Module 20: Two Dimensional Rotational Kinematics

20.1 Introduction

The physical objects that we encounter in the world consist of collections of atoms that are bound together to form systems of particles. When forces are applied, the shape of the body may be stretched or compressed like a spring, or sheared like jello. In some systems the constituent particles are very loosely bound to each other as in fluids and gasses, and the distances between the constituent particles will vary. We shall begin our study of extended objects by restricting ourselves to an ideal category of objects, rigid bodies, which do not stretch, compress, or shear.

A body is called a *rigid body* if the distance between any two points in the body does not change in time. Rigid bodies, unlike point masses, can have forces applied at different points in the body. For most objects, treating as a rigid body is an idealization, but a very good one. In addition to forces applied at points, forces may be distributed over the entire body. Distributed forces are difficult to analyze; however, for example, we regularly experience the effect of the gravitational force on bodies. Based on our experience observing the effect of the gravitational force on rigid bodies, we note that the gravitational force can be concentrated at a point in the rigid body called the *center of gravity*, which for small bodies (so that $\vec{\mathbf{g}}$ may be taken as constant within the body) is identical to the *center of mass* of the body (we shall prove this fact in Appendix 20.A).

Let's consider a rigid rod thrown in the air (Figure 20.1) so that the rod is spinning as its center of mass moves with velocity $\vec{\mathbf{v}}_{\text{cm}}$. Rigid bodies, unlike point-like objects, can have forces applied at different points in the body. We have explored the physics of translational motion; now, we wish to investigate the properties of rotation exhibited in the rod's motion, beginning with the notion that every particle is rotating about the center of mass with the same angular (rotational) velocity.

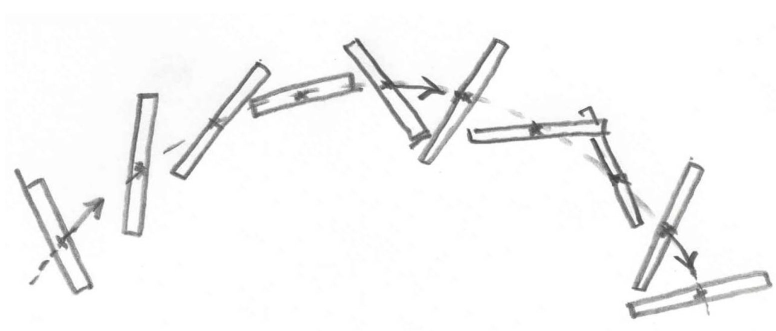


Figure 20.1 The center of mass of a thrown rigid rod follows a parabolic trajectory while the rod rotates about the center of mass.

We can use Newton's Second Law to predict how the center of mass will move. Since the only external force on the rod is the gravitational force (neglecting the action of air resistance), the center of mass of the body will move in a parabolic trajectory.

How was the rod induced to rotate? In order to spin the rod, we applied a torque with our fingers and wrist to one end of the rod as the rod was released. The applied torque is proportional to the angular acceleration. The constant of proportionality is called the *moment of inertia*. When external forces and torques are present, the motion of a rigid body can be extremely complicated while it is translating and rotating in space. We shall begin our study of rotating objects by considering the simplest example of rigid body motion, rotation about a fixed axis.

20.2 Fixed Axis Rotation: Rotational Kinematics

Fixed Axis Rotation

When we studied static equilibrium, we demonstrated the need for two conditions: The total force acting on an object is zero, as is the total torque acting on the object. If the total torque is non-zero, then the object will start to rotate.

A simple example of rotation about a fixed axis is the motion of a compact disc in a CD player, which is driven by a motor inside the player. In a simplified model of this motion, the motor produces angular acceleration, causing the disc to spin. As the disc is set in motion, resistive forces oppose the motion until the disc no longer has any angular acceleration, and the disc now spins at a constant angular velocity. Throughout this process, the CD rotates about an axis passing through the center of the disc, and is perpendicular to the plane of the disc (see Figure 20.2). This type of motion is called *fixed-axis rotation*.

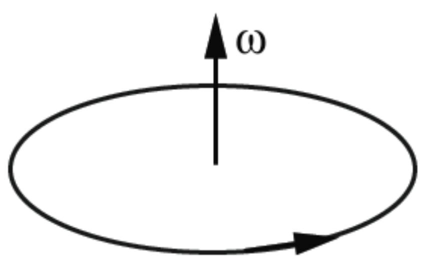


Figure 20.2 Rotation of a compact disc about a fixed axis.

When we ride a bicycle forward, the wheels rotate about an axis passing through the center of each wheel and perpendicular to the plane of the wheel (Figure 20.3). As long as the bicycle does not turn, this axis keeps pointing in the same direction. This motion is more complicated than our spinning CD because the wheel is both moving (translating) with some center of mass velocity, $\vec{\mathbf{v}}_{cm}$, and rotating.

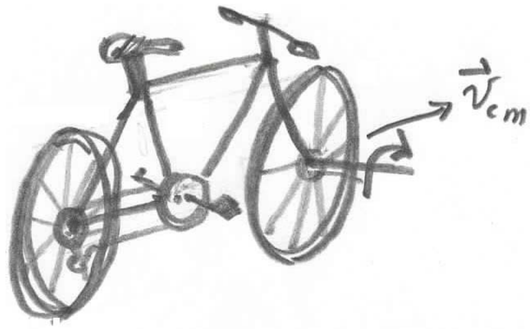


Figure 20.3 Fixed axis rotation and center of mass translation for a bicycle wheel.

When we turn the bicycle's handlebars, we change the bike's trajectory and the axis of rotation of each wheel changes direction. Other examples of non-fixed axis rotation are the motion of a spinning top, or a gyroscope, or even the change in the direction of the earth's rotation axis. This type of motion is much harder to analyze, so we will restrict ourselves in this chapter to considering fixed axis rotation, with or without translation.

