = V/d$ (downward) between the plates of each.

(c) When we disconnect the battery before inserting the dielectric, each capacitor carries a charge $Q = CV$, giving a total charge of $2CV$ on the parallel combination. After inserting the dielectric, charge will flow between the left and right capacitors so as to equalize the voltages across them at some new value V'. The *total* charge on the two will remain constant, though. The capacitance of the right-hand capacitor is unchanged, so its new charge is $Q_{\mathrm{R}} = CV'$. The capacitance of the left-hand capacitor has increased by a factor of κ, thus its charge is now $Q_{\mathrm{L}} = \kappa CV'$. Enforcing charge conservation as required, we have

$$Q_{\mathrm{R}} + Q_{\mathrm{L}} = (1+\kappa)CV' = 2CV \quad \Rightarrow \quad V' = \frac{2V}{(1+\kappa)} \quad .$$

The electric field between the plates of each capacitor is then

$$E = V'/d = \frac{2V}{(1+\kappa)d} = 2V/3d \quad \text{(directed downward)}.$$

Problem 6

(a) If the coil lies in the x-y plane at time $t = 0$, then the angle between the coil and the x-y plane at later times will be $2\pi ft$. The magnetic flux through the plane surface bounded by the coil will be $\Phi_B = nSB\cos(2\pi ft)$. The induced EMF is $\mathcal{E} = -d\Phi_B/dt = 2\pi fnSB\sin(2\pi ft)$, and the induced current is $I = \mathcal{E}/R = (2\pi fnSB/R)\sin(2\pi ft)$. The maximum value of this induced current is thus $I_{\mathrm{max}} = 2\pi fnSB/R$, or using the given numerical value for B, $I_{\mathrm{max}} = \pi fnS/R$ Amperes. This maximum occurs when the plane of the coil is at a right angle to the x-y plane.

(b) The time-averaged mechanical power supplied to maintain rotation must equal the time-averaged power $\overline{P}$ dissipated in the resistor:

$$\overline{P} = \overline{I}^2 R = I_{\mathrm{max}}^2 R\,\overline{\sin^2(2\pi ft)} = \frac{1}{2} I_{\mathrm{max}}^2 R = \frac{(\pi fnS)^2}{2R} \text{ Watts.}$$

Problem 7

(a) The phase is of the form $(ky + \omega t)$, so propagation is in the $-\hat{y}$ direction.

(b) In order for $\vec{E}_0 \times \vec{B}_0$ to be in the $-\hat{y}$ direction given $\vec{E}_0$ in the $+\hat{z}$ direction, $\vec{B}_0$ must be in the $-\hat{x}$ direction. $|\vec{E}_0|/|\vec{B}_0| = v = c/n$ is the wavespeed in the medium, so

$$|\vec{B}_0| = \frac{n|\vec{E}_0|}{c} = \frac{(1.5)(2)}{3\times 10^8} = 10^{-8} \quad \Rightarrow \quad \vec{B}_0 = -10^{-8}\hat{x} \quad .$$

(c)

$$v = \frac{c}{n} = \lambda f = \frac{\lambda\omega}{2\pi} \quad \Rightarrow \quad \lambda = \frac{2\pi c}{n\omega} = \frac{2\pi(3\times 10^8)}{(1.5)(4\pi\times 10^{15})} = 10^{-7}\,\mathrm{m} \quad .$$

Problem 8

(a)

$$\frac{Q}{C} + R\frac{dQ}{dt} = V_0$$

(b) For the given $Q(t) = CV_0(1 - e^{-t/\tau})$, we have

$$\frac{Q}{C} + R\frac{dQ}{dt} = V_0(1 - e^{-t/\tau}) + (RCV_0/\tau)e^{-t/\tau} = V_0 \quad \textbf{if} \quad \tau = RC \quad .$$

(c) The current in the resistor is

$$I(t) = \frac{dQ}{dt} = (CV_0/\tau)e^{-t/\tau} = (V_0/R)e^{-t/\tau} \quad .$$

(d)

$$U_C(t_1) = \frac{1}{2}Q(t_1)V_C(t_1) = \frac{Q^2(t_1)}{2C} = \frac{1}{2}CV_0^2(1 - e^{-t_1/\tau})^2$$

