Angular Momentum: Rotation and Translation for Fixed Axis Rotation

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Constants of the Motion

When are the conserved quantities, momentum, energy, and angular momentum about a point S, constant for a system?

- No external torques about point S: angular momentum about S is constant

$$\vec{\mathbf{0}} = \vec{\boldsymbol{\tau}}_S^{ext} = \frac{d\vec{\mathbf{L}}_S^{system}}{dt}$$

- No external work: mechanical energy constant

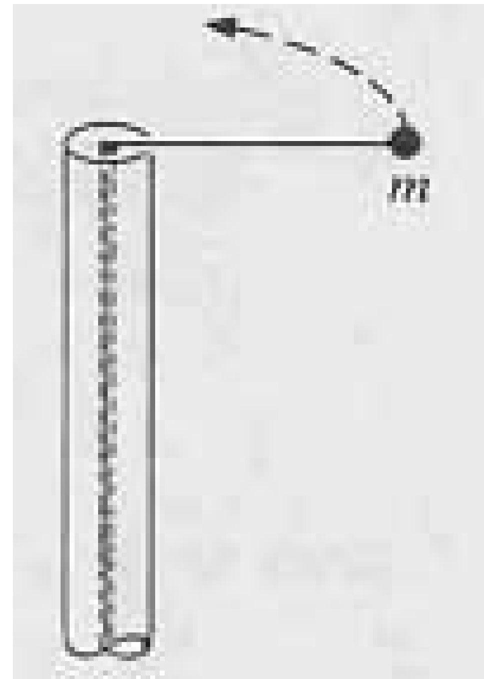
$$0 = W_{ext} = \Delta E_{mechanical}$$

- No external forces: momentum constant

$$\vec{\mathbf{F}}^{ext} = \frac{d}{dt}\vec{\mathbf{p}}^{system}$$

PRS Question

A tetherball of mass m is attached to a post of radius by a string. Initially it is a distance r_0 from the center of the post and it is moving tangentially with a speed v_0 . The string passes through a hole in the center of the post at the top. The string is gradually shortened by drawing it through the hole. Ignore gravity. Until the ball hits the post,



1. The energy and angular momentum about the center of the post are constant.

2. The energy of the ball is constant but the angular momentum about the center of the post changes.

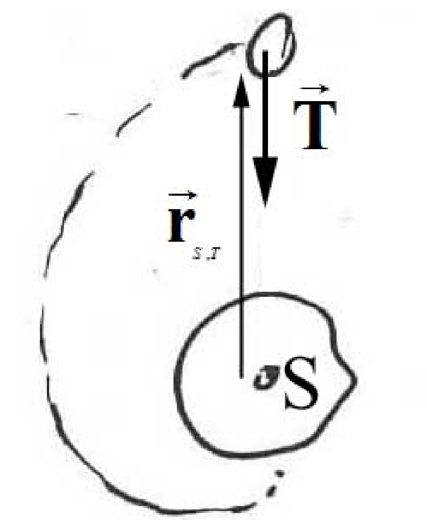
3. Both the energy and the angular momentum about the center of the post, change.

4. The energy of the ball changes but the angular momentum about the center of the post is constant.

PRS Question: Solution

- The tension force points radially in; so torque about central point is zero: angular momentum about central point is constant.

- All radial forces, for example gravitation, have angular momentum about central point constant



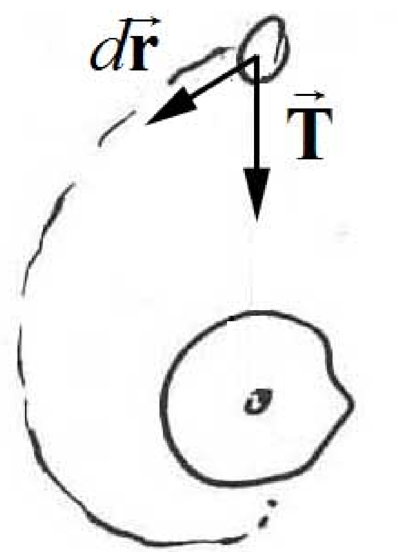
$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,T} \times \vec{\mathbf{T}} = \vec{\mathbf{0}}$$

$$\vec{\mathbf{L}}_{S,f}^{system} = \vec{\mathbf{L}}_{S,0}^{system}$$

PRS Question: Solution

- Small displacement has radially component inward so work done by tension is not zero.

