Adam S. Bolton
bolton@mit.edu

May 1, 2002

MIT 8.02 Spring 2002
Assignment #9 Solutions

Problem 9.1

Wavelength of radio waves. (Giancoli 32-37.)

Channel 2:

$$\lambda_2 = \frac{c}{f_2} = \frac{3.00 \times 10^8}{54.0 \times 10^6} = 5.56\,\text{m} \quad .$$

Channel 69:

$$\lambda_{69} = \frac{c}{f_{69}} = \frac{3.00 \times 10^8}{806 \times 10^6} = 0.372\,\text{m} \quad .$$

Problem 9.2

Traveling electromagnetic waves.

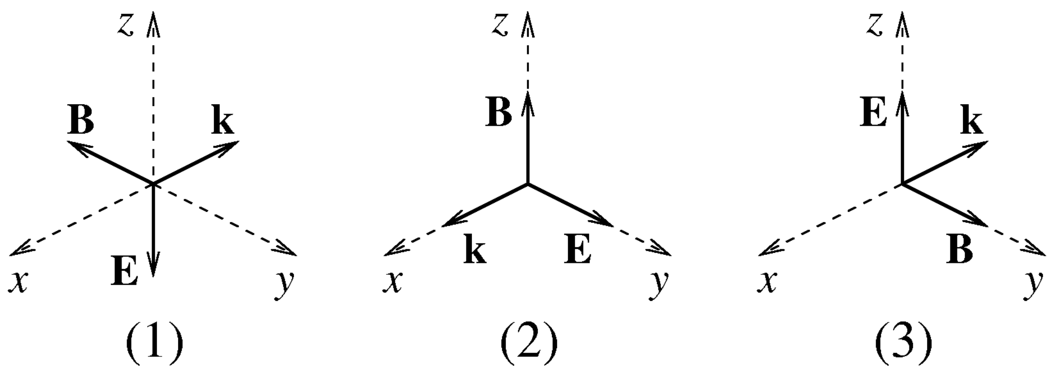
The given electric field is in all three cases of the form

$$\mathbf{E}(x,t) = \mathbf{E}_0 \sin(kx \pm \omega t + \alpha) \quad ,$$

with $\mathbf{E}_0$ perpendicular to the direction of propagation (the x-direction) and $\alpha = 0$ or $\pi/2$ (recall that $\sin(\theta + \pi/2) = \cos\theta$). For such a wave, the propagation direction is $+\hat{x}$ if the argument is $(kx - \omega t + \alpha)$ and $-\hat{x}$ if the argument is $(kx + \omega t + \alpha)$. k is the wavenumber, $\lambda = 2\pi/k$ is the wavelength, $f = \omega/2\pi$ is the frequency in Hertz, $v = \omega/k$ is the speed, and $n = c/v$ is the index of refraction. From the given expressions and these definitions, we can read off the answers to **(a)**–**(e)**:

	prop. direct.	λ (m)	k (m^{-1})	f (Hz)	v (m/s)	n
case (1)	$-\hat{x}$	4.00	1.57	7.50×10^7	3.00×10^8	1.0
case (2)	$+\hat{x}$	2.00	3.14	1.50×10^8	3.00×10^8	1.0
case (3)	$-\hat{x}$	1.00	6.28	2.13×10^8	2.13×10^8	1.4

(f) In order to construct the corresponding equations for $\mathbf{B}$, we must remember two features of a traveling EM plane wave: (i) $\mathbf{B}$ is in phase with $\mathbf{E}$, and (ii) $\mathbf{B}$ is perpendicular to both $\mathbf{E}$ and the propagation direction such that $\mathbf{E} \times \mathbf{B}$ points in the direction of propagation. If the vector $\mathbf{k}$ indicates the direction of propagation, then our three cases must have the following orientations:



As for magnitudes: $B = E/v = nE/c$. The full expressions for the magnetic fields of our three cases are thus (with B in Tesla)

$$\begin{aligned}
\text{case (1):}\quad B_y &= (-8.33\times10^{-8})\sin(1.57x + 4.71\times10^{8}t)\ ,\quad B_x = B_z = 0\\
\text{case (2):}\quad B_z &= (1.67\times10^{-7})\cos(3.14x - 9.42\times10^{8}t)\ ,\quad B_x = B_y = 0\\
\text{case (3):}\quad B_y &= (1.87\times10^{-7})\cos(6.28x + 1.34\times10^{9}t)\ ,\quad B_x = B_z = 0
\end{aligned}$$

