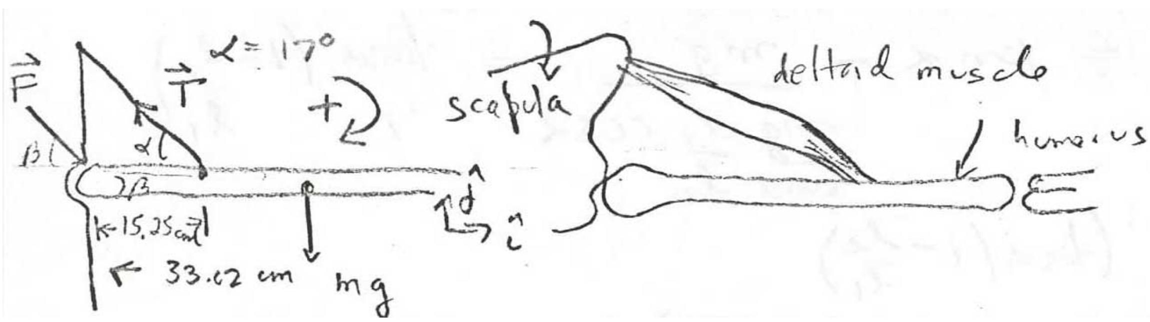
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01T **Fall Term 2004**

Practice Exam 2 Solutions

Problem 1: static equilibrium

You are able to hold out your arm in an outstretched horizontal position thanks to the action of the deltoid muscle. Assume the humerus bone has a mass of m, the center of mass of the humerus is a distance d from the scapula, the deltoid muscle attaches to the humerus a distance a distance s from the scapula and the angle the deltoid muscle makes with the horizontal is α. A schematic representation of this action looks as follows:



a) What is the tension T in the deltoid muscle?

b) What are the vertical and horizontal components of the force exerted by the scapula (shoulder blade) on the humerus?

Solution:

Choose the unit vectors $\hat{\mathbf{i}}$ to point horizontally to the right and $\hat{\mathbf{j}}$ vertically upwards. Let F_x and F_x be the horizontal and vertical components of the force exerted by the scapula (shoulder blade) on the humerus ? The condition that the sum of the forces acting on the rigid body is zero,

$$\vec{\mathbf{F}}_{total} = \vec{\mathbf{T}} + m\vec{\mathbf{g}} + \vec{\mathbf{F}} = \vec{\mathbf{0}},$$

becomes

$$\hat{\mathbf{i}} : F_x - T\cos\alpha = 0$$
$$\hat{\mathbf{j}} : F_y + T\sin\alpha - mg = 0.$$

Choose the contact point of the scapula (shoulder blade) on the humerus to compute the torque. Then the total torque about this point S is

Then torque equilibrium becomes

$$\vec{\boldsymbol{\tau}}_S^{total} = \vec{\boldsymbol{\tau}}_{S,T} + \vec{\boldsymbol{\tau}}_{S,cm} = \vec{\mathbf{0}}\,.$$

The torque about S due to the center of mass of the humerus is

$$\vec{\boldsymbol{\tau}}_{S,cm} = \vec{\mathbf{r}}_{S,cm} \times m_1\vec{\mathbf{g}} = d\hat{\mathbf{i}} \times \left(-mg\hat{\mathbf{j}}\right) = -dmg\hat{\mathbf{k}}\,.$$

The torque about S due to the deltoid muscle is

$$\vec{\boldsymbol{\tau}}_{S,T} = \vec{\mathbf{r}}_{S,T} \times \vec{\mathbf{T}} = s\hat{\mathbf{i}} \times \left(-T\cos\alpha\hat{\mathbf{i}} + T\sin\alpha\hat{\mathbf{j}}\right) = sT\sin\alpha\hat{\mathbf{k}}\,.$$

So the total component of the torque in the $\hat{\mathbf{k}}$ direction must vanish

$$sT\sin\alpha - dmg = 0\,.$$

We can solve the torque equation for the magnitude of the deltoid muscular force

$$T = \frac{dmg}{s\sin\alpha}\,.$$

The horizontal force equation can now be solved for the horizontal component of the force of the scapula on the humerus

$$F_x = T\cos\alpha = \frac{dmg\cos\alpha}{s\sin\alpha}\,.$$

The vertical force equation can now be solved for the vertical component of the force of the scapula on the humerus,

$$F_y = mg - T\sin\alpha = mg - \frac{dmg}{s} = mg\left(1 - d/s\right) < 0\,.$$

Notice that the vertical component points downward since $d/s > 1$.

