Simple and Physical Pendulums
Challenge Problem Solutions

Problem 1 Solutions:

For this problem, the answers to parts a) through d) will rely on an analysis of the pendulum motion. There are two conventional methods of analyzing the pendulum, which will be presented here. Unconventional methods are not in the current plan.

The two methods are consideration of torque and angular acceleration and use of Newton's Second Law.

Method I: Torque and Angular Acceleration

Choose the origin for calculating torques and angular momentum as the pivot point of the pendulum, and let θ be the angular displacement of the pendulum string from the vertical. The only forces on the pendulum are the gravitational force on the pendulum bob (the pointlike object) and the tension in the string. With respect to the pivot, the tension exerts no torque. The moment arm for the gravitational torque is $r_{\perp} = l\sin\theta$, for a net torque

$$\tau_{\text{net}} = -mgl\sin\theta . \tag{1.1}$$

The minus sign in Equation (1.1) is crucial; the torque will act to restore the angle θ to its equilibrium value $\theta = 0$. If $\theta > 0$, $\tau_{\text{net}} < 0$ and if $\theta < 0$, $\tau_{\text{net}} > 0$ (this assumes that the angle θ is restricted to the range $-\pi < \theta < \pi$, implied by the small-angle approximation $\sin\theta_0 \simeq \theta_0$).

With the assumption of a massless string, the moment of inertia about the pivot point is

$$I_{\text{pivot}} = ml^2 . \tag{1.2}$$

The angular acceleration α is related kinematically to the displacement angle θ by

$$\alpha = \frac{d^2\theta}{dt^2} . \tag{1.3}$$

The torque, moment of inertia and angular acceleration are related by

$$\tau_{\text{net}} = I_{\text{pivot}}\,\alpha . \tag{1.4}$$

Combining Equations (1.1), (1.2), (1.3) and (1.4) yields

$$
\begin{aligned}
-mgl\sin\theta &= ml^2\frac{d^2\theta}{dt^2} \\
\frac{d^2\theta}{dt^2}+\frac{g}{l}\sin\theta &= 0.
\end{aligned} \tag{1.5}
$$

Method II: Newton's Second Law

The object will move in a circular arc centered at the pivot point. The forces on the object are the tension in the string and gravity. The tension in the string could be found from the speed of the object and the angle from the vertical, but that's a different problem, solved many times in these practice problems. Our concern is with the tangential force, which is the component of the gravitational force along the arc of the circle,

$$F_{\text{tangential}} = -mg\sin\theta\,. \tag{1.6}$$

The sign in Equation (1.6) is crucial; the tangential force tends to restore the pendulum to the equilibrium value $\theta = 0$. If $\theta > 0$, $F_{\text{tangential}} < 0$ and if $\theta < 0$, $F_{\text{tangential}} > 0$. As in Method I above, this assumes that the angle θ is restricted to the range $-\pi < \theta < \pi$, implied by the small-angle approximation $\sin\theta_0 \simeq \theta_0$.

The tangential component of acceleration is

$$a_{\text{tangential}} = l\alpha = l\frac{d^2\theta}{dt^2}\,; \tag{1.7}$$

using this in Newton's Second Law, $F_{\text{tangential}} = m\,a_{\text{tangential}}$ reproduces the second expression in (1.5).

In the limit of small oscillations, $\sin\theta \simeq \theta$, this expression becomes

$$\frac{d^2\theta}{dt^2}+\frac{g}{l}\theta = 0. \tag{1.8}$$

The solutions to (1.8) are well-known. With the initial condition that the pendulum is released from rest at a small angle θ_0,

$$\theta(t) = \theta_0\cos\left(\sqrt{\frac{g}{l}}\,t\right) = \theta_0\cos\left(\Omega\, t\right) = \theta_0\cos\left(\frac{2\pi}{T}t\right) \tag{1.9}$$

where Ω is the angular frequency of oscillation (denoted Ω to distinguish from the kinematic variable $\omega = \frac{d\theta}{dt}$) and T is the period of oscillation.

a) From inspection of the expression in (1.9), the period is $T = 2\pi\sqrt{l/g}$.

