MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01T **Fall Term 2004**

Class Problem 1: Solution

Problem 1 A car is driving at a constant but unknown velocity, v_0, on a straightaway. A motorcycle is a distance d behind the car. Initially, they are both traveling at the same velocity. The motorcycle starts to pass the car by speeding up at a constant acceleration a_m. When the motorcyclist is side by side with the car, the motorcycle stops accelerating and is traveling at twice the velocity of the car. How long does the motorcycle accelerate? What was the initial velocity of the car and motorcycle? How far did the motorcycle travel while accelerating? Express all your answers in terms of the given quantities in the problem.

I. Understand – get a conceptual grasp of the problem

We assume you've recognized the problem is in the domain of kinematics – the quantitative description of motion. What is the problem asking? What are the given conditions and assumptions? What is to be found and how is this determined or constrained by the given conditions?

In particular: how many objects are there, is the motion in 1, 2, or 3 dimensions, is the motion relative or is there some logical absolute reference frame? Qualitatively describe the motion (in each coordinate and of each body if there are several).

Model: Look for the three most common **model motions –** either the velocity or the acceleration is constant or the motion is uniform circular. Often these apply only to part of the time (or to only one body) – i.e. might the acceleration be constant but different before and after the rocket engine stops or between the two racing cars? Is the motion an example of uniform circular motion.

Advice: Write *your own* representation of the problem's stated data: draw a motion diagram (strobe picture), a graph of position or velocity or acceleration vs. time, or a diagram. Make a table of these quantities vs. time if it's numerical. What are the initial conditions and how do you represent conditions mathematically (e.g. until car A passes car B). A great many problems will involve special motion, perhaps in one or another coordinate: constant velocity, constant acceleration, uniform circular motion, relative motion – learn to recognize these motions. Get the problem into your brain!

Question: Describe the strategy you have chosen for solving this problem. You may want to consider the following issues. What does a sketch of the problem look like? What type of coordinate system will you choose? What information can you deduce from a plot of distance vs. time for both the car and the motorcycle? What conditions must be satisfied when the person just catches up to the streetcar?

Answer: Note: Your answer should include a sketch and coordinate system for the system where you clearly indicated your choice of origin, positive directions and reference frame; a single graph showing qualitatively the position of the motorcycle and the driver as a function of time.

Modeling the problem:

This is a two body problem with each body undergoing one dimensional motion.

First I sketch the problem and introduce a Cartesian coordinate. My first question is where should I choose my origin. I will choose $t=0$ to depict the moment the motorcycle is at the origin and the car is at a positive distance d. At an arbitrary time, I introduce the position functions $x_1(t)$ for the motorcycle and $x_2(t)$ for the car. I denote the time the car overtakes the motorcycle as t_f.

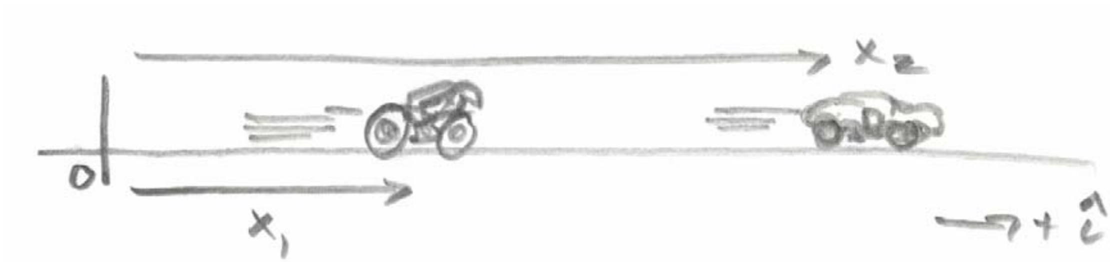

Figure 1: Coordinate System

I am specifically interested in the time period that the motorcycle is accelerating at a constant rate while the car is moving with a uniform velocity. This means that I will need two sets of kinematic equations for position and velocity for the motorcycle and car noting that the acceleration of the car is zero, $a_c = 0$. I can relate these sets of equations by two extra conditions that are stated in the problem. At the moment the motorcycle overtakes the car, the positions are equal and the motorcycle velocity is twice the car. (figure 2).

