Module 21: Two Dimensional Rotational Dynamics

21.1 Torque

In order to understand the dynamics of a rotating rigid body we must introduce a new quantity, the torque. Let a force $\vec{\mathbf{F}}_P$ with magnitude $F = \left|\vec{\mathbf{F}}_P\right|$ act at a point P. Let $\vec{\mathbf{r}}_{S,P}$ be the vector from the point S to a point P, with magnitude $r = \left|\vec{\mathbf{r}}_{S,P}\right|$. The angle between the vectors $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$ is θ with $[0 \le \theta \le \pi]$ (Figure 21.1).

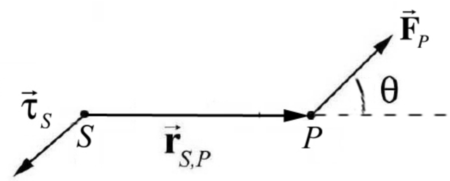


Figure 21.1 Torque about a point S due to a force acting at a point P

The torque about a point S due to force $\vec{\mathbf{F}}_P$ acting at P, is defined by

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,P} \times \vec{\mathbf{F}}_P . \tag{21.1.1}$$

(See Module 3 for a review of the definition of the cross product of two vectors). The magnitude of the torque about a point S due to force $\vec{\mathbf{F}}_P$ acting at P, is given by

$$\tau_S = r\,F \sin\theta . \tag{21.1.2}$$

The SI units for torque are $[\mathrm{N} \cdot \mathrm{m}]$. The direction of the torque is perpendicular to the plane formed by the vectors $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$ (for $[0 < \theta < \pi]$), and by definition points in the direction of the unit normal vector to the plane $\hat{\mathbf{n}}_{RHR}$ as shown in Figure 21.2.

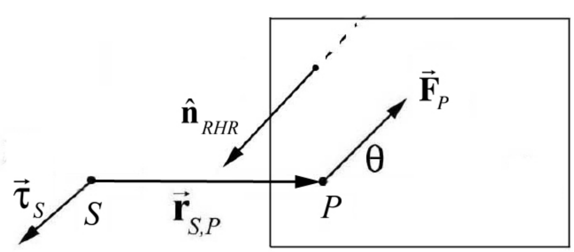


Figure 21.2 Vector direction for the torque

Recall that the magnitude of a cross product is the area of the parallelogram (the height times the base) defined by the two vectors. Figure 21.3 shows the two different ways of defining height and base for a parallelogram defined by the vectors $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$.

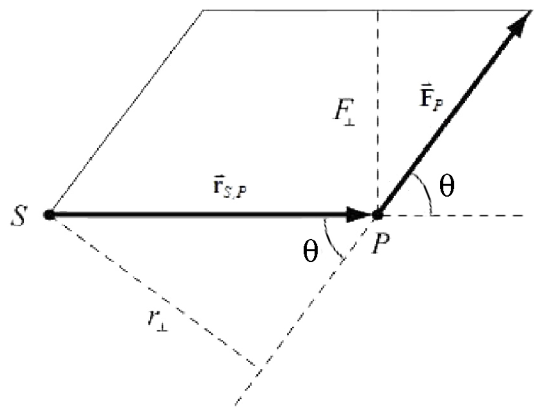


Figure 21.3 Area of the torque parallelogram.

Let $r_\perp = r\sin\theta$ and let $F_\perp = F\sin\theta$ be the component of the force $\vec{\mathbf{F}}_P$ that is perpendicular to the line passing from the point S to P. (Recall the angle θ has a range of values $0 \le \theta \le \pi$ so both $r_\perp \ge 0$ and $F_\perp \ge 0$.) Then the area of parallelogram defined by $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$ is given by

$$\text{Area} = \tau_S = r_\perp F = r\,F_\perp = r\,F\sin\theta\,. \tag{21.1.3}$$

We can interpret the quantity $r_\perp$ as follows. We begin by drawing the *line of action of the force* $\vec{\mathbf{F}}_P$. This is a straight line passing through P, parallel to the direction of the force $\vec{\mathbf{F}}_P$. Draw a perpendicular to this line of action that passes through the point S (Figure 21.4). The length of this perpendicular, $r_\perp = r\sin\theta$, is called *the moment arm about the point S of the force* $\vec{\mathbf{F}}_P$.

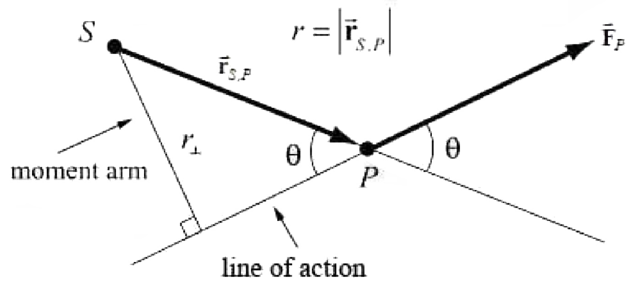


Figure 21.4 The moment arm about the point S associated with a force acting at the point P is the perpendicular distance from S to the line of action of the force passing through the point P

You should keep in mind three important properties of torque:

1. The torque is zero if the vectors $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$ are parallel $(\theta = 0)$ or anti-parallel $(\theta = \pi)$.

2. Torque is a vector whose direction and magnitude depend on the choice of a point S about which the torque is calculated.

3. The direction of torque is perpendicular to the plane formed by the two vectors, $\vec{\mathbf{F}}_P$ and $r = \left|\vec{\mathbf{r}}_{S,P}\right|$ (the vector from the point S to a point P).

Alternative Approach to Assigning a Sign Convention for Torque

In the case where all of the forces $\vec{\mathbf{F}}_i$ and position vectors $\vec{\mathbf{r}}_{i,P}$ are coplanar (or zero), we can, instead of referring to the direction of torque, assign a purely algebraic positive or negative sign to torque according to the following convention. We note that the arc in Figure 21.5a circles in counterclockwise direction. (Figures 21.5a and 21.5b use the simplifying assumption, for the purpose of the figure only, that the two vectors in question, $\vec{\mathbf{F}}_P$ and $\vec{\mathbf{r}}_{S,P}$ are perpendicular. The point S about which torques are calculated is not shown.) We can associate with this counterclockwise orientation a unit normal vector $\hat{\mathbf{n}}$ according to the right-hand rule: curl your right hand fingers in the counterclockwise direction and your right thumb will then point in the $\hat{\mathbf{n}}_1$ direction. The arc in Figure 21.5b circles in the clockwise direction, and we associate this orientation with the unit normal $\hat{\mathbf{n}}_2$.