(g) The instantaneous Poynting vector for case (3) is (Giancoli Equation (32-18), p. 801):

$$\begin{aligned}
\mathbf{S} &= \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = -\frac{1}{\mu_0}E_z B_y \hat{x}\\
&= -\frac{(40)(1.87\times10^{-7})}{(4\pi\times10^{-7})}\cos^2(6.28x + 1.34\times10^{9}t)\hat{x}\\
&= (-6.0)\cos^2(6.28x + 1.34\times10^{9}t)\hat{x}\quad .
\end{aligned}$$

The time average of $\cos^2(A+Bt)$ is $\frac{1}{2}$ for any A and B, so the time-averaged Poynting vector for *all* positions (including the two specified) is

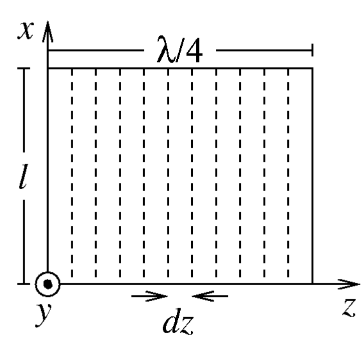
$$\overline{\mathbf{S}} = (-3.0)\hat{x}\quad \text{(units: Joules per square meter per second).}$$

Thus we find that this traveling electromagnetic wave transmits energy in the $-\hat{x}$ direction through space.

Problem 9.3

EM waves – Maxwell's equations and the "speed of light".

We want to apply Faraday's law to the given plane surface (area A_1) and the rectangular loop that bounds it. For definiteness, we'll take the normal to the surface to be in the $+\hat{y}$ direction. To calculate Φ_B, we divide the surface up into many strips of thickness dz as shown in the diagram. Each strip will make a differential contribution to the flux of



$$d\Phi_B = B_y\,dA = B_0\cos(kz - \omega t)\,l\,dz\quad .$$

The total flux will then be given by

$$\Phi_B = \int d\Phi_B = B_0 l \int_0^{\lambda/4} \cos(kz - \omega t)\, dz = \frac{B_0 l}{k}[\sin(k\lambda/4 - \omega t) - \sin(-\omega t)] \ .$$

Since $k\lambda/4 = k(2\pi/k)/4 = \pi/2$, this becomes

$$\Phi_B = \frac{B_0 l}{k}[\cos(\omega t) + \sin(\omega t)] \implies -\frac{d\Phi_b}{dt} = \frac{B_0 l \omega}{k}[\sin(\omega t) - \cos(\omega t)] \ .$$

Now to calculate $\oint \mathbf{E}\cdot d\boldsymbol{l}$. Our choice of $+\hat{y}$ (as opposed to $-\hat{y}$) for the normal to our surface dictates that our line integral be taken counterclockwise when viewed as in the diagram. Since $\mathbf{E}$ is purely in the $\hat{x}$ direction, $\mathbf{E}\cdot d\boldsymbol{l} = 0$ along the top and bottom edges of the integration curve. This leaves us with

$$\begin{aligned}\oint \mathbf{E}\cdot d\boldsymbol{l} &= \int_0^l E_x(z=\lambda/4, t)\, dx + \int_l^0 E_x(z=0,t)\, dx \\ &= E_0 \sin(\omega t) \int_0^l dx + E_0 \cos(\omega t) \int_l^0 dx \\ &= E_0 l[\sin(\omega t) - \cos(\omega t)] \ .\end{aligned}$$

Faraday's law asserts that $\oint \mathbf{E}\cdot d\boldsymbol{l} = -d\Phi_B/dt$. For the case under consideration, this gives

$$E_0 l[\sin(\omega t) - \cos(\omega t)] = \frac{B_0 l \omega}{k}[\sin(\omega t) - \cos(\omega t)] \ .$$

This will be satisfied for *all* time only if $E_0 = B_0\omega/k$. Given that $c = \omega/k$ is the wave speed, we have the result $B_0 = E_0/c$ as a consequence of Faraday's law. Combining this with $B_0 = \epsilon_0\mu_0 c E_0$ as obtained in lecture from Ampère's law, we conclude that $c = 1/\sqrt{\epsilon_0\mu_0}$ is the speed of light in vacuum.

Problem 9.4

A standing electromagnetic wave.