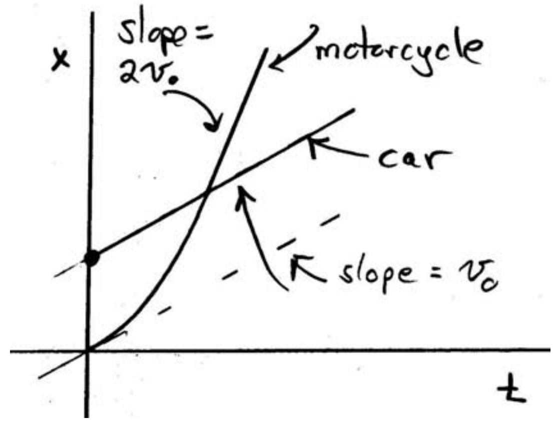

Figure 2: plot of position vs. time for car and motorcycle

The key is to translate all this information into specific equations.

II. Devise a Plan - set up a procedure to obtain the desired solution

General - Have you seen a problem like this – i.e. does the problem fit in a schema you already know? Is a part of the problem a known schema; could you simplify this problem so that it is? Can you find *any* useful results from the given initial conditions and other data even if it is not the solution? Can you imagine a route to the solution if only you know some apparently not given information? Count the unknowns and check that you have that many independent equations.

In particular: choose the best type of coordinate system to simplify the problem, pick the orientation and location of the origin of the coordinate system in accord with the initial conditions. Warning: almost always it is best to pick positive to the right or up and represent downwards acceleration, for example, as –g. Given that the problem involves some particular type of motion (constant acceleration, circular motion) think over *all* the equations that involve this concept.

Question: Devise a plan for solving for: how long does the motorcycle accelerate?; what was the initial velocity of the car and motorcycle?; how far did the motorcycle travel while accelerating?

Answer:

I can write down the kinematic equations for the motorcycle, with the initial position corresponding to my choice of origin, $x_{1,0} = 0$; the initial velocity, the unknown $v_{1,0} = v_0$; and the acceleration of the motorcycle $a_1 = a_m$; then

$$x_1(t) = v_0 t + \frac{1}{2} a_m t^2$$

$$v_{x,1}(t) = v_0 + a_m t \, .$$

The initial position of the car is $x_{2,0} = d$; the initial velocity of the car is the same as the motorcycle $v_{2,0} = v_0$; and the acceleration of the car is $a_2 = 0$; so the kinematic equations for position and velocity of the car are

$$x_2(t) = d + v_0 t \, .$$

Note that the velocity of the car is constant,

$$v_{x,2}(t) = v_0 \, .$$

At the overlap time, t_f, I can now state the two extra conditions mathematically. The first is that the position of the car and the motorcycle are equal,

$$x_1(t_f) = x_2(t_f) \, .$$

This becomes, using our kinematic equations for position,

$$v_0 t_f + \frac{1}{2} a_m t_f^{\ 2} = d + v_0 t_f \, .$$

I notice an immediately simplification, so I rewrite this condition as

$$\frac{1}{2} a_m t_f^{\ 2} = d$$

My second condition at the overtake time is that the velocity of the motorcycle is twice the velocity of the car,

$$v_{x,1}(t_f) = 2 v_0 \, .$$

This equation becomes using the equation for the velocity of the motorcycle

$$v_0 + a_m t_f = 2 v_0 \, .$$

This equation simplifies to

$$a_m t_f = v_0 \, .$$

Summary: I now collect my two equations,

$$\frac{1}{2}a_m t_f^{\,2} = d$$

$$a_m t_f = v_0\ .$$

I now see that I have two equations and two unknowns, the time of overlap, t_f, and the unknown initial velocity, v_0. Although the distance and acceleration are not given specific values they are specified as given constants hence I will treat the representing symbols as known quantities. Two independent equations with two unknowns are solvable. So I just need to decide which quantity I will solve for first. Since the first question asks how long does the motorcycle accelerate; I will solve for the time of overlap, t_f. Then I will solve for the unknown velocity, and finally for the distance the motorcycle travels using

$$x_1(t_f) = v_0 t_f + \frac{1}{2}a_m t_f^{\,2} = v_0 t_f + d = x_2(t_f)\ .$$

III. Carry our your plan – solve the problem!

> This generally involves mathematical manipulations. Try to keep them as simple as possible by not substituting in lengthy algebraic expressions until the end is in sight, make your work as neat as you can to ease checking and reduce careless mistakes. Keep a clear idea of where you are going and have been (label the equations and what you have now found), if possible, check each step as you proceed.