It's important to note that the terms "clockwise" and "counterclockwise" might be different for different observers. For instance, if the plane containing $\vec{\mathbf{F}}_P$ and $\vec{\mathbf{r}}_{S,P}$ is horizontal, an observer above the plane and an observer below the plane would disagree on the two terms. For a vertical plane, the directions that two observers on opposite sides of the plane would be mirror images of each other, and so again the observers would disagree.

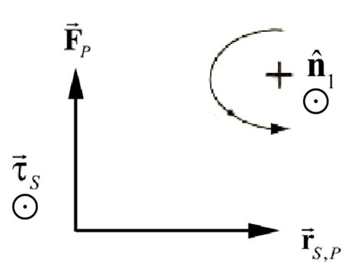


Figure 21.5a Positive torque

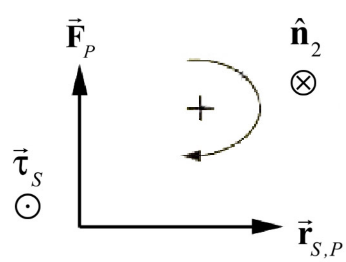


Figure 21.5b Negative torque

1. Suppose we choose counterclockwise as positive. Then we assign a positive sign for the component of the torque when the torque is in the same direction as the unit normal $\hat{\mathbf{n}}_1$, i.e. $\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,P} \times \vec{\mathbf{F}}_P = +|\vec{\mathbf{r}}_{S,P}||\vec{\mathbf{F}}_P|\hat{\mathbf{n}}_1$ (Figure 21.5a).

2. Suppose we choose clockwise as positive. Then we assign a negative sign for the component of the torque in Figure 21.5b since the torque is directed opposite to the unit normal $\hat{\mathbf{n}}_2$, i.e. $\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,P} \times \vec{\mathbf{F}}_P = -|\vec{\mathbf{r}}_{S,P}||\vec{\mathbf{F}}_P|\hat{\mathbf{n}}_2$.

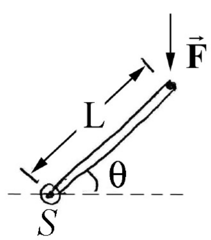
21.1.1 Example: Consider two vectors $\vec{\mathbf{r}}_{P,F} = x\hat{\mathbf{i}}$ with $x > 0$ and $\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_z\hat{\mathbf{k}}$ with $F_x > 0$ and $F_z > 0$. Calculate the torque $\vec{\mathbf{r}}_{P,F} \times \vec{\mathbf{F}}$.

Answer. We calculate the cross product noting that in a right handed choice of unit vectors, $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \vec{\mathbf{0}}$ and $\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$,

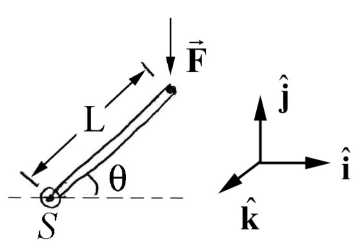
$$\begin{aligned}\vec{\mathbf{r}}_{P,F} \times \vec{\mathbf{F}} &= x\hat{\mathbf{i}} \times (F_x\hat{\mathbf{i}} + F_z\hat{\mathbf{k}}) = (x\hat{\mathbf{i}} \times F_x\hat{\mathbf{i}}) + (x\hat{\mathbf{i}} \times F_z\hat{\mathbf{k}}) \\ &= -xF_z\,\hat{\mathbf{j}}\end{aligned}$$

Since $x > 0$ and $F_z > 0$, the direction of the cross product is in the $-y$-direction.

21.1.2 Example In the figure, a force of magnitude F is applied to one end of a lever of length L. What is the magnitude and direction of the torque about the point S?



Answer. Choose units vectors such that $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$, with $\hat{\mathbf{i}}$ pointing to the right and $\hat{\mathbf{j}}$ pointing up.

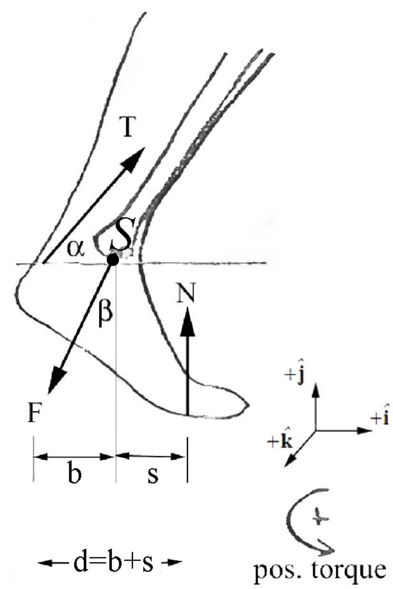


The torque about the point S is given by $\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,F} \times \vec{\mathbf{F}}$, where $\vec{\mathbf{r}}_{SF} = L\cos\theta\hat{\mathbf{i}} + L\sin\theta\hat{\mathbf{j}}$ and $\vec{\mathbf{F}} = -F\hat{\mathbf{j}}$ then

$$\vec{\boldsymbol{\tau}}_S = (L\cos\theta\,\hat{\mathbf{i}} + L\sin\theta\,\hat{\mathbf{j}}) \times -F\,\hat{\mathbf{j}} = -FL\cos\theta\,\hat{\mathbf{k}}\,.$$

21.1.3 Example Torque and the Ankle

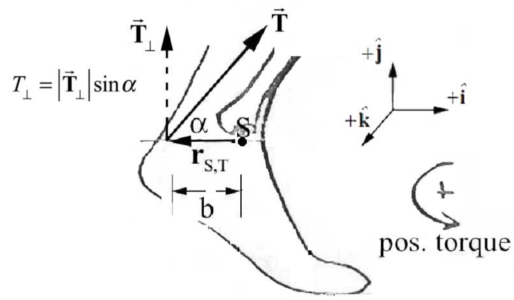
A person of mass $m = 75\text{ kg}$ is crouching with their weight evenly distributed on both tiptoes. The forces on the skeletal part of the foot are shown in the diagram.



The normal force $\vec{\mathbf{N}}$ acts at the contact point between the foot and the ground. In this position, the tibia acts on the foot at the point S with a force $\vec{\mathbf{F}}$ of an unknown magnitude $F = |\vec{\mathbf{F}}|$ and makes an unknown angle β with the vertical. This force acts on the ankle a horizontal distance $s = 4.8\text{ cm}$ from the point where the foot contacts the floor. The Achilles tendon also acts on the foot and is under considerable tension with magnitude $T \equiv |\vec{\mathbf{T}}|$ and acts at an angle $\alpha = 37^0$ with the horizontal as shown in the figure. The tendon acts on the ankle a horizontal distance $b = 6.0\text{ cm}$ from the point S where the tibia acts on the foot. You may ignore the weight of the foot. Let $g = 9.8\text{ m}\cdot\text{s}^{-2}$ be the gravitational constant. Compute the torque about the point S due to

a) the normal force of the floor on the foot;
b) the tendon force on the foot;
c) the force of the tibia on the foot.