$$dW^{ext} = \vec{\mathbf{T}} \cdot d\vec{\mathbf{r}} \neq \vec{\mathbf{0}}$$

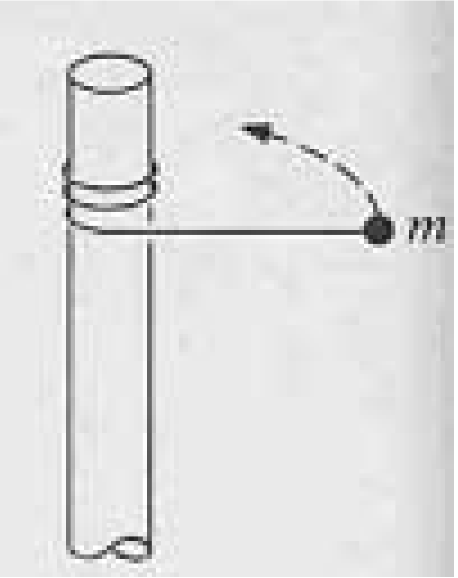


- Mechanical energy is not constant

$$W_{ext} = \Delta K \neq 0$$

PRS Question

A tetherball of mass m is attached to a post of radius R by a string. Initially it is a distance r_0 from the center of the post and it is moving tangentially with a speed v_0. The string wraps around the outside of the post. Ignore gravity. Until the ball hits the post,



1. The energy and angular momentum about the center of the post are constant.

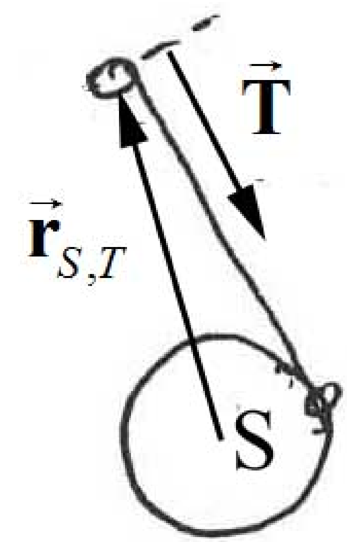
2. The energy of the ball is constant but the angular momentum about the center of the post changes.

3. Both the energy of the ball and the angular momentum about the center of the post, change.

4. The energy of the ball changes but the angular momentum about the center of the post is constant.

PRS Question: Solution

- The tension force points towards contact point; so torque about central point is not zero: angular momentum about central point is not constant.

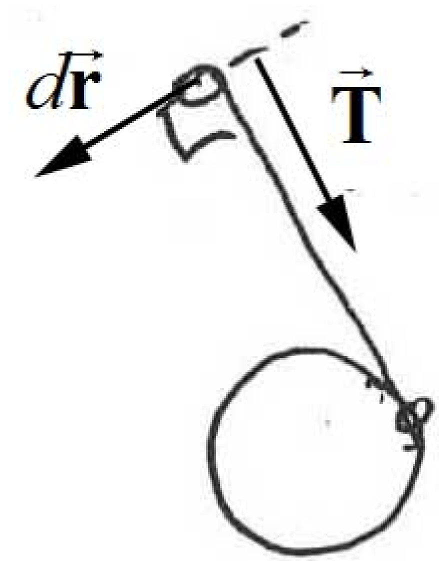


$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,T} \times \vec{\mathbf{T}} \neq \vec{\mathbf{0}}$$

$$\vec{\mathbf{L}}_{S,f}^{system} \neq \vec{\mathbf{L}}_{S,0}^{system}$$

PRS Question: Solution

- Small displacement is always perpendicular to string since at each instant in time ball undergoes instantaneous circular motion about string contact point with pole



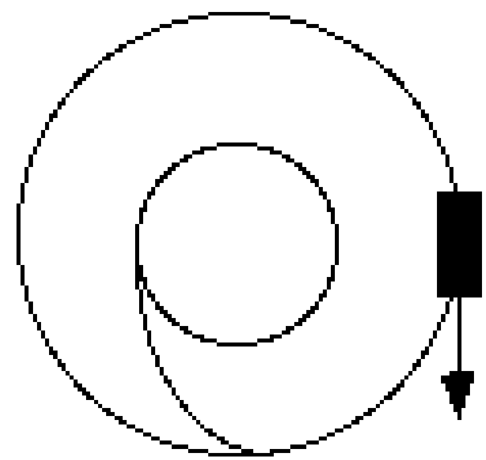
$$dW^{ext} = \vec{\mathbf{T}} \cdot d\vec{\mathbf{r}} = \vec{\mathbf{0}}$$