Problem 2: water bucket

A water bucket attached to a rope and spun in a vertical circle of radius r. Suppose the bucket has mass m and that the bucket can be approximated as a point mass at the end of the rope. The rope breaks when the bucket is moving vertically upwards. The bucket rises to a height h above the release point.

a) What was the velocity of the bucket when it was released?
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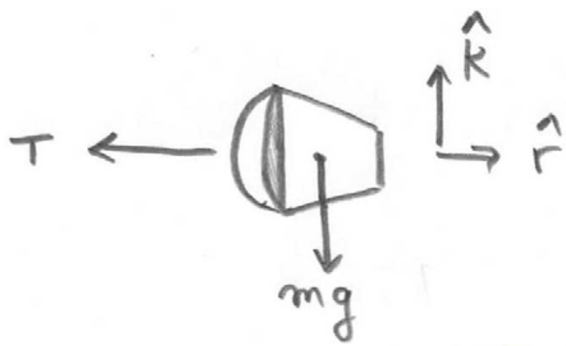
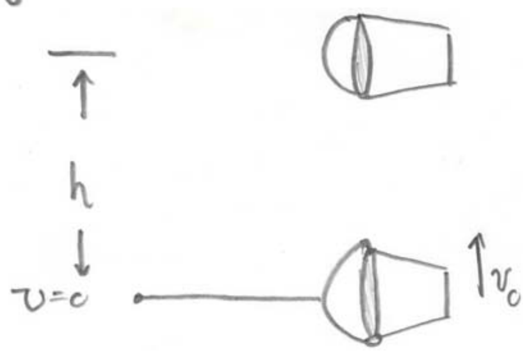
b) What was the tension in the cord when it broke?

Solution:

The force diagram on the bucket is shown in the figure.



Choose the initial state, at time t_0, when the string has just broken. Choose the final state, at time t_f, when the bucket has reached it's maximum height. Choose zero for potential energy at the point where the string breaks.



Initial Energy: The initial kinetic energy is $K_0 = \frac{1}{2}mv_0^{\ 2}$. The initial potential energy is $U_0 = 0$. So the initial mechanical energy is

$$E_0 = K_0 + U_0 = \frac{1}{2}mv_0^{\ 2}.$$

Final Energy: The final kinetic energy is $K_f = 0$. The final potential energy is $U_f = mgh$. So the final mechanical energy is

$$E_f = K_f + U_f = mgh.$$

Non-conservative work: There is no non-conservative work.

Change in Mechanical Energy:

The change in mechanical energy

$$0 = W_{nc} = \Delta E_{mech},$$

is then

$$0 = mgh - \frac{1}{2} m v_0^{\,2}$$

This equation can be solved for the escape velocity, v_0,

$$v_0 = \sqrt{2gh} \; .$$

Just before the string broke the radial force equation is

Therefore Newton's Second Law in the radial direction, $\hat{\mathbf{r}}$, is

$$-T = -m v_0^{\,2} / r \,.$$

So the tension in the string is

$$T = 2mgh / r \,.$$

Problem 3: spring and loop

A mass m is pushed against a spring with spring constant k and held in place with a catch. The spring compresses an unknown distance x. When the catch is removed, the mass leaves the spring and slides along a frictionless circular loop of radius r. When the mass reaches the top of the loop, the force of the loop on the mass (the normal force) is equal to twice the weight of the mass.

