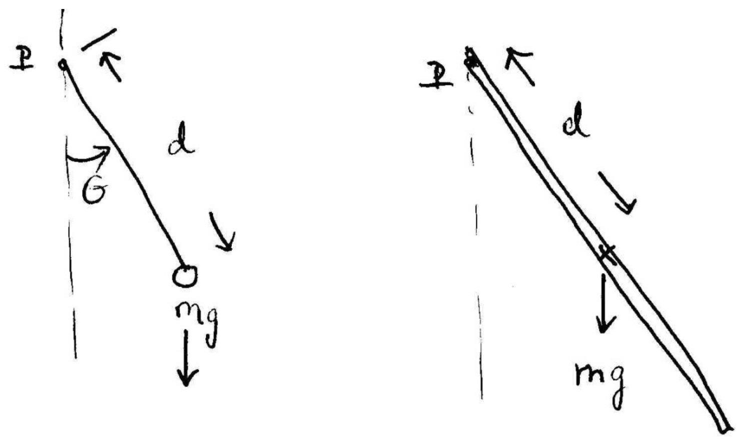
Simple Harmonic Oscillators and Torque

Concept Questions

Question 1: The center of gravity of a simple pendulum of mass m and length d is located at the position of the pendulum bob, a distance d from the pivot point. The center of gravity of a uniform rod of mass m and length $2d$ pivoted at one end is also a distance d from the pivot point. How does the period of this uniform rod compare to the period of the simple pendulum for small oscillations?



1. The rod has a longer period;
2. the rod has a shorter period;
3. the rod has the same period.

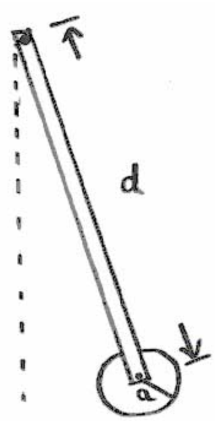
Answer 1. In both cases, when $\theta > 0$, the torque about the pivot point has the same magnitude, $|\tau_P| = mgd\sin\theta$ and is restoring. The magnitude of the angular acceleration is inversely proportional to the moment of inertia about the pivot point, $|\alpha| = |\tau_P| / I_P$. The moment of inertial of the simple pendulum about the pivot point is $I_{P,simple} = md^2$. The moment of inertial of the uniform rod about the pivot point is $I_{P,rod} = (1/3)m(2d)^2 = (4/3)md^2$. The rod has a greater moment of inertia about the pivot point and hence has a smaller angular acceleration. If the angular acceleration is smaller it takes longer to complete a cycle. Therefore the rod has a longer period. The period of a simple pendulum of length d is $T_{simple} = 2\pi\sqrt{d/g}$ while the period of a uniform rod of length $2d$ is

$$T_{rod} = 2\pi\sqrt{I_P / mgd} = 2\pi\sqrt{(4/3)md^2 / mgd} = 2\pi\sqrt{(4/3)d / g} > T_{simple} \qquad (1.1)$$

in agreement with our above reasoning.

Question2 Physical Pendulum A physical pendulum consists of a uniform rod of length d and mass m pivoted at one end. A disk of mass m_1 and radius a is fixed to the other end. Suppose the disk is now mounted to the rod by a frictionless bearing so that is perfectly free to spin. Does the period of the pendulum

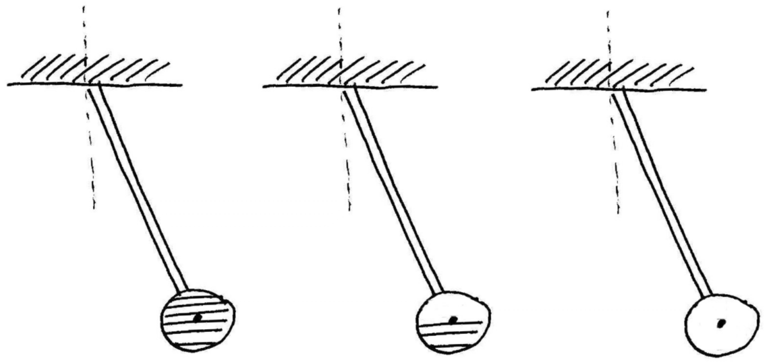
1. increase?
2. stay the same?
3. decrease?



Answer 3. When the disk is fixed to the rod, an internal torque will cause the disk to rotate about its center of mass. When the pendulum reaches the bottom of its swing, the decrease in potential energy will be result in an increase in the rotational kinetic energy of both the rod and the disk and the center of mass translation kinetic energy of the rod-disk system. When the disk is mounted on the frictionless bearing there is no internal torque that will make the disk start to rotate about its center of mass when the pendulum is released. Therefore when the pendulum reaches the bottom of its swing, the same decrease in potential energy will be transferred into a larger smaller in rotational kinetic energy of just the rod since the disc is not rotating and a greater increase in the center of mass translation kinetic energy of the rod-disk system.
So when the disk bearings are frictionless, the center of mass of the rod-disk system is traveling faster at the bottom of its arc hence will take less time to complete one cycle and so the period is shorter compared to the fixed disk.

You might be tempted to argue that the moment of inertia about the pivot point is the same in both cases, the torque is the same so the period should be the same. But the disk with frictionless bearings is not a rigid body which means that the disk has a different angular acceleration than the rod and hence you must treat each part of the system separately when applying $\tau_P = I_P \alpha$.

Question 3 A physical pendulum consists of a rod and bob that is completely filled with water. As the pendulum oscillates through a very small angle, the water leaks out until the bob is empty. Describe how the angular frequency changes as the water is leaking out. You may treat the rod as massless.



Answer: When the bob is completely full of water, the center of mass of the bob-water system is at the center of the bob, and the moment of inertia about the pivot point of the system is

$$I_P = (m_b + m_w)d^2 + I_{cm} \tag{3.1}$$

where d is the distance form the center of the bob to the pivot point, m_b is the mass of the bob and m_w is the mass of the water. The moment of inertia about the center of mass is given b y

$$I_{cm} = \frac{2}{5}(m_b + m_w)R^2 \tag{3.2}$$

where R is the radius of the bob. So the moment of inertia about the pivot point is

$$I_P = (m_b + m_w)(d^2 + (2/5)R^2). \tag{3.3}$$

Before the water leeks out, when $\theta > 0$, the magnitude of torque about the pivot point is due to the gravitational force which acts at the center of mass of the system

$$\tau_P = (m_b + m_w)dg\sin\theta. \tag{3.4}$$

So the torques equation (the torque is a restoring torque)

$$-(m_b + m_w)dg\sin\theta = (m_b + m_w)(d^2 + (2/5)R^2)\frac{d^2\theta}{dt^2}. \tag{3.5}$$

Thus the differential equation and hence the period is independent of the total mass of the system

$$-dg\sin\theta = (d^2 + (2/5)R^2)\frac{d^2\theta}{dt^2}. \tag{3.6}$$

When the water totally leaks out, the mass of the center of mass is at the center of the bob so the above Eq. still holds and the period is the same. When the water is leeking the center of mass moves further away from the pivot point. The moment of inertia about the moving center of mass is changing as well making a detailed calculation impossible especially since the water is a non-rigid body. The fact that the center of mass moves further from the pivot will make the period longer and the angular frequency smaller.