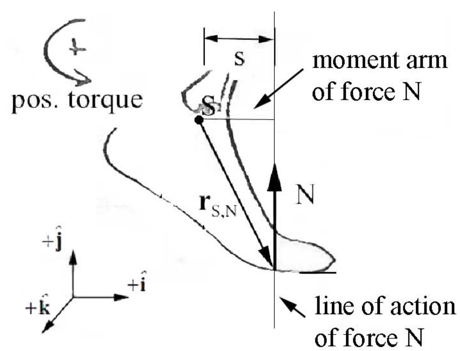
Solution: We shall first calculate the torque due to the force of the Achilles tendon on the ankle. The tendon force has the vector decomposition $\vec{\mathbf{T}} = T\cos\alpha\,\hat{\mathbf{i}} + T\sin\alpha\,\hat{\mathbf{j}}$.



The vector from the point S to the point of action of the force is given by $\vec{\mathbf{r}}_{S,T} = -b\hat{\mathbf{i}}$. Therefore the torque due to the force of the tendon $\vec{\mathbf{T}}$ on the ankle about the point S is then

$$\vec{\tau}_{S,T} = \vec{\mathbf{r}}_{S,T} \times \vec{\mathbf{T}} = -b\hat{\mathbf{i}} \times (T\cos\alpha\,\hat{\mathbf{i}} + T\sin\alpha\,\hat{\mathbf{j}}) = -b\,T\sin\alpha\,\hat{\mathbf{k}}$$

The torque diagram for the normal force is shown in the figure below;



The vector from the point S to the point where the normal force acts on the foot is given by $\vec{\mathbf{r}}_{S,N} = (s\hat{\mathbf{i}} - h\hat{\mathbf{j}})$. Since the weight is evenly distributed on the two feet, the normal force on one foot is equal to half the weight, or $N = (1/2)mg$. The normal force is therefore given by $\vec{\mathbf{N}} = N\,\hat{\mathbf{j}} = (1/2)\,mg\,\hat{\mathbf{j}}$. Therefore the torque of the normal force about the point S is

$$\vec{\tau}_{S,N} = \vec{\mathbf{r}}_{S,N} \times N\,\hat{\mathbf{j}} = (s\hat{\mathbf{i}} - h\hat{\mathbf{j}}) \times N\,\hat{\mathbf{j}} = s\,N\,\hat{\mathbf{k}} = (1/2)\,s\,mg\,\hat{\mathbf{k}}$$

The force $\vec{\mathbf{F}}$ that the tibia exerts on the ankle will make no contribution to the torque about this point S since the tibia force acts at the point S and therefore the vector $\vec{\mathbf{r}}_{S,F} = \vec{\mathbf{0}}$.

21.2 Torque, Angular Acceleration, and Moment of Inertia

For fixed-axis rotation, there is a direct relation between the component of the torque along the axis of rotation and angular acceleration.

Consider the forces that act on the rotating body. Most generally, the forces on different volume elements will be different, and so we will denote the force on the volume element of mass Δm_i by $\vec{\mathbf{F}}_i$.

Choose the z-axis to lie along the axis of rotation. As in Section 20.1, divide the body into volume elements of mass Δm_i. Let the point S denote a specific point along the axis of rotation (Figure 21.6). Each volume element undergoes a tangential acceleration as the volume element moves in a circular orbit of radius $r_{\perp,i} = \left|\vec{\mathbf{r}}_{\perp,i}\right|$ about the fixed axis.

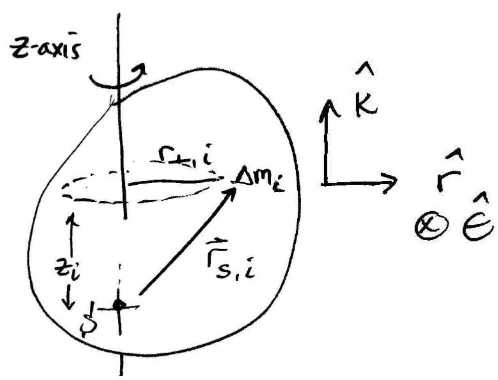


Figure 21.6: Volume element undergoing fixed-axis rotation about the z-axis.

The vector from the point S to the volume element is given by

$$\vec{\mathbf{r}}_{S,\,i} = z_i\,\hat{\mathbf{k}} + \vec{\mathbf{r}}_{\perp,i} = z_i\,\hat{\mathbf{k}} + r_{\perp,i}\,\hat{\mathbf{r}} \tag{21.2.1}$$

where z_i is the distance along the axis of rotation between the point S and the volume element. The torque about S due to the force $\vec{\mathbf{F}}_i$ acting on the volume element is given by

$$\vec{\boldsymbol{\tau}}_{S,\,i} = \vec{\mathbf{r}}_{S,\,i} \times \vec{\mathbf{F}}_i\,. \tag{21.2.2}$$

Substituting Equation (21.2.1) into Equation (21.2.2) gives

$$\vec{\boldsymbol{\tau}}_{S,\,i} = (z_i\,\hat{\mathbf{k}} + r_{\perp,i}\,\hat{\mathbf{r}}) \times \vec{\mathbf{F}}_i\,. \tag{21.2.3}$$

For fixed-axis rotation, we are interested in the z-component of the torque, which must be the term

$$(\tau_{S,\,i})_z = (r_{\perp,i}\,\hat{\mathbf{r}} \times \vec{\mathbf{F}}_i)_z \tag{21.2.4}$$

since the cross product $z_i\,\hat{\mathbf{k}} \times \vec{\mathbf{F}}_i$ must be directed perpendicular to the plane formed by the vectors $\hat{\mathbf{k}}$ and $\vec{\mathbf{F}}_i$, hence perpendicular to the z-axis.

The total force acting on the volume element has components

$$\vec{\mathbf{F}}_i = F_{\text{radial},\,i}\,\hat{\mathbf{r}} + F_{\tan,\,i}\,\hat{\boldsymbol{\theta}} + F_{z,\,i}\,\hat{\mathbf{k}}\,. \tag{21.2.5}$$

The z-component $F_{z,i}$ of the force cannot contribute a torque in the z-direction, and so substituting Equation (21.2.5) into Equation (21.2.4) yields

$$(\tau_{S,i})_z = (r_{\perp,i}\,\hat{\mathbf{r}} \times (F_{\text{radial},\,i}\,\hat{\mathbf{r}} + F_{\tan,\,i}\,\hat{\boldsymbol{\theta}}))_z\,. \tag{21.2.6}$$

The radial force does not contribute to the torque about the z-axis, since

$$r_{\perp,i}\,\hat{\mathbf{r}} \times F_{\text{radial},i}\,\hat{\mathbf{r}} = \vec{\mathbf{0}}\,. \tag{21.2.7}$$

So, we are interested in the contribution due to torque about the z-axis due to the tangential component of the force on the volume element (Figure 21.7).