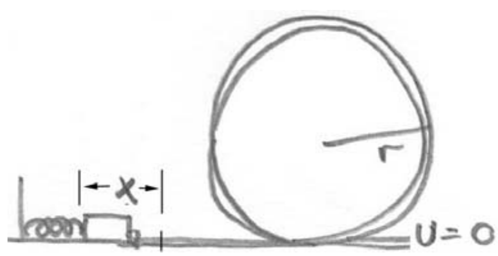
a) Using conservation of energy, find the kinetic energy at the top of the loop. Express your answer as a function of k, m, x, g, and R.

b) Using Newton's second law, derive the equation of motion for the mass when it is at the top of the loop.

c) How far was the spring compressed?

Solution:

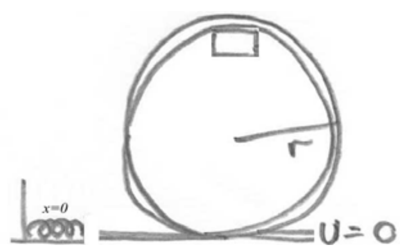
Choose polar coordinates with origin at the center of the loop. Let x denote the displacement of the spring from the equilibrium position. Choose the zero point for the gravitational potential energy $U_{grav} = 0$ at the bottom of the loop. Choose the zero point for the spring potential energy when the spring is at the equilibrium position. $U_{spring}(x=0)=0$.



Initial Energy: Choose for the initial state, the instant before the catch is released. The initial kinetic energy $K_0 = 0$. The initial potential energy is non-zero, $U_0 = (1/2)kx^2$. So the initial mechanical energy is

$$E_0 = K_0 + U_0 = (1/2)kx^2.$$

Final Energy: Choose for the final state, the instant the mass is at the top of the loop. The final kinetic energy $K_f = \frac{1}{2}mv_f^{\,2}$ since the mass is in motion. The final potential energy is non-zero, $U_f = mg2R$.



So the final mechanical energy is

$$E_f = K_f + U_f = 2mgR + \frac{1}{2}mv_f^{\,2}.$$

Non-conservative Work: Since we are assuming the track is frictionless, there is no non- conservative work.

Change in Mechanical Energy: The change in mechanical energy is therefore zero,

$$0 = W_{nc} = \Delta E_{mechanical} = E_f - E_0 .$$

Thus mechanical energy is conserved, $E_f = E_0$, or

$$2mgR + \frac{1}{2}mv_f^{\,2} = \frac{1}{2}kx^2.$$

So the kinetic energy at the top of the loop is

$$\frac{1}{2}mv_f^{\,2} = \frac{1}{2}kx^2 - 2mgR.$$

b) The force diagram is shown in the figure. Therefore Newton's Second Law in the radial direction, $\hat{\mathbf{r}}$, is

$$-mg - N = -mv_f^{\,2}/R.$$

When the mass reaches the top of the loop, the force of the loop on the object (the normal force) is equal to twice the weight of the mass, $N = 2mg$.

Therefore the Second Law becomes

$$3mg = mv_f^{\,2}/R.$$

We can rewrite this condition in terms of the kinetic energy as

$$\frac{3}{2}mgR = \frac{1}{2}mv_f^{\,2}$$

c) Substituting the force condition into the energy condition that we found in part a) yields,

$$\frac{7}{2}mgR = \frac{1}{2}kx^2 .$$

Thus the displacement of the spring from equilibrium is

$$x = \sqrt{\frac{7mgR}{k}} .$$

Problem 4: inclined plane, friction, spring

An object of mass m slides down a plane which is inclined at an angle θ. The object starts out at rest a distance d from an unstretched spring that lies at the bottom of the plane. The spring has a constant k.

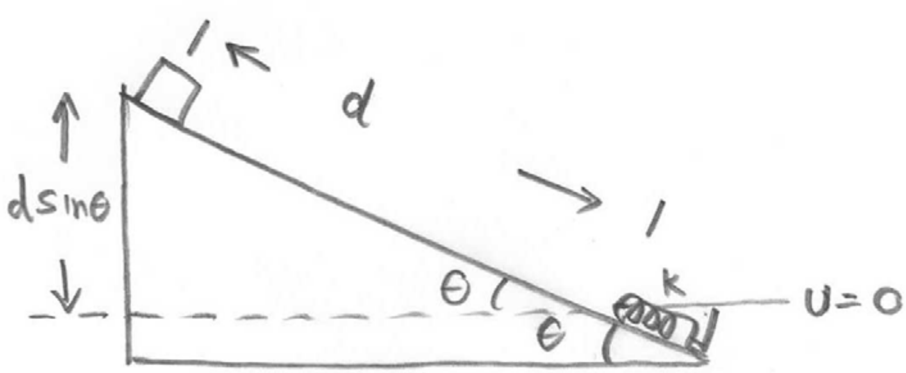
a) Assume the incline plane is frictionless. How far will the spring compress when the mass first comes to rest?