Angular Velocity and Angular Acceleration

When we considered the rotational motion of a point-like object, we introduced an angle coordinate θ, and then defined the angular velocity as

$$\omega \equiv \frac{d\theta}{dt}, \tag{20.2.1}$$

and angular acceleration as

$$\alpha \equiv \frac{d^2\theta}{dt^2}. \tag{20.2.2}$$

For a rigid body undergoing fixed-axis rotation, we can divide the body up into small volume elements with mass Δm_i. Each of these volume elements is moving in a circle of radius $r_{\perp,i}$ about the axis of rotation (Figure 20.4).

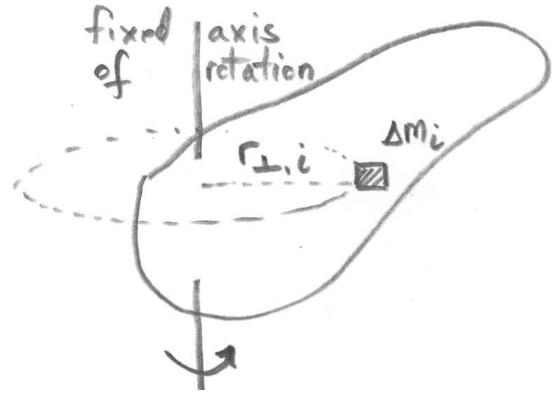


Figure 20.4 Coordinate system for fixed-axis rotation.

We will adopt the notation implied in Figure 20.4, and denote the vector from the axis to the point where the mass element is located as $\vec{\mathbf{r}}_{\perp,i}$, with $r_{\perp,i} = \left|\vec{\mathbf{r}}_{\perp,i}\right|$.

Because the body is rigid, all the volume elements will have the same angular velocity ω and hence the same angular acceleration α. If the bodies did not have the same angular velocity, the volume elements would "catch up to" or "pass" each other, precluded by the rigid-body assumption.

The angular velocity is in fact a vector quantity. Define the angular velocity vector to be directed along the z-axis with z-component equal to the time derivative of an angle θ,

$$\vec{\boldsymbol{\omega}} = \frac{d\theta}{dt}\,\hat{\mathbf{k}} = \omega\,\hat{\mathbf{k}}\,. \tag{20.2.3}$$

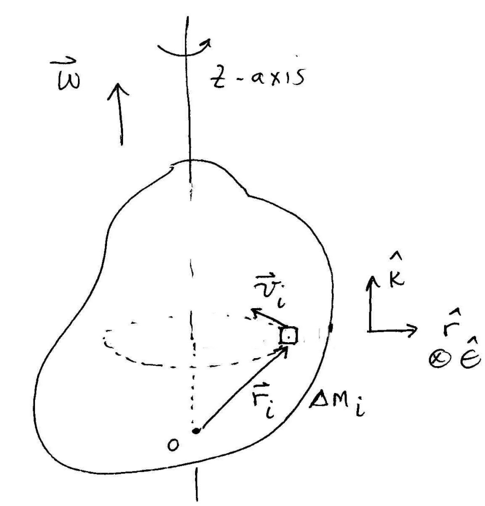


Figure 20.4a: Angular velocity vector for a volume element for fixed axis rotation

For a rigid body rotating with angular velocity $\vec{\boldsymbol{\omega}}$, the velocity $\vec{\mathbf{v}}_i$ of any point in the rigid body located at position $\vec{\mathbf{r}}_i$ is given by (Figure 20.4a)

$$\vec{\mathbf{v}}_i = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}_i \qquad (20.2.4)$$

In a similar fashion, all points in the rigid body have the same angular acceleration, $\alpha = d^2\theta / dt^2$. Define the angular acceleration vector to also be directed along the z axis,

$$\vec{\boldsymbol{\alpha}} = \frac{d^2\theta}{dt^2}\,\hat{\mathbf{k}} = \alpha\,\hat{\mathbf{k}}\,. \qquad (20.2.5)$$

Sign Convention: **Angular Velocity and Angular Acceleration**

For rotational problems we shall always choose a right handed cylindrical coordinate system. If the positive z-axis points up, then we choose θ to be increasing in the counterclockwise direction as shown in Figure 20.5a.

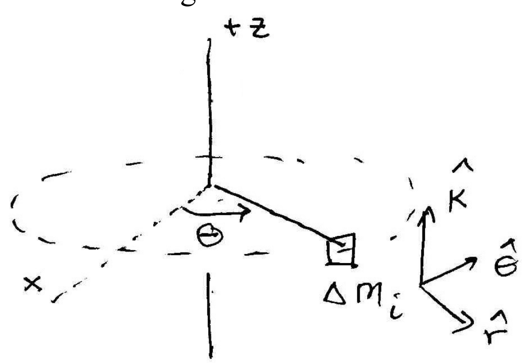


Figure 20.5a Coordinate system for fixed axis rotation.

If the rigid body rotates in the counterclockwise direction, then the z-component of the angular velocity is positive, $\omega \equiv d\theta/dt > 0$. The angular velocity vector then points in the $+\hat{\mathbf{k}}$ -direction as shown in Figure 20.5b.

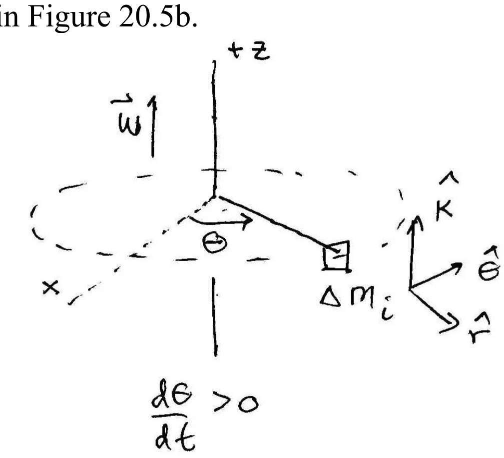


Figure 20.5b Angular velocity vector for $\omega \equiv d\theta/dt > 0$

If the rigid body rotates in the clockwise direction, then the z-component of the angular velocity angular velocity is negative, $\omega \equiv d\theta/dt < 0$. The angular velocity vector then points in the $-\hat{\mathbf{k}}$-direction as shown in Figure 20.5c.