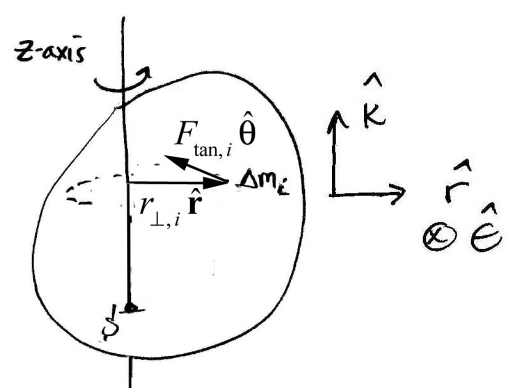


Figure 21.7 Tangential force acting on a volume element.

The component of the torque about the z-axis is given by

$$(\tau_{S,i})_z = (r_{\perp,i}\,\hat{\mathbf{r}} \times F_{\tan,i}\,\hat{\boldsymbol{\theta}})_z = r_{\perp,i}\,F_{\tan,i}\,. \tag{21.2.8}$$

The z-component of the torque is directed out of the page in Figure 21.7, where $F_{\tan,i}$ is positive (the tangential force is directed counterclockwise, as in the figure).

Applying Newton's Second Law in the tangential direction,

$$F_{\tan,\,i} = \Delta m_i\, a_{\tan,\,i}\,. \tag{21.2.9}$$

Using our kinematics result that the tangential acceleration is $a_{\tan,\,i} = r_{\perp,\,i}\,\alpha$, we have that

$$F_{\tan,\,i} = \Delta m_i\, r_{\perp,\,i}\,\alpha\,. \tag{21.2.10}$$

From Equation (21.2.8), the component of the torque about the z-axis is then given by

$$(\tau_{S,i})_z = r_{\perp,\,i}\,F_{\tan,\,i} = \Delta m_i (r_{\perp,\,i})^2 \alpha\,. \tag{21.2.11}$$

The total component of the torque about the z-axis is the summation of the torques on all the volume elements,

$$\begin{aligned}(\tau_S^{\text{total}})_z &= (\tau_{S,1})_z + (\tau_{S,2})_z + \cdots = \sum_{i=1}^{i=N} (\tau_{S,i})_z = \sum_{i=1}^{i=N} r_{\perp,\,i}\,F_{\tan,\,i} \\ &= \sum_{i=1}^{i=N} \Delta m_i (r_{\perp,\,i})^2 \alpha.\end{aligned} \tag{21.2.12}$$

Since each element has the same angular acceleration, α, the summation becomes

$$(\tau_S^{\text{total}})_z = \left(\sum_{i=1}^{i=N} \Delta m_i (r_{\perp,\,i})^2\right)\alpha\,. \tag{21.2.13}$$

Recalling our definition of the moment of inertia, **(add link)** the z-component of the torque is proportional to the angular acceleration,

$$(\tau_S^{\text{total}})_z = I_S\,\alpha\,, \tag{21.2.14}$$

and the moment of inertia, I_S, is the constant of proportionality.

This is very similar to Newton's Second Law: the total force is proportional to the acceleration,

$$\vec{\mathbf{F}}^{\text{total}} = m^{\text{total}}\vec{\mathbf{a}}\,. \tag{21.2.15}$$

where the total mass, m^{total}, is the constant of proportionality.

21.2.1 Example: Turntable

The turntable in Example 14.1.1, of mass $1.2\ \text{kg}$ and radius $1.3\times 10^{1}\ \text{cm}$, has a moment of inertia $I_S = 1.01\times 10^{-2}\ \text{kg}\cdot\text{m}^2$ about an axis through the center of the disc and perpendicular to the disc. The turntable is spinning at an initial constant frequency of $f_0 = 33\ \text{cycles}\cdot\text{min}^{-1}$. The motor is turned off and the turntable slows to a stop in $8.0\ \text{s}$ due to frictional torque. Assume that the angular acceleration is constant. What is the magnitude of the frictional torque acting on the disc?

Answer:

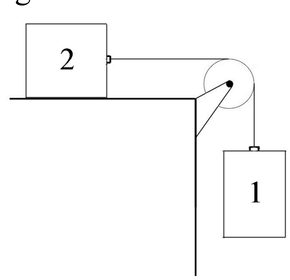
We have already calculated the angular acceleration of the disc in Example 14.1.1, where we found that the angular acceleration is

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_0}{t_f - t_0} = \frac{-3.5\ \text{rad}\cdot\text{s}^{-1}}{8.0\ \text{s}} = -4.3\times 10^{-1}\ \text{rad}\cdot\text{s}^{-2} \qquad (21.2.16)$$

and so the magnitude of the frictional torque is

$$\begin{aligned} \left|\tau^{\text{total}}_{\text{friction}}\right| &= I_S\left|\alpha\right| = (1.01\times 10^{-2}\ \text{kg}\cdot\text{m}^2)(4.3\times 10^{-1}\ \text{rad}\cdot\text{s}^{-2}) \\ &= 4.3\times 10^{-3}\ \text{N}\cdot\text{m}. \end{aligned} \qquad (21.2.17)$$

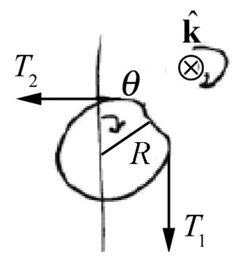
21.2.2 Example: A pulley of mass m_{p}, radius R, and moment of inertia about its center of mass I_{cm}, is attached to the edge of a table. An inextensible string of negligible mass is wrapped around the pulley and attached on one end to block 1 that hangs over the edge of the table. The other end of the string is attached to block 2 which slides along a table. The coefficient of sliding friction between the table and the block 2 is μ_k. Block 1 has mass m_1 and block 2 has mass m_2, with $m_1 > \mu_k m_2$. At time $t = 0$, the blocks are released from rest and the string does not slip around the pulley. At time $t = t_1$, block 1 hits the ground. Let g denote the gravitational constant.



a) Find the magnitude of the acceleration of each block. Express your answer in terms of m_p, I_{cm}, R, m_1, m_2, μ_k, and t_1 as needed.

b) How far did the block 1 fall before hitting the ground?