b) Now assume that the incline plane has a coefficient of kinetic friction μ_k. How far will the spring compress when the mass first comes to rest?

c) In case b), how much energy has been lost to heat?

Solution:

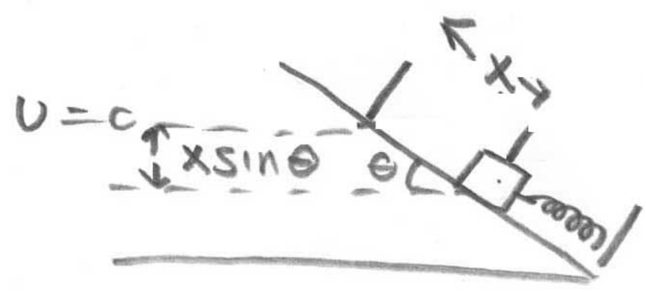
Let x denote the displacement of the spring from the equilibrium position. Choose the zero point for the gravitational potential energy $U_{grav} = 0$ at the bottom of the inclined plane but at the location of the end of the unstretched spring. Choose the zero point for the spring potential energy when the spring is at the equilibrium position, $U_{spring}(x=0) = 0$.



Initial Energy: Choose for the initial state, the instant the object is released at the top of the inclined plane. The initial kinetic energy $K_0 = 0$. The initial potential energy is non-zero, $U_0 = mgd\sin\theta$. So the initial mechanical energy is

$$E_0 = K_0 + U_0 = mgd\sin\theta .$$

Final Energy: Choose for the final state, when the object first comes to rest and the spring is compressed a distance x at the bottom of the inclined plane. The final kinetic energy $K_f = 0$ since the mass is in motion. The final potential energy is non-zero, $U_f = \frac{1}{2}kx^2 - xmg\sin\theta$. Notice that the gravitational potential energy is negative because the object has dropped below the height of the zero point of gravitational potential energy.



So the final mechanical energy is

$$E_f = K_f + U_f = \frac{1}{2}kx^2 - xmg\sin\theta.$$

Non-conservative Work: Since we are assuming the track is frictionless, there is no non- conservative work.

Change in Mechanical Energy: The change in mechanical energy is therefore zero,

$$0 = W_{nc} = \Delta E_{mechanical} = E_f - E_0.$$

Thus mechanical energy is conserved, $E_f = E_0$, or

$$dmg\sin\theta = \frac{1}{2}kx^2 - xmg\sin\theta.$$

This is a quadratic equation

$$x^2 - \frac{2mg\sin\theta}{k}x - \frac{2dmg\sin\theta}{k} = 0$$

with positive choice of square root for the solution to insure a positive displacement of the spring from equilibrium,

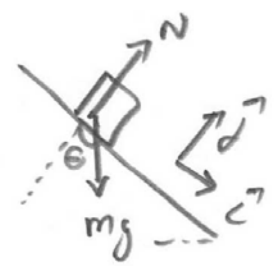
$$x = \frac{mg\sin\theta}{k} + \left(\frac{m^2g^2\sin^2\theta}{k^2} + \frac{2dmg\sin\theta}{k}\right)^{1/2}.$$

b) The effect of kinetic friction is that there is now a non-zero non-conservative work done on the object that has moved a distance, $d+x$, given by

$$W_{nc} = -f_k\left(d+x\right) = -\mu_k N\left(d+x\right) = -\mu_k mg\cos\theta\left(d+x\right).$$