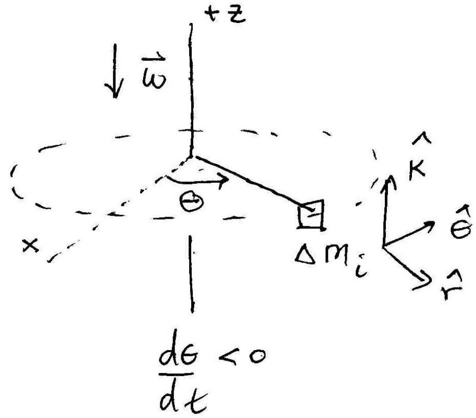


Figure 20.5c Angular velocity vector for $\omega \equiv d\theta/dt < 0$

If the rigid body *increases* its rate of rotation in the counterclockwise (positive) direction then the z-component of the angular acceleration is positive, $\alpha \equiv d^2\theta/dt^2 = d\omega/dt > 0$. The angular acceleration vector then points in the $+\hat{\mathbf{k}}$-direction as shown in Figure 20.5d.

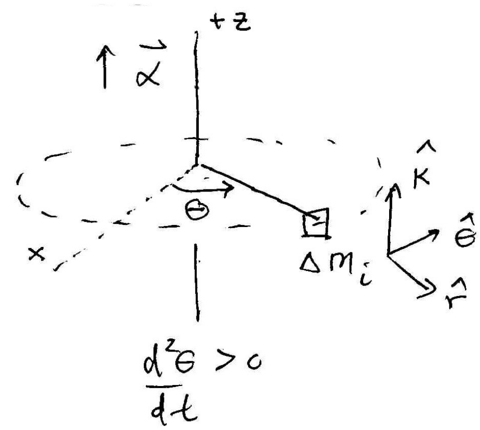


Figure 20.5d Angular acceleration vector for $\alpha \equiv d^2\theta/dt^2 = d\omega/dt > 0$

If the rigid body *decreases* its rate of rotation in the counterclockwise (positive) direction then the z-component of the angular acceleration is negative, $\alpha \equiv d^2\theta/dt^2 = d\omega/dt < 0$. The angular acceleration vector then points in the $-\hat{\mathbf{k}}$-direction as shown in Figure 20.5e.

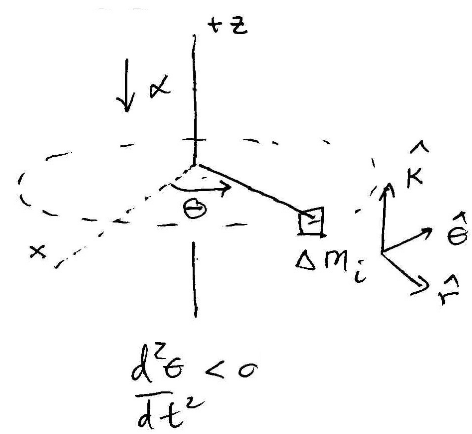


Figure 20.5e Angular acceleration vector for $\alpha \equiv d^2\theta/dt^2 = d\omega/dt < 0$

To phrase this more generally, if α and ω have the same sign, the body is speeding up; if opposite signs, the body is slowing down. This general result is independent of the choice of positive direction of rotation.

Note that in Figure 20.2, the CD has the angular velocity vector points downward (in the $-\hat{\mathbf{k}}$ -direction); CDs do not operate the same way record player turntables do.

Tangential Velocity and Tangential Acceleration

Since the small volume Δm_i element of mass is moving in a circle of radius $r_{\perp,i} = |\vec{\mathbf{r}}_{\perp,i}|$ with angular velocity ω, the element has a tangential velocity component

$$v_{\tan,\,i} = r_{\perp,\,i}\,\omega\,. \tag{20.2.6}$$

If the magnitude of the tangential velocity is changing, the volume element undergoes a tangential acceleration given by

$$a_{\tan,\,i} = r_{\perp,\,i}\,\alpha\,. \tag{20.2.7}$$

Recall from Chapter 6.3 Equation (6.3.14) that the volume element is always accelerating inward with magnitude

$$\left|a_{\mathrm{rad},\,i}\right| = \frac{v_{\tan,\,i}^2}{r_{\perp,\,i}} = r_{\perp,\,i}\,\omega^2\,. \tag{20.2.8}$$

20.2.1 Example: Turntable, Part I

A turntable is a uniform disc of mass $1.2\ \text{kg}$ and a radius $1.3\times10^{1}\ \text{cm}$. The turntable is spinning initially in a counterclockwise direction when seen from above at a constant rate of $f_0 = 33\ \text{cycles}\cdot\text{min}^{-1}$ ($33\ \text{rpm}$). The motor is turned off and the turntable slows to a stop in $8.0\ \text{s}$. Assume that the angular acceleration is constant.

a) What is the initial angular velocity of the turntable?

b) What is the angular acceleration of the turntable?

Answer:

Choose a coordinate system shown in Figure 20.5f.

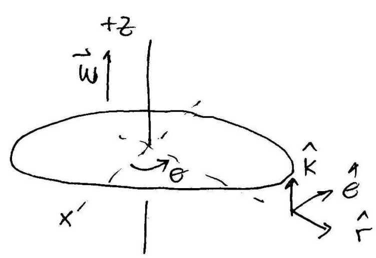


Figure 20.5f Angular velocity of turntable that is slowing down.