Solution: The torque diagram for the pulley is shown in the figure below



where we choose $\hat{\mathbf{k}}$ pointing into the page. Note that the tensions in the string on either side of the pulley are not equal. The reason is that the pulley is massive. To understand why, remember that the difference in the magnitudes of the torques due to the tension on either side of the pulley is equal to the moment of inertia times the magnitude of the angular acceleration which is non-zero for a massive pulley. So the tensions cannot be equal.

The torque principle states that for fixed axis rotation, the torque about that axis satisfies

$$\tau_z = I_z \alpha_z . \tag{21.2.18}$$

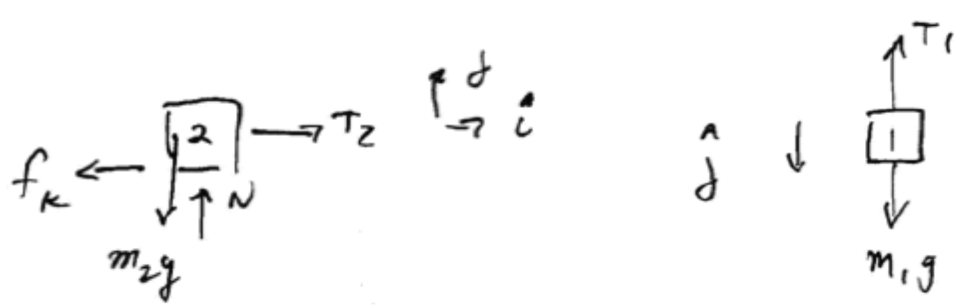
From our torque diagram, the torque about the point O at the center of the pulley is given by

$$\vec{\boldsymbol{\tau}}_O = \vec{\mathbf{r}}_{O,1} \times \vec{\mathbf{T}}_1 + \vec{\mathbf{r}}_{O,2} \times \vec{\mathbf{T}}_2 = R(T_1 - T_2)\hat{\mathbf{k}} . \tag{21.2.19}$$

Therefore the torque equation becomes

$$R(T_1 - T_2) = I_z \alpha_z . \tag{21.2.20}$$

The free body force diagrams on the two blocks are shown in the figure below.



Newton's Second Law on block 1 yields

$$m_1 g - T_1 = m_1 a_{y1} . \qquad (21.2.21)$$

Newton's Second Law on block 2 in the $\hat{\mathbf{j}}$ direction yields

$$N - m_2 g = 0 . \qquad (21.2.22)$$

Newton's Second Law on block 2 in the $\hat{\mathbf{i}}$ direction yields

$$T_2 - f_k = m_2 a_{x2} . \qquad (21.2.23)$$

The kinetic friction force is given by

$$f_k = \mu_k N = \mu_k m_2 g \qquad (21.2.24)$$

Therefore Eq. (21.2.23) becomes

$$T_2 - \mu_k m_2 g = m_2 a_{x2} . \qquad (21.2.25)$$

Constraints: Block 1 and block 2 have the same acceleration so

$$a \equiv a_{x1} = a_{x2} . \qquad (21.2.26)$$

We can solve Eqs. (21.2.21) and (21.2.25) for the two tensions yielding

$$T_1 = m_1 g - m_1 a , \qquad (21.2.27)$$

$$T_2 = \mu_k m_2 g + m_2 a . \qquad (21.2.28)$$

At point on the rim of the pulley has a tangential acceleration that is equal to the acceleration of the blocks so

$$a = R\alpha_z . \qquad (21.2.29)$$

The torque equation (Eq. (21.2.20)) then becomes

$$T_1 - T_2 = \frac{I_z}{R^2} a . \qquad (21.2.30)$$

Substituting Eqs. (21.2.27) and (21.2.28) into Eq. (21.2.30) yields

$$m_1 g - m_1 a - (\mu_k m_2 g + m_2 a) = \frac{I_z}{R^2} a \qquad (21.2.31)$$

which we can now solve for the accelerations of the blocks

$$a = \frac{m_1 g - \mu_k m_2 g}{m_1 + m_2 + I_z / R^2}. \qquad (21.2.32)$$

Block 1 hits the ground at time t_1, therefore it traveled a distance

$$y_1 = \frac{1}{2}\left(\frac{m_1 g - \mu_k m_2 g}{m_1 + m_2 + I_z / R^2}\right) t_1^{\,2}. \qquad (21.2.33)$$

Torque acts at the Center of Gravity

Suppose a rigid body in static equilibrium consists of N particles labeled by the index $i = 1, 2, 3, ..., N$. Choose a coordinate system with a choice of origin O such that mass m_i has position $\vec{\mathbf{r}}_i$. Each point particle experiences a gravitational force $\vec{\mathbf{F}}_{\text{gravity},i} = m_i \vec{\mathbf{g}}$. The total torque about the origin is then zero,

$$\vec{\boldsymbol{\tau}}_{O,\text{total}} = \sum_{i=1}^{i=N} \vec{\boldsymbol{\tau}}_{O,i} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_i \times \vec{\mathbf{F}}_{\text{gravity},i} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_i \times m_i \vec{\mathbf{g}} = \vec{\mathbf{0}}. \qquad (21.2.34)$$

If in Equation (21.2.34) the gravitational acceleration $\vec{\mathbf{g}}$ is assumed constant we can rearrange the summation by pulling the constant vector $\vec{\mathbf{g}}$ out of the summation ($\vec{\mathbf{g}}$ appears in each term in the summation);

$$\vec{\boldsymbol{\tau}}_{O,\text{total}} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_i \times m_i \vec{\mathbf{g}} = \left(\sum_{i=1}^{i=N} m_i \vec{\mathbf{r}}_i\right) \times \vec{\mathbf{g}} = \vec{\mathbf{0}}. \qquad (21.2.35)$$

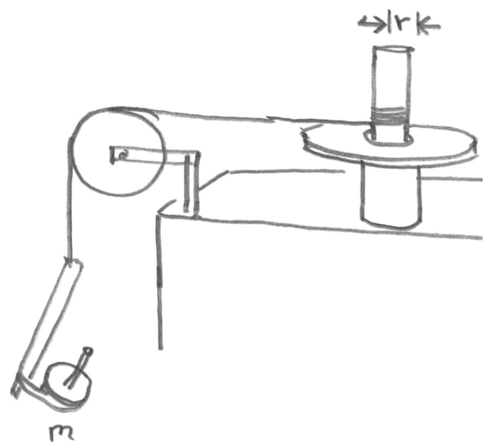
We now use our definition of the center of the center of mass, Equation (9.2.17) of the text, to rewrite Equation (21.2.35) as

$$\vec{\boldsymbol{\tau}}_{O,\text{total}} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_i \times m_i \vec{\mathbf{g}} = M_{\text{T}} \vec{\mathbf{R}}_{\text{cm}} \times \vec{\mathbf{g}} = \vec{\mathbf{R}}_{\text{cm}} \times M_{\text{T}} \vec{\mathbf{g}} = \vec{\mathbf{0}}. \qquad (21.2.36)$$