(a) Any standing wave of the form $\cos(kz)\cos(\omega t)$ has a wavelength of $2\pi/k$ and a frequency in Hertz of $\omega/2\pi$. For our wave, $k = 2\sqrt{3}\,\text{cm}^{-1}$ and $\omega = 7.0\times 10^{10}\,\text{rad/s}$, so

$$\lambda = 1.814\,\text{cm} \ , \quad f = 1.114\times 10^{10}\,\text{Hz} \ .$$

(b) The index of refraction of the medium is

$$n = \frac{c}{v} = \frac{c}{\omega/k} = \frac{(3.00\times 10^{10}\,\text{cm/s})}{(7.0\times 10^{10}\,\text{s}^{-1})/(2\sqrt{3}\,\text{cm}^{-1})} = 1.48$$

(c) To find $\mathbf{B}$, we picture our $\mathbf{E}$-field as the linear superposition of two traveling waves, one traveling in the $+\hat{z}$ direction and one in the $-\hat{z}$ direction. Using the trigonometric identity

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta) \ ,$$

we can rewrite our **E**-field as

$$\mathbf{E} = \frac{1}{2}E_0\hat{x}[\cos(2\sqrt{3}z - 7.0\times 10^{10}t) + \cos(2\sqrt{3}z + 7.0\times 10^{10}t)] \ .$$

Now, using the rule discussed in problem 9.2(f), the **B**-field associated with the traveling wave propagating in the $+\hat{z}$ direction must point in the $+\hat{y}$ direction (assuming E_0 is positive) so that $\mathbf{E}\times\mathbf{B}$ points in the $+\hat{z}$ direction. Similarly, for the wave propagating in the $-\hat{z}$ direction, **B** must point in the $-\hat{y}$ direction. Thus our total **B**-field must be

$$\mathbf{B} = \frac{1}{2}B_0\hat{y}[\cos(2\sqrt{3}z - 7.0\times 10^{10}t) - \cos(2\sqrt{3}z + 7.0\times 10^{10}t)] \ .$$

Using the identity

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta) \ ,$$

we can write this as

$$\mathbf{B} = B_0\hat{y}\sin(2\sqrt{3}z)\sin(7.0\times 10^{10}t) \ .$$

We see that in a standing wave, **B** is 90° out of phase relative to **E** both in space and in time. The value of B_0 is related to E_0 by

$$B_0 = \frac{k}{\omega}E_0 = \frac{E_0}{v} = \frac{nE_0}{c} \ .$$

(d) The instantaneous Poynting vector $\mathbf{S} = (\mathbf{E}\times\mathbf{B})/\mu_0$ for this wave at any point in space will have a time dependence of the form $\mathbf{S} \propto \sin(\omega t)\cos(\omega t)$. The average of $\sin(\omega t)\cos(\omega t)$ over one full period in time is zero, so $\overline{\mathbf{S}} \equiv 0$ at *all* points. This result tells us that standing elecromagnetic waves do not transmit energy through space. Compare this with the result of 9.2(g), where we found that a *traveling* electromagnetic wave *does* transmit energy.

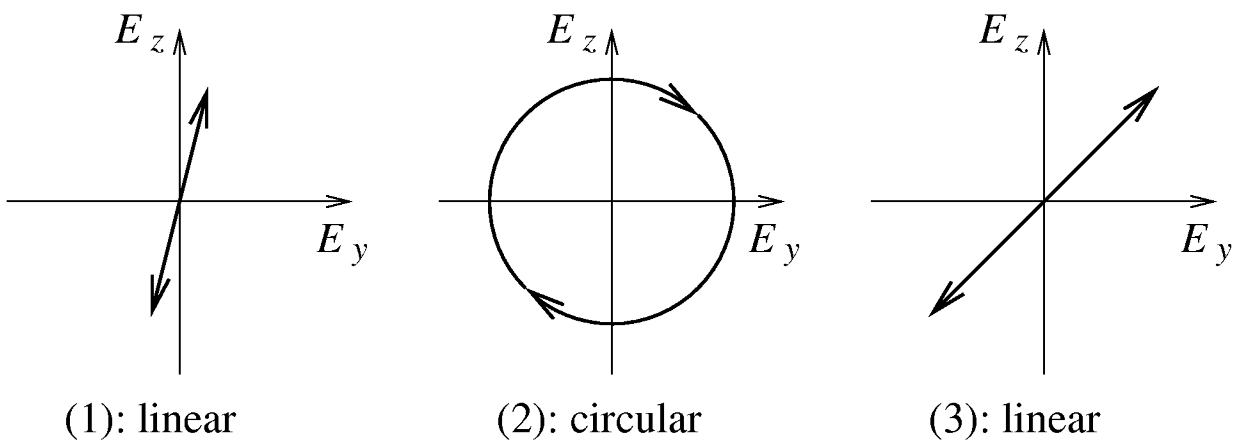
Problem 9.5

Polarization of electromagnetic radiation.