Initially, the disc is spinning with a frequency

$$f_0 = \left(33\ \frac{\text{cycles}}{\text{min}}\right)\left(\frac{1\,\text{min}}{60\ \text{s}}\right) = 0.55\ \text{cycles}\cdot\text{s}^{-1} = 0.55\ \text{Hz}, \tag{20.2.9}$$

so the initial angular velocity has magnitude

$$\omega_0 = 2\pi f_0 = \left(2\pi\ \frac{\text{radian}}{\text{cycle}}\right)\left(0.55\ \frac{\text{cycles}}{\text{s}}\right) = 3.5\ \text{rad}\cdot\text{s}^{-1}. \tag{20.2.10}$$

The angular velocity vector points in the $+\hat{\mathbf{k}}$-direction as shown in Figure 20.5b. The final angular velocity is zero, so the magnitude of the angular acceleration is

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_0}{t_f - t_0} = \frac{-3.5\ \text{rad}\cdot\text{s}^{-1}}{8.0\ \text{s}} = -4.3\times10^{-1}\ \text{rad}\cdot\text{s}^{-2}. \tag{20.2.11}$$

The z-component of the angular acceleration is negative, the disc is slowing down and so the angular acceleration vector then points in the $-\hat{\mathbf{k}}$-direction as shown in Figure 20.5g.

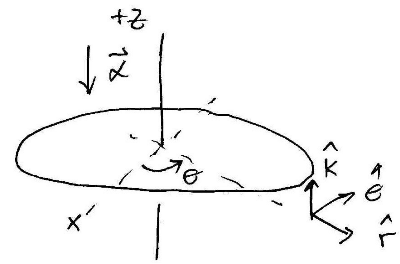


Figure 20.5g Angular acceleration of turntable that is slowing down.

20.3 Rotational Kinetic Energy

The general motion of a rigid body consists of a translation of the center of mass with velocity $\vec{\mathbf{v}}_{\text{cm}}$ and a rotation about the center of mass with angular velocity $\vec{\boldsymbol{\omega}}_{\text{cm}}$.

Having defined translational kinetic energy in Chapter 7.2, Equation 7.2.1 **(add correct reference)**, we now define the rotational kinetic energy for a rigid body about its center of mass.

Choose the z -axis to lie along the axis of rotation. As in Section 20.2, divide the body into volume elements of mass Δm_i. Let the point S denote a specific point along the axis of rotation (Figure 20.6). Each volume element undergoes a tangential acceleration as the volume element moves in a circular orbit of radius $r_{\perp,i} = |\vec{\mathbf{r}}_{\perp,i}|$ about the fixed axis.

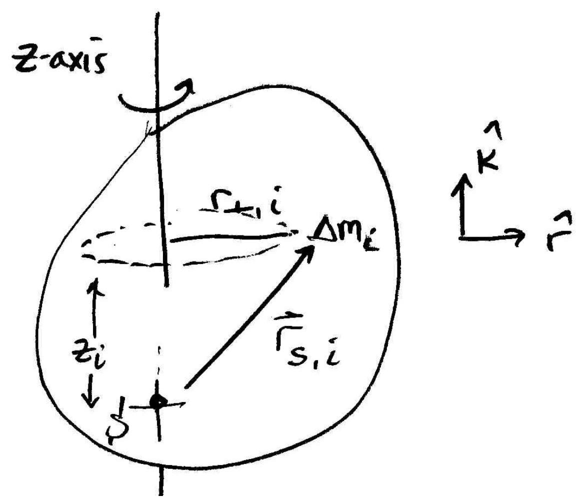


Figure 20.6: Volume element undergoing fixed-axis rotation about the z -axis.

Each individual mass element Δm_i undergoes circular motion about the center of mass with z-component of angular velocity ω_{cm} in a circle of radius $(r_{\text{cm},\,i})_\perp$. Therefore the velocity of each element is given by $\vec{\mathbf{v}}_{\mathrm{cm},i} = (r_{\mathrm{cm},\,i})_{\perp}\,\omega_{\mathrm{cm}}\,\hat{\boldsymbol{\theta}}$. The rotational kinetic energy is then

$$K_{\mathrm{cm},\,i} = \frac{1}{2}\Delta m_i\, v_{\mathrm{cm},\,i}^2 = \frac{1}{2}\Delta m_i (r_{\mathrm{cm},\,i})_{\perp}^2\,\omega_{\mathrm{cm}}^2\,. \tag{20.3.1}$$

We now add up the kinetic energy for all the mass elements,

$$\begin{aligned} K_{\mathrm{cm}} &= \lim_{\substack{i\to\infty \\ \Delta m_i \to 0}} \sum_{i=1}^{i=N} K_{\mathrm{cm},\,i} = \lim_{\substack{i\to\infty \\ \Delta m_i \to 0}} \sum_{i=1}^{i=N} \left(\sum_i \frac{1}{2}\Delta m_i (r_{\mathrm{cm},\,i})_{\perp}^2 \right) \omega_{\mathrm{cm}}^2 \\ &= \left(\frac{1}{2} \int_{\mathrm{body}} dm (r_{\mathrm{cm}})_{\perp}^2 \right) \omega_{\mathrm{cm}}^2 \end{aligned} \tag{20.3.2}$$

Definition: **Moment of Inertia about a Fixed Axis**

The quantity

$$I_{cm} = \int_{body} dm (r_{cm})_{\perp}^2\,. \tag{20.3.3}$$

is called the *moment of inertia* of the rigid body about a fixed axis passing through the center of mass, and is a physical property of the body. The SI units for moment of inertia are $\left[\mathrm{kg}\cdot\mathrm{m}^2\right]$.