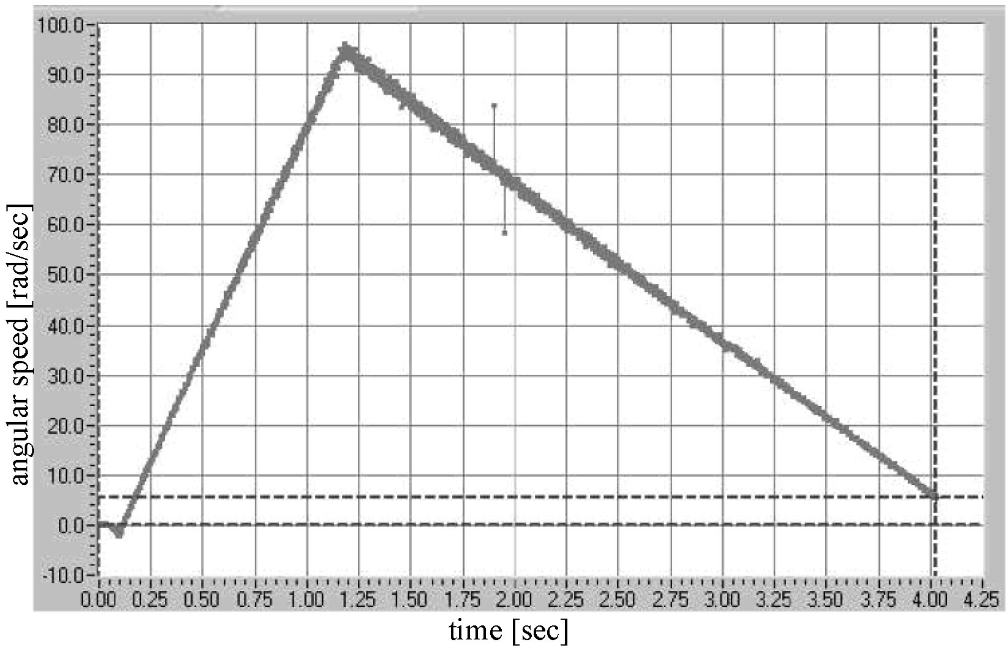
Thus the torque due to the gravitational force acting on each point-like particle is equivalent to the torque due to the gravitational force acting on a point-like particle of mass M_{T} located at a point in the body called the *center of gravity*, which is equal to the center of mass of the body in the typical case in which the gravitational acceleration $\vec{\mathbf{g}}$ is constant throughout the body.

21.2.3 Example

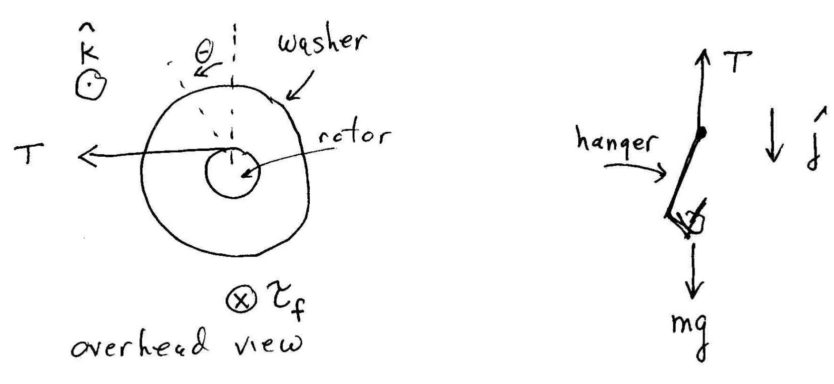
A steel washer is mounted on a cylindrical rotor of radius $r = 12.7\ \text{mm}$. A massless string, with an object of mass $m = 0.055\ \text{kg}$ attached to the other end, is wrapped around the side of the rotor and passes over a massless pulley.



Assume that there is a constant frictional torque about the axis of the rotor. The object is released and falls. As the mass falls, the rotor undergoes an angular acceleration of magnitude α_1. After the string detaches from the rotor, the rotor coasts to a stop with an angular acceleration of magnitude α_2. Let $g = 9.8\ \text{m}\cdot\text{s}^{-2}$ denote the gravitational constant. Based on the data in the figure below, what is the moment of inertia I_R of the rotor assembly (including the washer) about the rotation axis?



Solution: We begin by drawing free body diagrams for the rotor and hanger show in the figures below. (The choice of positive directions are indicated on the figures.) The frictional torque on the rotor is then given by $\vec{\boldsymbol{\tau}}_f = -\tau_f \hat{\mathbf{k}}$ where we use τ_f as the magnitude of the frictional torque. The torque about the center of the rotor due to the tension in the string is given by $\vec{\boldsymbol{\tau}}_T = rT\,\hat{\mathbf{k}}$ where r is the radius of the rotor. The angular acceleration of the rotor is given by $\vec{\boldsymbol{\alpha}}_1 = \alpha_1\,\hat{\mathbf{k}}$ and we expect that $\alpha_1 > 0$ because the rotor is speeding up.



While the hanger is falling, the rotor/washer combination has a net torque due to the tension in the string and the friction torque, and using the rotational equation of motion,

$$Tr - \tau_f = I_R \alpha_1 \qquad (21.2.37)$$

We apply Newton's Second Law to the hanger and find that

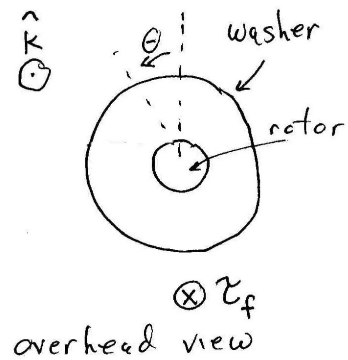
$$mg - T = ma_1 = m\alpha_1 r \tag{21.2.38}$$

where $a_1 = r\alpha_1$ has been used to express the linear acceleration of the falling hanger to the angular acceleration of the rotor; that is, the string does not stretch.

Before proceeding, it might be illustrative to multiply Equation (21.2.38) by r and add to Equation (21.2.37) to obtain

$$mgr - \tau_f = (I_R + mr^2)\alpha_1 \tag{21.2.39}$$

Equation (21.2.39) contains the unknown frictional torque, and this torque is determined by considering the slowing of the rotor/washer after the string has detached.



The torque on the system is just this frictional torque, and so

$$-\tau_f = I_R\alpha_2 \tag{21.2.40}$$

Note that in Equation (21.2.40), $\tau_f > 0$ and $\alpha_2 < 0$, consistent with Eq. (21.2.40).