b) Similarly, the angular frequency of oscillation is $\sqrt{g/l}$.

c) We could use energy considerations, with the initial gravitational potential energy relative to the bottom of the swing as

$$mgl\left(1-\cos\theta_0\right) \simeq mgl\frac{\theta_0^2}{2} \tag{1.10}$$

and set this equal to

$$K = \frac{1}{2}m v_{\max}^2 \tag{1.11}$$

to obtain

$$v_{\max} = \sqrt{gl}\,\theta_0 . \tag{1.12}$$

Or, we could use a standard result for harmonic oscillations,

$$v_{\max} = l\,\omega_{\max} = l\theta_0\sqrt{\frac{g}{l}} . \tag{1.13}$$

d) From the result of part c) (parts c) and d) are really the same),

$$\omega_{\max} = \sqrt{\frac{g}{l}}\,\theta_0 . \tag{1.14}$$

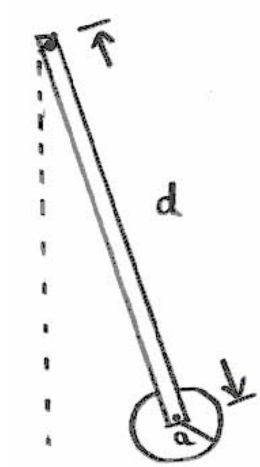
e) The angular velocity $\omega = \frac{d\theta}{dt}$ is a kinematic variable that changes with time in an oscillatory manner (sinusoidally in the limit of small oscillations). The angular frequency Ω is a parameter that describes the system. The angular velocity ω, besides being time-dependent, depends on the amplitude of oscillation θ_0. In the limit of small oscillations, Ω does not depend on the amplitude of oscillation.

f) Algebraically, the mass divided out from both the torque and the moment of inertia, or from the net force and the acceleration term in Newton's Second Law.

Consider also the argument that is attributed to Galileo: If a pendulum consisting of two identical masses joined together were set to oscillate, the two halve would not exert forces on each other. So, if the pendulum were split into two pieces, the pieces would oscillate the same as if they were one piece. This argument can be extended to simple pendula of arbitrary masses.

Problem 2: Physical Pendulum

A physical pendulum consists of two pieces: a uniform rod of length d and mass m pivoted at one end, and a disk of radius a, mass m_1, fixed to the other end. The pendulum is initially displaced to one side by a small angle θ_0 and released from rest. You can then approximate $\sin\theta \cong \theta$ (with θ measured in radians).



a) Find the period of the pendulum.

b) Suppose the disk is now mounted to the rod by a frictionless bearing so that is perfectly free to spin. Find the new period of the pendulum.

Problem 2 Solutions:

a) The physical pendulum consists of two pieces. A uniform rod of length d and a disk attached at the end of the rod. The moment of inertia about the pivot point P is the sum of the moments of inertia of the two pieces,

$$I_P^{total} = I_P^{rod} + I_P^{disc}$$

We calculated the moment of inertia of a rod about the end point P in class, and found that

$$I_P^{rod} = \frac{1}{3}md^2 .$$

We can use the parallel axis theorem to calculate the moment of inertia of the disk about the pivot point P,

$$I_P^{disc} = I_{cm}^{disc} + m_1 d^2 .$$

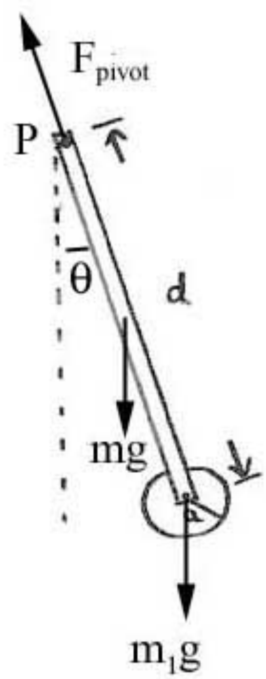
We calculated the moment of inertia of a disk about the center of mass in class, and found that

$$I_{cm}^{disk} = \frac{1}{2} m_1 a^2 .$$

So the total moment of inertia is

$$I_P^{total} = \frac{1}{3} m d^2 + m_1 d^2 + \frac{1}{2} m_1 a^2 .$$

The force diagram on the pendulum is shown below. In particular, there is an unknown pivot force, the gravitational force acting at the center of mass of the rod, and the gravitational force acting at the center of mass of the disk.