Thus

$$K_{\mathrm{cm}} = \left(\frac{1}{2} \int_{\mathrm{body}} dm (r_{\mathrm{cm}})_{\perp}^2 \right) \omega_{\mathrm{cm}}^2 \equiv \frac{1}{2} I_{cm}\omega_{\mathrm{cm}}^2 \tag{20.3.4}$$

20.3.1 Example: Moment of Inertia of a Rod of Uniform Mass Density

Consider a thin uniform rod of length L and mass m. In this problem, we will calculate the moment of inertia about an axis perpendicular to the rod that passes through the center of mass of the rod. A sketch of the rod, volume element, and axis is shown in Figure 20.8.

Choose Cartesian coordinates, with the origin at the center of mass of the rod, which is midway between the endpoints since the rod is uniform. Choose the x-axis to lie along the length of the rod, with the positive x-direction to the right, as in the figure.

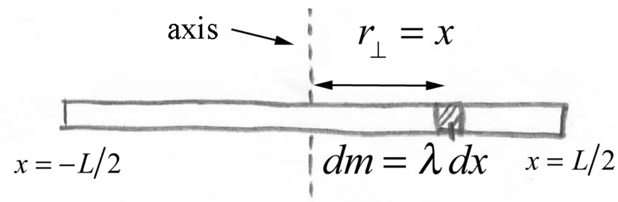


Figure 20.8 Moment of inertia of a uniform rod about center of mass.

Identify an infinitesimal mass element $dm = \lambda\, dx$, located at a displacement x from the center of the rod, where the mass per unit length $\lambda = m / L$ is a constant, as we have assumed the rod to be uniform.

When the rod rotates about an axis perpendicular to the rod that passes through the center of mass of the rod, the element traces out a circle of radius $r_\perp = x$.

We add together the contributions from each infinitesimal element as we go from $x = -L/2$ to $x = L/2$. The integral is then

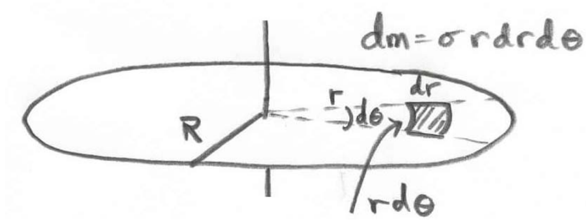
$$\begin{aligned} I_{\text{cm}} &= \int_{\text{body}} (r_{\text{cm}})_\perp^2\, dm = \lambda \int_{-L/2}^{L/2} (x^2)\, dx = \lambda \left. \frac{x^3}{3} \right|_{-L/2}^{L/2} \\ &= \frac{m}{L}\frac{(L/2)^3}{3} - \frac{m}{L}\frac{(-L/2)^3}{3} = \frac{1}{12} m L^2. \end{aligned} \tag{20.3.5}$$

By using a constant mass per unit length along the rod, we need not consider variations in the mass density in any direction other than the x- axis. We also assume that the width is the rod is negligible. (Technically we should treat the rod as a cylinder or a rectangle in the x-y plane if the axis is along the z- axis. The calculation of the moment of inertia in these cases would be more complicated.)

Example 20.3.2: Moment of Inertia: *Uniform Disc*

A thin uniform disc of mass M and radius R is mounted on an axle passing through the center of the disc, perpendicular to the plane of the disc. Calculate the moment of inertia about an axis that passes perpendicular to the disc through the center of mass of the disc

Solution:



As a starting point, consider the contribution to the moment of inertia from the mass element dm show in the figure above. Take the point S to be the center of mass of the disc. Choose cylindrical coordinates with the coordinates (r,θ) in the plane and the z-axis perpendicular to the plane. The area element

$$da = r\,dr\,d\theta \tag{20.3.6}$$

can be thought of as the product of arc length $r\,d\theta$ and the radial width dr. Since the disc is uniform, the mass per unit area is a constant,

$$\sigma = \frac{dm}{da} = \frac{m_{\text{total}}}{\text{Area}} = \frac{M}{\pi R^2}\,. \tag{20.3.7}$$

Therefore the mass in the infinitesimal area element as given in Equation (20.3.6), a distance r from the axis of rotation, is given by

$$dm = \sigma\, r\,dr\,d\theta = \frac{M}{\pi R^2} r\,dr\,d\theta\,. \tag{20.3.8}$$

When the disc rotates, the mass element traces out a circle of radius $(r_{cm})_{\perp}^2 = r^2$; that is, the distance from the center is the perpendicular distance from the axis.

The moment of inertia integral is now an integral in two dimensions; the angle θ varies from $\theta = 0$ to $\theta = 2\pi$, and the radial coordinate r varies from $r = 0$ to $r = R$. Thus the limits of the integral are

$$I_{\text{cm}} = \int_{\text{body}} (r_{cm})_{\perp}^2\,dm = \frac{M}{\pi R^2}\int_{r=0}^{r=R}\int_{\theta=0}^{\theta=2\pi} r^3\,d\theta\,dr\,. \tag{20.3.9}$$

The integral can now be explicitly calculated by first integrating the θ-coordinate

$$I_{\text{cm}} = \frac{M}{\pi R^2}\int_{r=0}^{r=R}\left(\int_{\theta=0}^{\theta=2\pi} d\theta\right) r^3 dr = \frac{M}{\pi R^2}\int_{r=0}^{r=R} 2\pi r^3 dr = \frac{2M}{R^2}\int_{r=0}^{r=R} r^3 dr \tag{20.3.10}$$

and then integrating the r-coordinate,

$$I_{\text{cm}} = \frac{2M}{R^2}\int_{r=0}^{r=R} r^3 dr = \frac{2M}{R^2}\frac{r^4}{4}\bigg|_{r=0}^{r=R} = \frac{2M}{R^2}\frac{R^4}{4} = \frac{1}{2}MR^2 . \qquad (20.3.11)$$