- Mechanical energy is constant

$$W_{ext} = \Delta K = 0$$

PRS Question

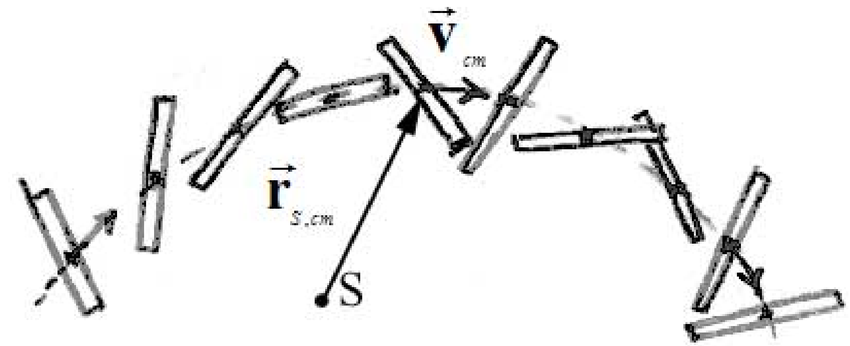
A streetcar is freely coasting (no friction) around a large circular track. It is then switched to a small circular track. When coasting on the smaller circle the streetcar's



1. mechanical energy is conserved and angular momentum about the center is conserved
2. mechanical energy is not conserved and angular momentum about the center is conserved
3. mechanical energy is not conserved and angular momentum about the center is not conserved
4. mechanical energy is conserved and angular momentum about the center is not conserved.

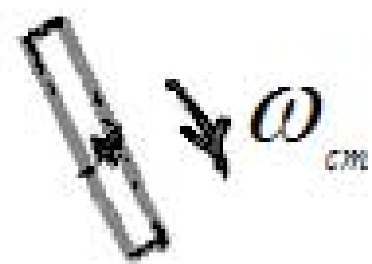
Total Angular Momentum about a Fixed Point

- Translation of center of mass: treat total mass as point particle: orbital angular momentum about point S



$$\vec{\mathbf{L}}_S^{orbital} = \vec{\mathbf{r}}_{S,cm} \times \vec{\mathbf{p}}^{total}$$

- Rotation about center of mass: spin angular momentum

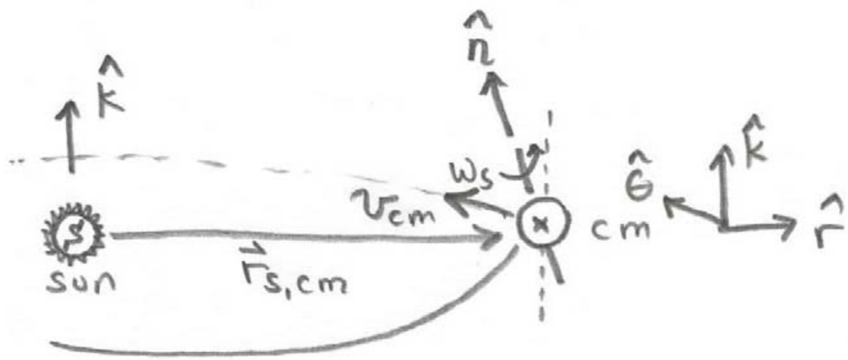


$$\vec{\mathbf{L}}_{cm}^{\ spin} = I_{cm}\vec{\boldsymbol{\omega}}_{spin}$$

- Total angular momentum about S:

$$\vec{\mathbf{L}}_S^{\ total} = \vec{\mathbf{r}}_{S,cm} \times m_T \vec{\mathbf{v}}_{cm} + \vec{\mathbf{L}}_{cm}^{\ spin}$$

Earth's Motion Orbital Angular Momentum about Sun



- Orbital angular momentum about center of sun

$$\vec{\mathbf{L}}_S^{orbital} = \vec{\mathbf{r}}_{S,cm} \times \vec{\mathbf{p}}^{total} = r_{s,e} m_e v_{cm} \hat{\mathbf{k}}$$