The torque about the pivot point is given by

$$\vec{\boldsymbol{\tau}}_P = \vec{\mathbf{r}}_{P,cm} \times m\vec{\mathbf{g}} + \vec{\mathbf{r}}_{P,disc} \times m_1 \vec{\mathbf{g}} \; .$$

$$\vec{\tau}_P = (d/2)\hat{\mathbf{r}} \times mg(-\sin\theta\,\hat{\theta} + \cos\hat{\mathbf{r}}) + d\hat{\mathbf{r}} \times m_1 g(-\sin\theta\,\hat{\theta} + \cos\hat{\mathbf{r}}) = -((d/2)m + dm_1) g \sin\theta \hat{\mathbf{k}}$$

.

The rotational dynamical equation is

$$\vec{\tau}_P = I_p^{total}\vec{\alpha}\,.$$

Therefore

$$-((d/2)m + dm_1)g\sin\theta\hat{\mathbf{k}} = (\frac{1}{3}md^2 + m_1 d^2 + \frac{1}{2}m_1 a^2)\frac{d^2\theta}{dt^2}\hat{\mathbf{k}}\,.$$

When the angle of oscillation is small, then we can use the small angle approximation

$$\sin\theta \cong \theta\,.$$

Then the pendulum equation becomes

$$\frac{d^2\theta}{dt^2} \cong -\frac{(d/2)m + dm_1}{\frac{1}{3}md^2 + m_1 d^2 + \frac{1}{2}m_1 a^2}\theta\,.$$

The angular frequency of oscillation for the pendulum is approximately

$$\omega_{pendulum} \cong \sqrt{\frac{(d/2)m + dm_1}{\frac{1}{3}md^2 + m_1 d^2 + \frac{1}{2}m_1 a^2}}\,,$$

with period

$$T = \frac{2\pi}{\omega_p} \cong 2\pi\sqrt{\frac{\frac{1}{3}md^2 + m_1 d^2 + \frac{1}{2}m_1 a^2}{(d/2)m + dm_1}}\,.$$

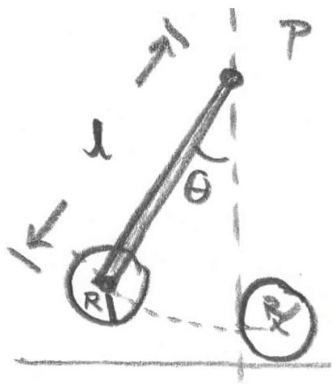
b) If the disk is not fixed to the rod, then it will not rotate as the pendulum oscillates. Therefore it does not contribute to the moment of inertia. Notice that the pendulum is no longer a rigid body. So the total moment of inertia is only due to the rod,

$$I_P^{total} = \frac{1}{3}md^2\,.$$

Therefore the period of oscillation is given by

$$T = \frac{2\pi}{\omega_p} \cong 2\pi\sqrt{\frac{\frac{1}{3}md^2}{(d/2)m + dm_1}}$$

Problem 3: A physical pendulum consists of a disc of radius R and mass m_1 fixed at the end of a massless rod. The other end of the rod is pivoted about a point P. The distance from the pivot point to the center of mass of the bob is l. Initially the bob is released from rest from a small angle θ_0 with respect to the vertical. At the bottom of the bob's trajectory, it collides completely inelastically with another less massive disc of radius R and mass m_2, $m_1 > m_2$.



a) What is the period of the bob before the collision?

b) What is the velocity of the bob just before the collision at the bottom of the bob's trajectory?

c) What is the velocity of the bob and disc immediately after the collision?