Subtracting Equation (21.2.40) from Equation (21.2.39) eliminates τ_f,

$$mgr = mr^2\alpha_1 + I_R(\alpha_1 - \alpha_2) \tag{21.2.41}$$

and solving for I_R yields

$$I_R = \frac{mr\left(g - r\alpha_1\right)}{\alpha_1 - \alpha_2} \tag{21.2.42}$$

For a numerical result, the values for α_1 and α_2 from the above figure are $\alpha_1 = (96\,\text{rad}\cdot\text{s}^{-1})/(1.15\,\text{s}) = 83\ \text{rad}\cdot\text{s}^{-2}$ and $\alpha_2 = -(89\,\text{rad}\cdot\text{s}^{-1})/(2.85\,\text{s}) = -\,31\,\text{rad}\cdot\text{s}^{-2}$. Inserting these expressions into Eq. (21.2.42) yields

$$I_R = 5.3\times 10^{-5}\,\text{kg}\cdot\text{m}^2\,. \tag{21.2.43}$$

21.3 Torque and Rotational Work

When a constant torque τ_S is applied to an object, and the object rotates through an angle $\Delta\theta$ about an axis through the center of mass, then the torque does an amount of work $\Delta W = \tau_S\,\Delta\theta$ on the object. By extension of the linear work-energy theorem, the amount of work done is equal to the change in the rotational kinetic energy of the object,

$$W_{\text{rot}} = \frac{1}{2}I_{\text{cm}}\omega_f^2 - \frac{1}{2}I_{\text{cm}}\omega_0^2 = K_{\text{rot},f} - K_{\text{rot},0}\,. \tag{21.3.1}$$

The rate of doing this work is the rotational power exerted by the torque,

$$P_{\text{rot}} \equiv \frac{dW_{\text{rot}}}{dt} = \lim_{\Delta t\to 0}\frac{\Delta W_{\text{rot}}}{\Delta t} = \tau_S\frac{d\theta}{dt} = \tau_S\,\omega\,. \tag{21.3.2}$$

Rotational Work

Consider a rigid body rotating about an axis. Each small element of mass Δm_i in the rigid body is moving in a circle of radius $(r_{S,i})_\perp$ about the axis of rotation passing through the point S. Each mass element undergoes a small angular displacement $\Delta\theta$ under the action of a tangential force, $\vec{\mathbf{F}}_{\tan,i} = F_{\tan,i}\,\hat{\boldsymbol{\theta}}$, where $\hat{\boldsymbol{\theta}}$ is the unit vector pointing in the tangential direction (Figure 21.7). The element will then have an associated displacement vector for this motion, $\Delta\vec{\mathbf{r}}_{S,i} = (r_{S,i})_\perp\,\Delta\theta\,\hat{\boldsymbol{\theta}}$ and the work done by the tangential force is

$$\Delta W_i = \vec{\mathbf{F}}_{\tan,i}\cdot\Delta\vec{\mathbf{r}}_{S,i} = (F_{\tan,i}\,\hat{\boldsymbol{\theta}})\cdot((r_{S,i})_\perp\,\Delta\theta\,\hat{\boldsymbol{\theta}}) = F_{\tan,i}(r_{S,i})_\perp\,\Delta\theta\,. \tag{21.3.3}$$

Applying Newton's Second Law to the element Δm_i in the tangential direction,

$$F_{\tan,i} = \Delta m_i\,a_{\tan,i}\,. \tag{21.3.4}$$

Using the expression in for tangential acceleration $a_{\tan,i} = (r_{S,i})_\perp\alpha$, we have that

$$F_{\tan,i} = \Delta m_i (r_{S,i})_\perp\alpha\,. \tag{21.3.5}$$

Thus the rotational work done on the mass element is

$$\Delta W_i = \Delta m_i (r_{S,i})_\perp^2 \alpha \, \Delta\theta \,. \qquad (21.3.6)$$

Summing the rotational work done on all of the mass elements, we obtain

$$\Delta W = \sum_i \Delta W_i = \left(\sum_i \Delta m_i (r_{S,i})_\perp^2 \right) \alpha \, \Delta\theta \,. \qquad (21.3.7)$$

In the limit that the discrete mass elements become infinitesimal continuous mass elements, $\Delta m_i \to dm$, the summation becomes an integral over the body:

$$\Delta W = \left(\sum_i \Delta m_i (r_{S,i})_\perp^2 \right) \alpha \, \Delta\theta \;\; \to \;\; \left(\int_{\text{body}} dm (r_S)_\perp^2 \right) \alpha \, \Delta\theta \,. \qquad (21.3.8)$$

Since the integral in this expression is just the moment of inertia about a fixed axis passing through the point S, we have for the rotational work

$$\Delta W = I_S \, \alpha \, \Delta\theta \,. \qquad (21.3.9)$$

Since the z-component of the torque (in the direction along the axis of rotation) about S is given by

$$(\tau_S)_z = I_S \, \alpha \,, \qquad (21.3.10)$$

the rotational work is the product of the torque and the angular displacement,

$$\Delta W = (\tau_S)_z \Delta\theta \,. \qquad (21.3.11)$$

Recall the result of Equation (21.2.8) that the component of the torque (in the direction along the axis of rotation) about S due to the tangential force, $\vec{\mathbf{F}}_{\tan,i}$, acting on the mass element Δm_i is

$$(\tau_{S,i})_z = F_{\tan,i} (r_{S,i})_\perp \,, \qquad (21.3.12)$$

and the total torque is the sum

$$(\tau_S)_z = \sum_i (\tau_{S,i})_z = \sum_i F_{\tan,i} (r_{S,i})_\perp \qquad (21.3.13)$$

and so the work done is

$$\Delta W = \sum_i \Delta W_i = \sum_i F_{\tan,i} (r_{S,i})_\perp \, \Delta\theta = (\tau_S)_z \Delta\theta . \tag{21.3.14}$$

In the limit of small angles, $\Delta\theta \to d\theta$, $\Delta W \to dW$ and the differential rotational work is

$$dW = (\tau_S)_z d\theta . \tag{21.3.15}$$

We can integrate this amount of rotational work as the angle coordinate of the rigid body changes from some initial value $\theta = \theta_0$ to some final value $\theta = \theta_f$,

$$W = \int dW = \int_{\theta_0}^{\theta_f} (\tau_S)_z d\theta . \tag{21.3.16}$$

Rotational Work-Kinetic Energy Theorem

We will now show that the rotational work is equal to the change in rotational kinetic energy. We begin by substituting our result from Equation (21.3.10) into Equation (21.3.15) for the infinitesimal rotational work,

$$dW_{\text{rot}} = I_S \alpha \, d\theta . \tag{21.3.17}$$

Recall that the rate of change of angular velocity is equal to the angular acceleration, $\alpha \equiv d\omega/dt$ and that the angular velocity is $\omega \equiv d\theta/dt$. Note that in the limit of small displacements,

$$\frac{d\omega}{dt} d\theta = d\omega \frac{d\theta}{dt} = d\omega\, \omega . \tag{21.3.18}$$