- Center of mass velocity and angular velocity

$$v_{cm} = r_{s,e} \omega_{orbit}$$

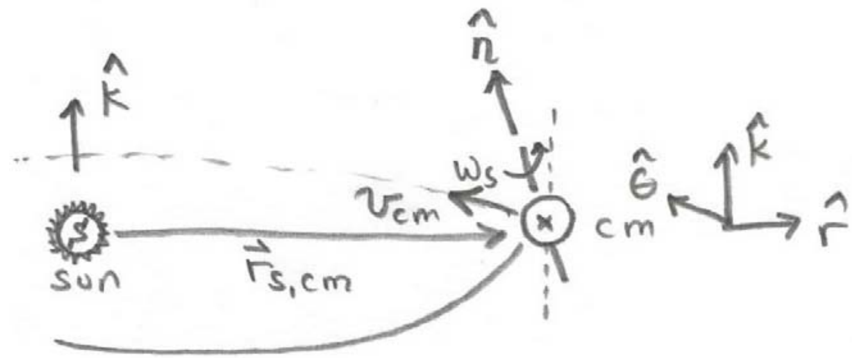
- Period and angular velocity

$$\omega_{orbit} = \frac{2\pi}{T_{orbit}} = 2.0 \times 10^{-7}\, rad \cdot s^{-1}$$

- Magnitude

$$\vec{\mathbf{L}}_S^{orbital} = m_e r_{s,e}{}^2 \omega_{orbit} \hat{\mathbf{k}} = \frac{m_e r_{s,e}{}^2 2\pi}{T_{orbit}} \hat{\mathbf{k}} \qquad \vec{\mathbf{L}}_S^{orbital} = 2.67 \times 10^{40}\, kg \cdot m^2 \cdot s^{-1} \hat{\mathbf{k}}$$

Earth's Motion
Spin Angular Momentum



- Spin angular momentum about center of mass of earth

$$\vec{\mathbf{L}}_{cm}^{spin} = I_{cm}\vec{\boldsymbol{\omega}}_{spin} = \frac{2}{5} m_e R_e^{\,2} \omega_{spin} \hat{\mathbf{n}}$$

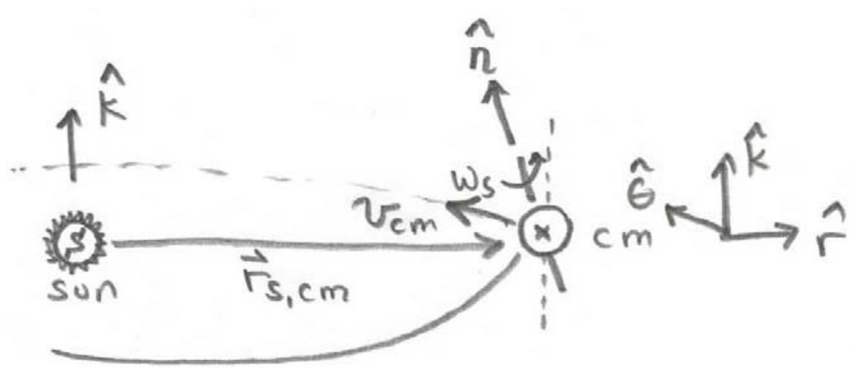
- Period and angular velocity

$$\omega_{spin} = \frac{2\pi}{T_{spin}} = 7.29 \times 10^{-5}\, rad \cdot s^{-1}$$

- Magnitude

$$\vec{\mathbf{L}}_{cm}^{spin} = 7.09 \times 10^{33}\, kg \cdot m^2 \cdot s^{-1} \hat{\mathbf{n}}$$

Total Angular Momentum for Earth's Motion about Sun



- Ratio: $$\frac{L_S^{orbital}}{L_{cm}^{spin}} = \frac{m_e {r_{s,e}}^2 \omega_{orbit}}{(2/5) m_e {R_e}^2 \omega_{spin}} = \frac{5}{2}\frac{{r_{s,e}}^2}{{R_e}^2}\frac{T_{spin}}{T_{orbit}} = 3.77 \times 10^6$$

- Total: $$\vec{\mathbf{L}}_S^{total} = \frac{m_e {r_{s,e}}^2 2\pi}{T_{orbit}}\hat{\mathbf{k}} + \frac{2}{5} m_e {R_e}^2 \omega_{spin} \hat{\mathbf{n}}$$