(e) The time-dependent power dissipated in the resistor is

$$P(t) = I^2(t)R = (V_0^2/R)e^{-2t/\tau} \quad .$$

The heat dissipated in the resistor between $t = 0$ and t_1 is

$$W = \int_0^{t_1} P(t)\,dt = \frac{V_0^2}{R}\int_0^{t_1} e^{-2t/\tau}dt = \frac{-\tau V_0^2}{2R}\, e^{-2t/\tau}\Big|_0^{t_1} = \frac{1}{2}CV_0^2(1 - e^{-2t_1/\tau}) \quad .$$

Problem 9

(a) The electric force $q\mathbf{E}$ is directed upward (remember, the electron is negatively charged) and has magnitude $F_E = (1.6 \times 10^{-19})(10^5) = 1.6 \times 10^{-14}\,\mathrm{N}$. The magnetic force $q\mathbf{v} \times \mathbf{B}$ is directed downward and has magnitude $F_B = (1.6 \times 10^{-19})(3 \times 10^6)(0.1) = 4.8 \times 10^{-14}\,\mathrm{N}$. $F_B > F_E$, so the net force is directed downwards and has magnitude $F = F_B - F_E = 3.2 \times 10^{-14}\,\mathrm{N}$.

(b) If we adjust the field strengths E and B to satisfy $E = vB$ (v = electron speed), the electric and magnetic forces will be of equal magnitude. Since the two forces are oppositely directed, they will balance one another, and the electron will be able to pass through the capacitor undeflected. (As an example, if we increase E to $3 \times 10^5\,\mathrm{V/m}$ while holding B fixed, both forces will have a magnitude of $4.8 \times 10^{-14}\,\mathrm{N}$.)

Problem 10

(a) Poynting flux: $S = 10^3\,\mathrm{W/m^2}$. Area: $A = 10^{-4}\,\mathrm{m^2}$. Time interval: $\Delta t = 30\,\mathrm{s}$.

$$\text{Energy absorbed} = SA\,\Delta t = (10^3)(10^{-4})(30) = 3\,\mathrm{J} \quad .$$

(b) Radiation pressure: $P = S/c$ for the case of total absorption. Force on surface:

$$F = PA = SA/c = \frac{(10^3)(10^{-4})}{(3 \times 10^8)} \simeq 3.3 \times 10^{-10}\,\mathrm{N} \quad .$$

(c) 100% Reflection $\Longrightarrow$ **no** energy is absorbed.

Problem 11

(a) In the $\omega = 0$ steady state, all currents are constant in time. The self-inductor acts as a wire with zero resistance, shorting out the capacitor and ensuring $I_C = 0$ (there would be no current in this branch even if we replaced the capacitor with a resistor). Thus we effectively have a resistor R in series with a zero-resistance wire, and $I_L = I_R = V_0/R$. (Notice, even though it is irrelevant here, that the reactance of the capacitor $(1/\omega C)$ is infinitely high.)

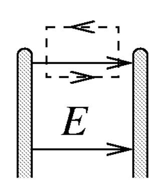
(b) With ω infinitely high, the reactance of the self-inductor (ωL) becomes infinitely large and thus $I_L = 0$. The reactance of the capacitor $(1/\omega C)$ goes to zero. We effectively have a resistor R in series with a zero-reactance capacitor, and our peak current values are simply $I_C = I_R = V_0/R$.

(c) We would expect the peak value of I_R to **decrease** from the value V_0/R found in part (a) as we increase ω to non-zero values. The effective impedance of the parallel combination of the capacitor and the self-inductor was zero with $\omega = 0$. For $\omega \neq 0$ it can only increase, as the self-inductor will no longer act as zero-Ohm wire and the reactance of the capacitor $(1/\omega C)$ will also be non-zero. (This expectation can be confirmed by an exact calculation, if you are feeling ambitious.)

(d) The frequency is $\omega = 1\big/\sqrt{LC}$. (What else could it be?) This is the natural oscillation frequency of the LC sub-circuit alone. We will have an oscillating current in the LC sub-circuit that dissipates no energy. The voltage across the capacitor will be exactly offset by the power-supply voltage at all times, so that there will never be any voltage across the resistor. Hence, $I_R = 0$. (None of this is very intuitive.)

Problem 12 *(See homework problem 5.7 and Exam 2, problem 4)*

Assuming an abrupt drop to zero, integrating around the dashed path shown gives $\oint \mathbf{E} \cdot d\boldsymbol{l} \neq 0$, for there is an E-field between the plates. Since this is a static situation, we have $d\Phi_B/dt = 0$ for the rate of change of magnetic flux through the open surface bounded by our loop, and thus Faraday's law asserts $\oint \mathbf{E} \cdot d\boldsymbol{l} = 0$. Therefore the E-field *cannot* drop abruptly to zero outside the capacitor.