Remark: Instead of taking the area element as a small patch $da = r\,dr\,d\theta$, choose a ring of radius r and width dr. Then the area of this ring is given by

$$da_{\text{ring}} = \pi(r+dr)^2 - \pi r^2 = \pi r^2 + 2\pi r\,dr + \pi(dr)^2 - \pi r^2 = 2\pi r\,dr + \pi(dr)^2 . \qquad (20.3.12)$$

In the limit that $dr \to 0$, the term proportional to $(dr)^2$ can be ignored and the area is $da = 2\pi r dr$. This equivalent to first integrating the $d\theta$ variable

$$da_{\text{ring}} = r\,dr\left(\int_{\theta=0}^{\theta=2\pi} d\theta\right) = 2\pi r\,dr . \qquad (20.3.13)$$

Then the mass element is

$$dm_{\text{ring}} = \sigma da_{\text{ring}} = \frac{M}{\pi R^2} 2\pi r\,dr . \qquad (20.3.14)$$

The moment of inertia integral is just an integral in the variable r,

$$I_{\text{cm}} = \int_{\text{body}} (r_\perp)^2 dm = \frac{2\pi M}{\pi R^2}\int_{r=0}^{r=R} r^3 dr = \frac{1}{2}MR^2 . \qquad (20.3.15)$$

20.4 Parallel Axis Theorem

Consider a rigid body of mass m undergoing fixed-axis rotation. Consider two parallel axes. The first axis passes through the center of mass of the body, and the moment of inertia about this first axis is I_{cm}. The second axis passes through some other point S in the body. Let $d_{S,\text{cm}}$ denote the perpendicular distance between the two parallel axes (Figure 20.9). Then the moment of inertia I_S about an axis passing through a point S is related to I_{cm} by

$$I_S = I_{\text{cm}} + m\,d_{S,\text{cm}}^2 . \qquad (20.4.1)$$

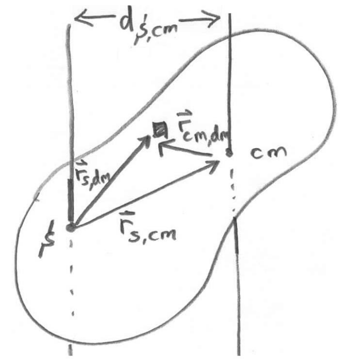


Figure 20.9 Geometry of the parallel axis theorem.

Proof of the Parallel Axis Theorem

Identify an infinitesimal volume element of mass dm. The vector from the point S to the mass element is $\vec{\mathbf{r}}_{S,dm}$, the vector from the center of mass to the mass element is $\vec{\mathbf{r}}_{\mathrm{cm},dm}$, and the vector from the point S to the center of mass is $\vec{\mathbf{r}}_{S,\mathrm{cm}}$. From Figure 20.9, we see that

$$\vec{\mathbf{r}}_{S,dm} = \vec{\mathbf{r}}_{S,\mathrm{cm}} + \vec{\mathbf{r}}_{\mathrm{cm},dm}. \tag{20.4.2}$$

The notation gets complicated at this point. We are interested in distances from the respective axes, so denote the following vectors as motivated in Section 14.2.
As in Figure 20.9 and Equation (20.4.2), $\vec{\mathbf{r}}_{\mathrm{cm},\,dm}$ is the vector from the center of mass to the position of the mass element of mass dm. This vector has a component vector $\vec{\mathbf{r}}_{\mathrm{cm},\|,dm}$ parallel to the axis through the center of mass and a component vector $\vec{\mathbf{r}}_{\mathrm{cm},\perp,dm}$ perpendicular to the axis through the center of mass. The magnitude of the perpendicular component vector is

$$\left|\vec{\mathbf{r}}_{\mathrm{cm},\perp,dm}\right| = r_{\mathrm{cm},\perp,dm}. \tag{20.4.3}$$

As in Figure 20.9 and Equation (20.4.2), $\vec{\mathbf{r}}_{S,\,dm}$ is the vector from the point S to the position of the mass element of mass dm. This vector has a component vector $\vec{\mathbf{r}}_{S,\|,dm}$ parallel to the axis through the point S and a component vector $\vec{\mathbf{r}}_{S,\perp,dm}$ perpendicular to the axis through the point S. The magnitude of the perpendicular component vector is

$$\left|\vec{\mathbf{r}}_{S,\perp,dm}\right| = r_{S,\perp,dm}. \tag{20.4.4}$$

As in Figure 20.9 and Equation (20.4.2), $\vec{\mathbf{r}}_{S,\mathrm{cm}}$ is the vector from the point S to the center of mass. This vector has a component vector $\vec{\mathbf{r}}_{S,\|,\mathrm{cm}}$ parallel to *both* axes and a perpendicular component vector of $\vec{\mathbf{r}}_{S,\text{cm}}$ perpendicular to *both* axes (the axes are parallel, of course) is $\vec{\mathbf{r}}_{S,\perp,\text{cm}}$. The magnitude of the perpendicular component vector is

$$\left|\vec{\mathbf{r}}_{S,\perp,\text{cm}}\right| = d_{S,\text{cm}} . \tag{20.4.5}$$

Equation (20.4.2) is now expressed as two equations,

$$\begin{aligned}\vec{\mathbf{r}}_{S,\perp,dm} &= \vec{\mathbf{r}}_{S,\perp,\text{cm}} + \vec{\mathbf{r}}_{\text{cm},\perp,dm} \\ \vec{\mathbf{r}}_{S,\|,dm} &= \vec{\mathbf{r}}_{S,\|,\text{cm}} + \vec{\mathbf{r}}_{\text{cm},\|,dm}.\end{aligned} \tag{20.4.6}$$

At this point, note that if we had simply decided that the two parallel axes are parallel to the z-direction, we could have saved some steps and perhaps spared some of the notation with the triple subscripts. However, we want a more general result, one valid for cases where the axes are not fixed, or when different objects in the same problem have different axes. For example, consider the turning bicycle, for which the two wheel axes will not be parallel, or a spinning top that *precesses* (wobbles). Such cases will be considered in Chapter 15, and we will show the general case of the parallel axis theorem in anticipation of use for more general situations.