(a) If we let $x = 0$, our electric fields vary with time as

$$\begin{aligned}
(1):&\quad E_y = -E_0\sin(\omega t) \quad E_z = -4E_0\sin(\omega t)\\
(2):&\quad E_y = -E_0\cos(\omega t) \quad E_z = E_0\sin(\omega t)\\
(3):&\quad E_y = 2E_0\sin(\omega t) \quad E_z = 2E_0\sin(\omega t)
\end{aligned}$$

We can now plot a trace of **E** as a function of time at $x = 0$ and see the polarization easily (note: plots axes not to scale from one to the next):



(b) The amplitude of **B** is given by the electric field amplitude divided by c, since we are in a vacuum. We obtain the direction by requiring that $\mathbf{E}\times\mathbf{B}$ be in the direction of propagation. (It is helpful to note that, for **E** and **B** purely in the y-z direction, $\mathbf{E}\times\mathbf{B} = \hat{x}(E_yB_z - E_zB_y)$.) Using these prescriptions, we find

$$\begin{array}{lll} (1): & B_y = (-4E_0/c)\sin(kx-\omega t) & B_z = (E_0/c)\sin(kx-\omega t) \\ (2): & B_y = (E_0/c)\sin(kx+\omega t) & B_z = (E_0/c)\cos(kx+\omega t) \\ (3): & B_y = (2E_0/c)\sin(kx-\omega t) & B_z = (2E_0/c)\cos(kx-\omega t+\pi/2) \end{array}$$

($B_x = 0$ for all cases.)

Problem 9.6

Radiation pressure due to the sun. (Giancoli 32-29.)

Let $P = 3.8\times10^{26}$ W be the Sun's total power output. Assuming negligible absorption in the intervening space, the amount of energy per unit time crossing a spherical surface of radius r centered on the Sun will also be P. The time-averaged Poynting flux (energy per unit area per unit time) at a distance r from the center of the Sun will therefore be

$$\overline{S}(r) = \frac{P}{4\pi r^2} \quad .$$

Assuming full absorption, the dust particles will feel a radiation pressure of $p_{\rm rad} = \overline{S}/c$ (see Giancoli section 32-8, pp. 802-803). If the particles have a radius a, they will feel an outward (i.e. away from the Sun) force given by

$$F_{\rm rad} = \pi a^2 p_{\rm rad} = \pi a^2\overline{S}/c = \frac{a^2P}{4r^2c} \quad .$$

The particles also feel a gravitational force directed towards the Sun. If ρ is the mass density of the dust and M is the Sun's mass, the magnitude of this force is

$$F_G = \frac{G(\frac{4}{3}\pi a^3\rho)M}{r^2} \quad .$$

The magnitude of the radiation pressure force grows as a^2, while the magnitude of the gravitational force grows as a^3. So for very small particles, the outward radiation force should dominate, while for larger particles, the inward gravitational force will dominate. The scale is set by the particle size a_0 for which the two forces exactly balance one another:

$$\frac{a_0^2 P}{4r^2 c} = \frac{G(\frac{4}{3}\pi a_0^3 \rho)M}{r^2} \quad \Longrightarrow \quad a_0 = \frac{3P}{16\pi G\rho M c} \quad .$$

Plugging in the (many!) numbers, we get

$$a_0 = \frac{3(3.8\times 10^{26})}{16\pi(6.67\times 10^{-11})(2.0\times 10^{3})(1.99\times 10^{30})(3.00\times 10^{8})} = 2.85\times 10^{-7}\,\mathrm{m} \quad .$$

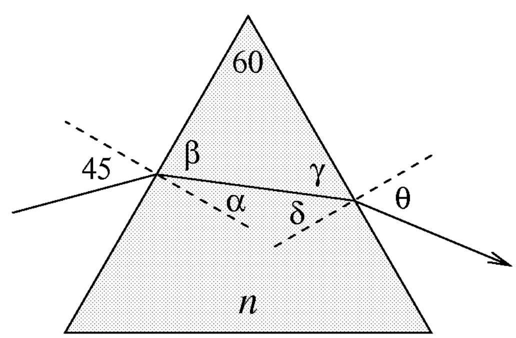
Dust particles with a radius smaller than this would have been ejected by radiation pressure.