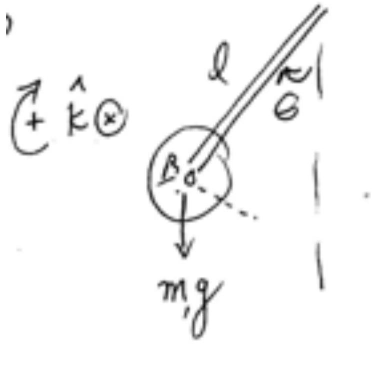
d) What is the new period of the pendulum after the collision?

e) What angle does the pendulum rise to when it next comes to rest?

Problem 3 Solutions:

a)

$$I_P = \frac{1}{2} m_1 R^2$$



$$\vec{\tau}_P = I_p \alpha$$

$$-\ell m g \sin\theta \hat{k} = I_p \frac{d^2\theta}{dt^2} \hat{k}$$

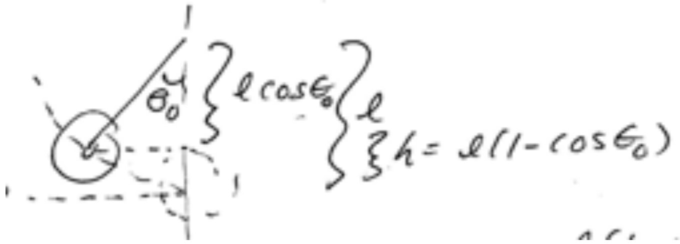
$$\frac{d^2\theta}{dt^2} + \frac{\ell m_1 g}{I_p} \sin\theta = 0$$

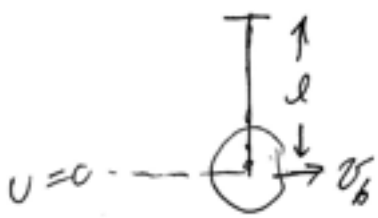
$$\sin\theta \approx \theta$$

$$\frac{d^2\theta}{dt^2} + \frac{\ell m_1 g}{I_p} \theta = 0$$

$$\omega = \sqrt{\frac{\ell m_1 g}{I_p}} = \sqrt{\frac{\ell m_1 g}{\frac{1}{2} m_1 R^2}} = \sqrt{\frac{2\ell}{R^2} g}$$

b)





$$E_0 = m_1 g \ell \left(1-\cos\theta_0\right)$$

$$E_b = \frac{1}{2} I_p \omega_b^2$$

$$v_b = \ell \omega_b$$

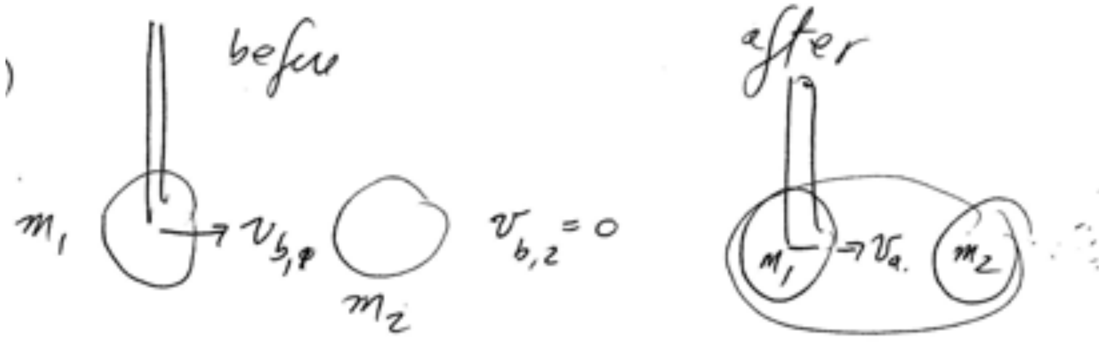
$$E_b = \frac{1}{2} I_p \quad \frac{v_b^2}{\ell^2}$$

$$E_0 = E_b \Rightarrow m_1 g \ell \left(1-\cos\theta_0\right) = \frac{1}{2} I_p \frac{v_b^2}{\ell^2}$$

$$m_1 g \ell \left(1-\cos\theta_0\right) = \frac{1}{2} I_p \frac{v_b^2}{\ell^2}$$