The moment of inertia about the point S is

$$I_S = \int_{\text{body}} dm (r_{S,\perp,dm})^2 . \tag{20.4.7}$$

From (20.4.6) we have

$$\begin{aligned}(r_{S,\perp,dm})^2 &= \vec{\mathbf{r}}_{S,\perp,dm} \cdot \vec{\mathbf{r}}_{S,\perp,dm} \\ &= (\vec{\mathbf{r}}_{S,\perp,\text{cm}} + \vec{\mathbf{r}}_{\text{cm},\perp,dm}) \cdot (\vec{\mathbf{r}}_{S,\perp,\text{cm}} + \vec{\mathbf{r}}_{\text{cm},\perp,dm}) \\ &= d_{S,\text{cm}}^2 + (r_{\text{cm},\perp,dm})^2 + 2\vec{\mathbf{r}}_{S,\perp,\text{cm}} \cdot \vec{\mathbf{r}}_{\text{cm},\perp,dm}.\end{aligned} \tag{20.4.8}$$

Thus we have for the moment of inertia about S,

$$I_S = \int_{\text{body}} dm\, d_{S,\text{cm}}^2 + \int_{\text{body}} dm (r_{\text{cm},\perp,dm})^2 + 2\int_{\text{body}} dm (\vec{\mathbf{r}}_{S,\perp,\text{cm}} \cdot \vec{\mathbf{r}}_{\text{cm},\perp,dm}) . \tag{20.4.9}$$

In the first integral in Equation (20.4.9), $r_{S,\perp,\text{cm}} = d_{S,\text{cm}}$ is the distance between the parallel axes and is a constant and may be taken out of the integral, and

$$\int_{\text{body}} dm\, d_{S,\text{cm}}^2 = m\, d_{S,\text{cm}}^2 . \tag{20.4.10}$$

The second term in Equation (20.4.9) is the moment of inertia about the axis through the center of mass,

$$I_{\text{cm}} = \int_{\text{body}} dm\,(r_{\text{cm},\perp,dm})^2\,. \qquad (20.4.11)$$

The third integral in Equation (20.4.9) is zero. To see this, note that the term $\vec{\mathbf{r}}_{S,\perp,\text{cm}}$ is a constant and may be taken out of the integral,

$$2\int_{\text{body}} dm\,(\vec{\mathbf{r}}_{S,\perp,\text{cm}} \cdot \vec{\mathbf{r}}_{\text{cm},\perp,dm}) = \vec{\mathbf{r}}_{S,\perp,\text{cm}} \cdot 2\int_{\text{body}} dm\,\vec{\mathbf{r}}_{\text{cm},\perp,dm} \qquad (20.4.12)$$

The integral $\int_{\text{body}} dm\ \vec{\mathbf{r}}_{\text{cm},\perp,dm}$ is the perpendicular component of the position of the center of mass with respect to the center of mass, and hence $\vec{\mathbf{0}}$, with the result that

$$2\int_{\text{body}} dm\,(\vec{\mathbf{r}}_{S,\perp,\text{cm}} \cdot \vec{\mathbf{r}}_{\text{cm},\perp,dm}) = 0\,. \qquad (20.4.13)$$

Thus, the moment of inertia about S is just the sum of the first two integrals in Equation (20.4.9),

$$I_S = I_{\text{cm}} + m\,d_{S,\text{cm}}^2\,. \qquad (20.4.14)$$

20.4.1 Example: Uniform Rod

Let point S be the left end of the rod of Example 14.2.1 and Figure 20.8. Then the distance from the center of mass to the end of the rod is $d_{S,\text{cm}} = L/2$. The moment of inertia $I_S = I_{\text{end}}$ about an axis passing through the endpoint is related to the moment of inertia about an axis passing through the center of mass, $I_{\text{cm}} = (1/12)\,m L^2$, according to Equation (20.4.14),

$$I_S = \frac{1}{12} m L^2 + \frac{1}{4} m L^2 = \frac{1}{3} m L^2\,. \qquad (20.4.15)$$

In this case it's easy and useful to check by direct calculation. Use Equation (20.3.5) but with the limits changed to $x' = 0$ and $x' = L$, where $x' = x + L/2$;

$$
\begin{aligned}
I_{\text{end}} &= \int\limits_{\text{body}} r_\perp{}^2\, dm = \lambda \int_0^L x'^2\, dx' \\
&= \lambda \left.\frac{x'^3}{3}\right|_0^L = \frac{m}{L}\frac{(L)^3}{3} - \frac{m}{L}\frac{(0)^3}{3} = \frac{1}{3} m L^2.
\end{aligned}
\tag{20.4.16}
$$

20.5 Conservation of Energy Closed Systems

Consider a closed system ($\Delta E_{system} = 0$) such that the work done by all the non-conservative internal forces is zero. Then the change in the mechanical energy of the system is zero

$$\Delta E_{mechanical} = \Delta U + \Delta K = (U_f + K_f) - (U_0 + K_0) = 0 \tag{20.5.1}$$

For fixed axis rotation with a component of angular velocity ω about the fixed axis, the change in kinetic energy is given by

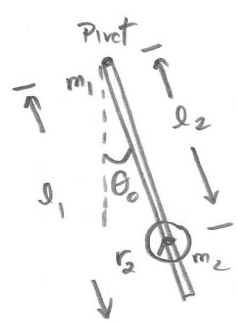
$$\Delta K \equiv K_f - K_0 = \frac{1}{2} I_S \omega_f^2 - \frac{1}{2} I_S \omega_0^2 \tag{20.5.2}$$

where S is a point that lies on the fixed axis. Then conservation of energy implies that

$$U_f + \frac{1}{2} I_S \omega_f^2 = U_0 + \frac{1}{2} I_S \omega_0^2 \tag{20.5.3}$$

Example 20.5.1.