Therefore the infinitesimal rotational work is

$$dW_{\text{rot}} = I_S \alpha \, d\theta = I_S \frac{d\omega}{dt} d\theta = I_S \, d\omega \frac{d\theta}{dt} = I_S \, d\omega\, \omega . \tag{21.3.19}$$

We can integrate this amount of rotational work as the angular velocity of the rigid body changes from some initial value $\omega = \omega_0$ to some final value $\omega = \omega_f$,

$$W_{\text{rot}} = \int dW_{\text{rot}} = \int_{\omega_0}^{\omega_f} I_S \, d\omega\, \omega = \frac{1}{2} I_S \, \omega_f^2 - \frac{1}{2} I_S \, \omega_0^2 . \tag{21.3.20}$$

When a rigid body is rotating about a fixed axis passing through a point S in the body, there is both rotation and translation about the center of mass unless S is the center of mass. If we choose the point S in the above equation for the rotational work to be the center of mass, then

$$W_{\text{rot}} = \frac{1}{2} I_{\text{cm}} \omega_{\text{cm},f}^2 - \frac{1}{2} I_{\text{cm}} \omega_{\text{cm},0}^2 = K_{\text{rot},f} - K_{\text{rot},0} \equiv \Delta K_{\text{rot}} . \tag{21.3.21}$$

Rotational Power

The rotational power is defined as the rate of doing rotational work,

$$P_{\text{rot}} \equiv \frac{dW_{\text{rot}}}{dt} . \tag{21.3.22}$$

We can use our result for the infinitesimal work to find that the rotational power is the product of the applied torque with the angular velocity of the rigid body,

$$P_{\text{rot}} \equiv \frac{dW_{\text{rot}}}{dt} = (\tau_S)_z \frac{d\theta}{dt} = (\tau_S)_z \omega . \tag{21.3.23}$$

21.3.1 Example Work Done by Frictional Torque

A steel washer is mounted on the shaft of a small motor. The moment of inertia of the motor and washer is I_0. The washer is set into motion. When it reaches an initial angular velocity ω_0, at $t = 0$, the power to the motor is shut off, and the washer slows down during a time interval $\Delta t_1 = t_a$ until it reaches an angular velocity of ω_a at time t_a. At that instant, a second steel washer with a moment of inertia I_w is dropped on top of the first washer. Assume that the second washer is only in contact with the first washer. The collision takes place over a time $\Delta t_{\text{int}} = t_b - t_a$ after which the two washers and rotor rotate with angular speed ω_b. Assume the frictional torque on the axle (magnitude τ_f) is independent of speed, and remains the same when the second washer is dropped.

a) What angle does the rotor rotate through during the collision?

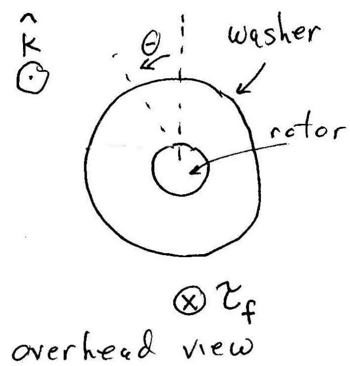
b) What is the work done by the friction torque from the bearings during the collision?

c) Write down an equation for conservation of energy. Can you solve this equation for ω_b?

c) What is the average rate that work is being done by the friction torque during the collision?

Solution:

We begin by solving for the frictional torque during the first stage of motion when the rotor is slowing down. We choose a coordinate system shown in the figure below.



The component of average angular acceleration is given by

$$\alpha_1 = \frac{\omega_a - \omega_0}{t_a} < 0$$

and using the rotational equation of motion, the frictional torque satisfies

$$-\tau_f = I_0\left(\frac{\omega_a - \omega_0}{\Delta t_1}\right).$$

During the collision, the component of the average angular acceleration of the rotor is given by

$$\alpha_2 = \frac{\omega_b - \omega_a}{(\Delta t_{\text{int}})} < 0.$$

The angle the rotor rotates through during the collision is (analogous to linear motion with constant acceleration)

$$\Delta\theta_2 = \omega_a \Delta t_{\text{int}} + \frac{1}{2}\alpha_2 \Delta t_{\text{int}}{}^2 = \omega_a \Delta t_{\text{int}} + \frac{1}{2}\left(\frac{\omega_b - \omega_a}{\Delta t_{\text{int}}}\right)\Delta t_{\text{int}}{}^2 = \frac{1}{2}(\omega_b + \omega_a)\Delta t_{\text{int}} > 0.$$

The non-conservative work done by the bearing friction during the collision is

$$W_{f,b} = -\tau_f \Delta\theta_{rotor} = -\tau_f \frac{1}{2}(\omega_a + \omega_b)\Delta t_{\text{int}}.$$

Using our result for the frictional torque, the work done by the bearing friction during the collision is

$$W_{f,b} = \frac{1}{2} I_0 \left(\frac{\omega_a - \omega_0}{\Delta t_1} \right) (\omega_a + \omega_b) \Delta t_{\text{int}} < 0 \, .$$

The negative work is consistent with the fact that the kinetic energy of the rotor is decreasing as the rotor is slowing down. Using the work energy theorem during the collision the kinetic energy of the rotor has deceased by

$$W_{f,b} = \frac{1}{2}(I_0 + I_w)\omega_b^2 - \frac{1}{2} I_0 \omega_a^2 \, .$$

Using our result for the work we have that

$$\frac{1}{2} I_0 \left(\frac{\omega_a - \omega_0}{\Delta t_1} \right) (\omega_a + \omega_b) \Delta t_{\text{int}} = \frac{1}{2}(I_0 + I_w)\omega_b^2 - \frac{1}{2} I_0 \omega_a^2 \, .$$

This is a quadratic equation for the angular speed ω_b of the rotor and washer immediately after the collision that we can in principle solve. However remember that we assumed that the frictional torque is independent of the speed of the rotor. Hence the best practice would be to measure ω_0, ω_a, ω_b, Δt_1, Δt_{int}, I_0, and I_w and then determine how closely our model agrees with conservation of energy.

The rate of work done by the frictional torque is given by

$$P_f = \frac{W_{f,b}}{\Delta t_{\text{int}}} = \frac{1}{2} I_0 \left(\frac{\omega_a - \omega_0}{\Delta t_1} \right) (\omega_a + \omega_b) < 0 \, .$$