Problem 13

(a) V_{right} will read -0.1 Volt, since it has the same internal resistance as V_{left} but has its terminals oppositely oriented with respect to the current in the circuit.

(b) If V_{left} reads $+0.1$ Volt, then the current must enter through the positive terminal. Thus the current flows counterclockwise. From Ohm's law,

$$I = V_{\text{left}}/R_{\text{left}} = (0.1)/(10^4) = 10^{-5}\,\text{Amperes}.$$

The induced EMF is counterclockwise as well, and is given by

$$\mathcal{E} = I(R + R_{\text{left}} + R_{\text{right}}) = (10^{-5})(2.5 \times 10^4) = 0.25\,\text{Volts}.$$

Problem 14

(a) TRUE: For light traveling in air (refractive index $\cong 1$), the Brewster (or polarizing) angle is given by $\tan\theta_{\rm p} = n$, where n is the refractive index of glass. n is a function of wavelength, so $\theta_{\rm p}$ will be slightly different for red light than for blue light.

(b) FALSE: Spectra of approaching stars show lines Doppler-shifted towards *shorter* wavelengths. *Lecture #35, Giancoli 37-12.*

(c) TRUE: The blue is on the outside of the secondary bow, and the light is highly polarized due to the two reflections inside the raindrop. *Lecture #31.*

(d) FALSE: Glare reflected from puddles is partially polarized in the horizontal direction, so the polarization of the glasses should be *vertical* in order to suppress it. *Lecture #31, Homework Problem 10.1c*

(e) TRUE: An exotic generator (as discussed in Lecture #23) based on the same principle as more familiar generators (*see also problem 7 of Exam #2*).

(f) TRUE: Identical currents in the two rings will generate identical magnetic fields, which will in turn result in identical magnetic fluxes. Self-inductance is simply the ratio of this flux to the current, therefore it will be the same for the two rings regardless of their differing composition.

(g) TRUE: Consider two parallel-plate capacitors in air: a smaller one with plate area A and plate separation d and a larger one with with plate area $2A$ and plate separation $2d$. Both have the same capacitance $C = \epsilon_0 A/d$. However, if both are charged to the same potential difference V, the electric field between the plates of the smaller capacitor will be $E_1 = V/d$ while the electric field between the plates of the larger one will be $E_2 = V/2d$. Thus the larger capacitor can be charged to twice as high a potential difference before reaching the breakdown E for air, and can hold more energy ($\mathrm{CV}^2/2$). If the dielectric constant, κ, is the same for both, the same reasoning holds. It is therefore very probable that the capacitor which is significantly larger in size than the other, can be charged to a higher potential. *In Lecture #8, I discussed this in some detail. I compared two capacitors, each 100μF; one was $10^4 cm^3$ in volume, the other only 1 cm^3. The first was rated for 4000 Volt, the second for 40 Volt.*

(h) FALSE: The spectrum is not generated by the wavelength-dependent index of refraction in plastic, but rather by the spaces between lines on the plastic acting as parallel slits to set up an interference pattern.

(i) TRUE: HST is not subject to the atmospheric blurring that limits the resolution of large ground-based telescopes. Its limiting angular resolution $\theta = (1.22)\lambda/D$ (D = aperture diameter) improves (gets smaller) as the wavelength of the light λ gets shorter. *Lecture #34; see also Problem 11.6.*

(j) TRUE: Even individual quantum particles (such as photons) exhibit interference effects.

(k) TRUE: *Lecture #31.*

(l) TRUE: The speed of sound in air does indeed depend on temperature. Musical tones are defined by their frequency $f = v/\lambda$, and this will change if the wavelength λ remains fixed but the sound speed v changes. (As an example of how λ remains fixed, recall that the wavelength of the lowest tone is twice the length of the flute.) *Lecture #26.*

(m) TRUE: The ice crystals act like prisms. *Lecture #31.*

(n) FALSE: *Any* unbalanced force on a particle causes an acceleration ($\mathbf{F} = m\mathbf{a}$). If the force is perpendicular to the particle's velocity (as is the case for magnetic forces), then the acceleration causes a change in the direction of velocity.

(o) FALSE: It is not the value of the flux itself that gives an induced EMF, but rather the time derivative of the flux. Φ_B can be zero at an instant, and $d\Phi_B/dt$ can still be non-zero.

ABSOLUTE END