$$v_b = 2 \frac{m_1 g \ell^3 \left(1-\cos\theta_0\right)}{\frac{1}{2} m_1 R^2} = 4 \frac{g \ell^3}{R^2} \left(1-\cos\theta_0\right)$$

c)



momentum is conserved

$$m_1 v_{b,1} = (m_1 + m_2) v_a$$

$$v_a = \frac{m_1 v_{b,1}}{m_1 + m_2} = \frac{m_1}{m_1 + m_2}\left(\frac{4g\ell^3}{R^2}(1 - \cos\theta_0)\right)$$

d)

Since the oscillation frequency is independent of the mass because $I_p^{final} = \frac{1}{2}(m_1 + m_2)R^2$

ω is the same, $\omega = \sqrt{\frac{2\ell}{R^2} g}$

$$E_a = E_f$$
$$E_a = \frac{1}{2} I_p^{after} \frac{v_a^2}{\ell^2} = \frac{1}{4}(m_1 + m_2)\frac{R^2 v_a^2}{\ell^2}$$
$$E_f = (m_1 + m_2)\ell(1 - \cos\theta_f)g$$

e) $E_a = E_f \Rightarrow$

$$\frac{1}{4}\frac{R^2 v_a^2}{\ell^2} = \ell(1 - \cos\theta_f)(m_1 + m_2)g$$
$$\frac{1}{4}\frac{R^2}{\ell^2}\left(\frac{m_1}{m_1 + m_2}\right) v_b^2 = \ell(1 - \cos\theta_f)(m_1 + m_2)g$$
$$\frac{m_1}{m_1 + m_2}\frac{1}{2} I_{p,before} v_0^2 / \ell^2 = \ell(1 - \cos\theta_f)(m_1 + m_2)g$$

$$\left(\frac{m_1}{m_1+m_2}\right)\left(m_1 g\ell\left(1-\cos\theta_0\right)\right)$$

$$=\left(m_1+m_2\right)g\left(\ell\right)\left(1-\cos\theta_f\right)$$

$$\Rightarrow\frac{m_1^2}{\left(m_1+m_2\right)^2}\left(1-\cos\theta_0\right)=\left(1-\cos\theta_f\right)$$

$$\cos\theta_f=1-\left(\frac{m_1^2}{\left(m_1+m_2\right)^2}\left(1-\cos\theta_0\right)\right)$$

Problem 4:

A wrench of mass m is pivoted a distance l_{cm} from its center of mass and allowed to swing as a physical pendulum. The period for small-angle-oscillations is T.

a) What is the moment of inertia of the wrench about an axis through the pivot?

b) If the wrench is initially displaced by an angle θ_0 from its equilibrium position, what is the angular speed of the wrench as it passes through the equilibrium position?

Problem 4 Solutions:

a) The period of the physical pendulum for small angles is $T = 2\pi\sqrt{I_{\text{P}} / m l_{\text{cm}} g}$; solving for the moment of inertia of the wrench we determine that

$$I_{\text{P}} = \frac{T^2 m l_{\text{cm}} g}{4\pi^2}.$$

b) For this part, we are not given a small-angle approximation, and should not assume that θ_0 is a small angle. We will need to use energy considerations, and assume that the pendulum is released from rest.

Taking the zero of potential energy to be at the bottom of the pendulum's swing, the initial potential energy is $U_{\text{initial}} = mgl_{\text{cm}}(1-\cos\theta_0)$ and the final kinetic energy at the bottom of the swing is $U_{\text{final}} = 0$. The initial kinetic energy is $K_{\text{initial}} = 0$ and the final kinetic energy is related to the angular speed ω_{final} at the bottom of the swing by $K_{\text{final}} = (1/2) I_{\text{P}} \omega_{\text{final}}^2$. Equating initial potential energy to final kinetic energy yields

$$\omega_{\text{final}}^2 = \frac{2mgl_{\text{cm}}(1-\cos\theta_0)}{I_{\text{P}}} = \frac{8\pi^2}{T^2}(1-\cos\theta_0).$$