Solution: From the first equation, time of overlap, t_f, is

$$t_f = \sqrt{2d / a_m}\ .$$

I will use this in the second equation to find the unknown velocity

$$v_0 = a_m t_f = a_m\sqrt{2d / a_m} = \sqrt{2da_m}\ .$$

The distance traveled by the motorcycle is then

$$x_1(t_f) = v_0 t_f + d = (\sqrt{2da_m})(\sqrt{2d / a_m}) + d = 2d + d = 3d$$

IV. Look Back – check your solution and method of solution

Can you see that the answer is correct now that you have it – often simply by retrospective inspection? Can you solve it a different way? Is the problem equivalent to one you've solved before if the variables have some specific values?

In particular: Check dimensions if analytic, units if numerical. Check special cases (i.e. if a = 0 does the solution simplify?), check that a general expressions reproduce the given initial conditions. Does it depend sensibly on the various quantities (e.g. is the time greater if the initial velocity is less?)? Is the scaling what you'd expect (time decreases with the square root of the acceleration, distance at some later time proportional to initial velocity)? Is the answer physically reasonable (especially if numbers are given or reasonable ones substituted).

Review the schema of the problem – what is the model, the physical approximations, the concepts needed, and any tricky math manipulation.

Question: Choose what you think are reasonable values for the distance d, and the constant acceleration a_m. What values do you then calculate for how long the motorcycle accelerates?; what is the initial velocity of the car and motorcycle; and how far does the motorcycle traveled while accelerating? Do your values make sense to you?

Answer:

Units: My equation for the time it took to overtake the car is $t_f = \sqrt{2d / a_m}$. The units (dimensions) of time are seconds, $[\dim t_f] = \sqrt{\mathrm{m} / \mathrm{m} \cdot \mathrm{s}^{-2}} = \mathrm{s}$, and velocity, $v_0 = \sqrt{2da_m}$, has unit (dimensions) $[\dim v_0] = \sqrt{(\mathrm{m})(\mathrm{m} \cdot \mathrm{s}^{-2})} = \mathrm{m} \cdot \mathrm{s}^{-1}$. Both quantities have the correct units.

Values: Suppose the car is moving at $v_0 = 20\ \mathrm{m} \cdot \mathrm{s}^{-1}$, and is a distance $d = 10\ \mathrm{m}$ ahead of the motorcycle. Then the acceleration of the motorcycle can be found from the equation $v_0 = \sqrt{2da_m}$ yielding

$$a_m = \frac{(v_0)^2}{2d} = \frac{(20\ \mathrm{m} \cdot \mathrm{s}^{-1})^2}{(2)(10\ \mathrm{m})} = 20\ \mathrm{m} \cdot \mathrm{s}^{-2}$$

or twice the gravitational acceleration. Note the final velocity of the motorcycle is $v_1(\mathrm{t_f}) = 40\ \mathrm{m} \cdot \mathrm{s}^{-1}$. It's really moving along!

The time it takes to overtake the car is then

$$t_f = \sqrt{2d / a_m} = \sqrt{(2)(10\text{ m})/(20\text{ m}\cdot\text{s}^{-2})} = 1\text{ s}.$$

This is not an unreasonable number for the time. Notice that a double negative is appropriate here because when we do these types of calculations, the best we can say is that our result does not contradict our experience.

Different Approaches to solving the Problem: I found that my two conditions for when the motorcycle overtakes the car simplifies to

$$\frac{1}{2} a_m t_f{}^2 = d$$

$$a_m t_f = v_0 .$$

These equations look suggestive, so can I find a nice physical reason for this equation? Suppose I move with the car. Then the car has zero velocity and the motorcycle has zero initial velocity, $v_0 = 0$, but the acceleration of the motorcycle relative to the car is still the same. So in this reference frame, the position of the motorcycle is

$$x_1'(t) = \frac{1}{2} a_m t^2$$

and the velocity is

$$v_{x,1}'(t) = a_m t .$$

The velocity of the motorcycle at t_f **is**

$$v_{x,1}'(t_f) = v_0$$

Thus the velocity equation is just

$$v_0 = a_m t .$$

So the reason the two conditions have such a simple form is that they are the result of solving the problem in a reference frame moving with the car.