A physical pendulum consists of a uniform rod of mass m_1 pivoted at one end about the point S. The rod has length l_1 and moment of inertia I_1 about the pivot point. A disc of mass m_2 and radius r_2 with moment of inertia I_{cm} about its center of mass is rigidly attached a distance l_2 from the pivot point. The pendulum is initially displaced to an angle θ_0 and then released from rest.



a) What is the moment of inertia of the physical pendulum about the pivot point S?

b) How far from the pivot point is the center of mass of the system?

c) What is the angular speed of the pendulum when the pendulum is at the bottom of its swing?

Solution:

a) The moment of inertia about the pivot point will be the sum of the moment of inertia of the rod, given as I_1, and the moment of inertia of the disc about the pivot point. The moment of inertia of the disc about the pivot point is found from the parallel axis theorem,

$$I_{\text{disc}} = I_{\text{cm}} + m_2 l_2^2 \tag{20.5.4}$$

The total moment of inertia about the pivot point S is then

$$I_S = I_1 + I_{\text{disc}} = I_1 + I_{\text{cm}} + m_2 l_2^2 . \tag{20.5.5}$$

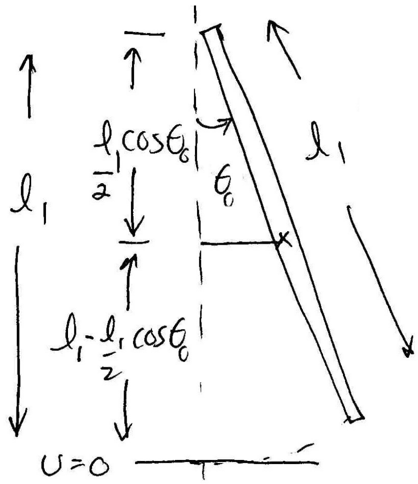
The center of mass of the compound system is located a distance from the pivot point

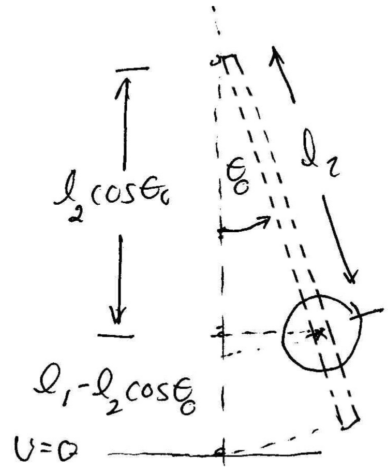
$$l_{\text{cm}} = \frac{m_1(l_1 / 2) + m_2 l_2}{m_1 + m_2} . \tag{20.5.6}$$

b) We can use conservation of mechanical energy, to find the angular speed of the pendulum at the bottom of its swing.

Take the zero point of gravitational potential energy to be the point where the bottom of the rod is at its lowest point, that is, $\theta = 0$. The initial mechanical energy is then

$$E_0 = U_0 = m_1 g(l_1 - \frac{l_1}{2}\cos\theta_0) + m_2 g(l_1 - l_2 \cos\theta_0) , \tag{20.5.7}$$





The final mechanical energy is

$$E_f = U_f + K_f = m_1 g\frac{l_1}{2} + m_2 g(l_1 - l_2) + \frac{1}{2} I_S \omega_f^2, \tag{20.5.8}$$

with I_S as found in Equation (20.5.5). There are no nonconservative forces acting, so the mechanical energy is constant therefore equating the expressions in (20.5.7) and (20.5.8) we get that

$$m_1 g(l_1 - \frac{l_1}{2}\cos\theta_0) + m_2 g(l_1 - l_2\cos\theta_0) = m_1 g\frac{l_1}{2} + m_2 g(l_1 - l_2) + \frac{1}{2} I_S \omega_f^2, \tag{20.5.9}$$

This simplifies to

$$\left(\frac{m_1 l_1}{2} + m_2 l_2\right) g(1 - \cos\theta_0) = \frac{1}{2} I_S \omega_f^2, \tag{20.5.10}$$

We now solve for ω_f (taking the positive square root to insure that we are calculating angular speed)

$$\omega_f = \sqrt{\frac{2\left(\frac{m_1 l_1}{2} + m_2 l_2\right) g(1 - \cos\theta_0)}{I_S}}, \tag{20.5.11}$$

Finally we substitute in Eq.(20.5.5) in to Eq. (20.5.11) and find

$$\omega_f = \sqrt{\frac{2\left(\dfrac{m_1 l_1}{2} + m_2 l_2\right) g(1-\cos\theta_0)}{I_1 + I_{\mathrm{cm}} + m_2 l_2^2}}\,. \qquad (20.5.12)$$

Note that we can rewrite Eq. (20.5.10), using Eq. (20.5.6) for the distance between the center of mass and the pivot point, to get

$$(m_1 + m_2) l_{cm} g(1-\cos\theta_0) = \frac{1}{2} I_S \omega_f^2, \qquad (20.5.13)$$

We can interpret this equation as follows. Treat the compound system as a point particle of mass $m_1 + m_2$ located at the center of mass l_{cm}. Take the zero point of gravitational potential energy to be the point where the center of mass is at its lowest point, that is, $\theta = 0$. Then

$$E_0 = (m_1 + m_2) l_{cm} g(1-\cos\theta_0), \qquad (20.5.14)$$

and

$$E_f = \frac{1}{2} I_S \omega_f^2, \qquad (20.5.15)$$

Thus conservation of energy reproduces Eq. (20.5.13).