Note the normal force is found by using Newton's Second Law in the direction perpendicular to the inclined plane,

$$N - mg\cos\theta = 0.$$



Change in Mechanical Energy: The change in mechanical energy is therefore ,

$$W_{nc} = \Delta E_{mechanical} = E_f - E_0,$$

which becomes

$$-\mu_k mg\cos\theta\left(d+x\right) = \left(\frac{1}{2}kx^2 - xmg\sin\theta\right) - dmg\sin\theta.$$

This simplifies to

$$0 = \left(\frac{1}{2}kx^2 - xmg\left(\sin\theta - \mu_k\cos\theta\right)\right) - dmg\left(\sin\theta - \mu_k\cos\theta\right).$$

This is the identical equation as in part a) above with

$$\sin\theta \to \sin\theta - \mu_k\cos\theta.$$

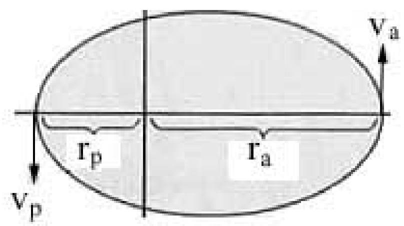
So the new displacement of the spring is

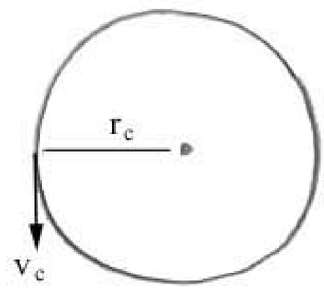
$$x = \frac{mg\left(\sin\theta - \mu_k\cos\theta\right)}{k} + \left(\frac{m^2g^2\left(\sin\theta - \mu_k\cos\theta\right)^2}{k^2} + \frac{2dmg\left(\sin\theta - \mu_k\cos\theta\right)}{k}\right)^{1/2}.$$

c) The heat lost is given by $W_{nc} = -\mu_k mg\cos\theta\left(d+x\right)$, where x is given in part b).

Problem 5: elliptic orbit

A satellite of mass m is in an elliptical orbit around a planet of mass m_p which is located at one focus of the ellipse. The satellite has a velocity v_a at the distance r_a when it is furthest from the planet. The distance of closest approach is r_p.





a) What is the magnitude of the velocity v_p of the satellite when it is closest to the planet?

b) If the satellite were in a circular orbit of radius $r_c = r_p$, is it's velocity v_c greater than, equal to, or less than the velocity v_p of the original elliptic orbit? Justify your answer.

Solution:

Choose zero for the gravitational potential energy to be when the satellite and planet are separated by an infinite distance, $U(r = \infty) = 0$. When the satellite is at closest approach the energy is

$$E_p = K_p + U_p = \frac{1}{2} m_1 v_p^{\ 2} - \frac{G m_1 m_2}{r_p}.$$

When the satellite is at the furthest distance from the planet, the energy is

$$E_a = K_a + U_a = \frac{1}{2} m_1 v_a^{\ 2} - \frac{G m_1 m_2}{r_a}.$$

Mechanical energy is conserved so

$$\frac{1}{2} m_1 v_p^{\ 2} - \frac{G m_1 m_2}{r_p} = \frac{1}{2} m_1 v_a^{\ 2} - \frac{G m_1 m_2}{r_a}.$$

Therefore the velocity of the satellite at closest approach is

$$v_p = \sqrt{v_a^2 - 2Gm_2\left(\frac{1}{r_a} - \frac{1}{r_p}\right)}.$$

b) The velocity of a satellite undergoing uniform circular motion can be found from the force equation,

$$-\frac{Gm_1m_2}{r_c^2} = -\frac{m_1v_c^2}{r_c}.$$

So the velocity is

$$v_c = \sqrt{\frac{Gm_2}{r_c}}.$$

Suppose we give the satellite that is in a circular motion a small increase in velocity in the tangential direction, $v_{new} = v_c + \Delta v$. This implies that the energy increases. The satellite will no longer travel in a circular orbit since for the same radius r_c ,

$$\frac{m_1v_{new}^2}{r_c} > \frac{Gm_1m_2}{r_c^2}.$$

The satellite will move away from the planet entering into an elliptical orbit. So any velocity v_p greater than v_c will form an elliptic orbit.