Problem 9.7

Snell's law in action ⇒ dispersion! (Giancoli 33-46.)

From Giancoli Figure 33-26 (p. 825), we can obtain approximate values for the index of refraction of silicate flint glass for the two wavelengths of interest:

$$\lambda_1 = 450\,\mathrm{nm}: \quad n_1 \simeq 1.64$$
$$\lambda_2 = 650\,\mathrm{nm}: \quad n_2 \simeq 1.62 \; .$$



Now, consider either of the two rays. Define the angles α, β, γ, and δ as shown in the diagram at right. Let n be either n_1 or n_2 and θ be either θ_1 or θ_2. We'll take the refractive index of the surrounding medium to be 1. Snell's law (Giancoli Equation (33-5), p. 823) tells us that

$$\sin(45^\circ) = 1/\sqrt{2} = n\sin\alpha \qquad \text{and} \qquad n\sin\delta = \sin\theta \quad .$$

Also, $\alpha = 90^\circ - \beta$ and $\delta = 90^\circ - \gamma$, so $\sin\alpha = \cos\beta$ and $\sin\delta = \cos\gamma$. Thus

$$1/\sqrt{2} = n\cos\beta \qquad \text{and} \qquad n\cos\gamma = \sin\theta \quad .$$

Finally, we have $\beta + \gamma + 60^\circ = 180^\circ \Rightarrow \gamma = 120^\circ - \beta$. Solving for θ now gives

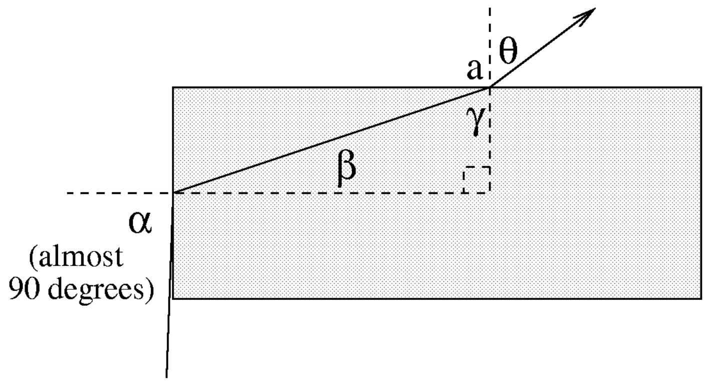
$$\begin{aligned} \theta &= \arcsin(n\cos\gamma) = \arcsin[n\cos(120^\circ - \beta)] \\ &= \arcsin\left\{ n\cos\left[120^\circ - \arccos\left(\frac{1}{\sqrt{2}n}\right)\right]\right\} \quad . \end{aligned}$$

For our two refractive indices of interest, this gives

$$\theta_1 = 68.1^\circ \;, \quad \theta_2 = 65.3^\circ \quad .$$

Problem 9.8

Snell's law in action ⇒ fiber optics! (Giancoli 33-53.)



The greatest test of our optic fiber's ability to guarantee total internal reflection will occur when the beam entrance angle $\alpha \to 90°$. So, let's consider that case in particular. Snell's law gives

$$\sin\alpha = \sin 90° = 1 = n\sin\beta = n\cos\gamma \quad .$$

Now suppose that total internal reflection does *not* necessarily occur at point "a". The angle θ that the emerging beam makes with the normal to the fiber's surface will be given by Snell's law:

$$\sin\theta = n\sin\gamma = n\sqrt{1-\cos^2\gamma} \quad .$$

Using $n\cos\gamma = 1$ from above, this becomes

$$\sin\theta = n\sqrt{1-1/n^2} = \sqrt{n^2-1} \quad .$$

So $\sin\theta$ increases as n gets bigger. $\sin\theta = 1$ (corresponding to $\theta = 90°$) is the critical value for the onset of total internal reflection at point "a". The condition on n for total internal reflection of all beams entering the fiber is therefore

$$\sqrt{n^2-1} > 1 \quad \Longrightarrow \quad n > \sqrt{2} \simeq 1.42 \quad ,$$

where we have rounded up just to be safe.

END