Module 6: Two and Three Dimensional Kinematics

6.1 Introduction to the Vector Description of Motion in Two and Three Dimensions

So far we have introduced the concepts of kinematics to describe motion in one dimension; however we live in a multidimensional universe. In order to explore and describe motion in this universe, we begin by looking at examples of two-dimensional motion, of which there are many; planets orbiting a star in elliptical orbits or a projectile moving under the action of uniform gravitation are two common examples.

We will now extend our definitions of position, velocity, and acceleration for an object that moves in two dimensions (in a plane) by treating each direction independently, which we can do with vector quantities by resolving each of these quantities into components. For example, our definition of velocity as the derivative of position holds for each component separately. In Cartesian coordinates, in which the directions of the unit vectors do not change from place to place, the position vector $\vec{\mathbf{r}}(t)$ with respect to some choice of origin for the object at time t is given by

$$\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}}. \tag{6.1.1}$$

The velocity vector $\vec{\mathbf{v}}(t)$ at time t is the derivative of the position vector,

$$\vec{\mathbf{v}}(t) = \frac{dx(t)}{dt}\,\hat{\mathbf{i}} + \frac{dy(t)}{dt}\,\hat{\mathbf{j}} \equiv v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}}, \tag{6.1.2}$$

where $v_x(t) \equiv dx(t)/dt$ and $v_y(t) \equiv dy(t)/dt$ denote the x- and y-components of the velocity respectively.

The acceleration vector $\vec{\mathbf{a}}(t)$ is defined in a similar fashion as the derivative of the velocity vector,

$$\vec{\mathbf{a}}(t) = \frac{dv_x(t)}{dt}\,\hat{\mathbf{i}} + \frac{dv_y(t)}{dt}\,\hat{\mathbf{j}} \equiv a_x(t)\,\hat{\mathbf{i}} + a_y(t)\,\hat{\mathbf{j}}, \tag{6.1.3}$$

where $a_x(t) \equiv dv_x(t)/dt$ and $a_y(t) \equiv dv_y(t)/dt$ denote the x- and y-components of the acceleration.

6.2 Reference Frames

In order to describe physical events that occur in space and time such as the motion of bodies, we introduced a coordinate system. A *space-time event* can now be specified by its spatial and temporal coordinates. In particular, the position of a moving body can be described by space-time events specified by the space-time coordinates. You can place an *observer* at the origin of coordinate system. The coordinate system with your observer acts as a *reference frame* for describing the position, velocity, and acceleration of bodies. The position vector of the body depends on the choice of origin (location of your observer) but the displacement, velocity, and acceleration vectors are independent of the location of the observer.

You can always choose a second reference frame that is moving with respect to the first reference frame. Then the position, velocity and acceleration of bodies as seen by the different observers do depend on the relative motion of the two reference frames. The relative motion can be described in terms of the relative position, velocity, and acceleration of the observer at the origin, O, in reference frame S with respect to a second observer located at the origin, O', in reference frame S'.

Let the vector $\vec{\mathbf{R}}$ point from the origin of frame S to the origin of reference frame S'. Suppose an object is located at a point 1. Denote the position vector of the object with respect to origin of reference frame S by $\vec{\mathbf{r}}$. Denote the position vector of the object with respect to origin of reference frame S' by $\vec{\mathbf{r}}'$ (Figure 6.1).

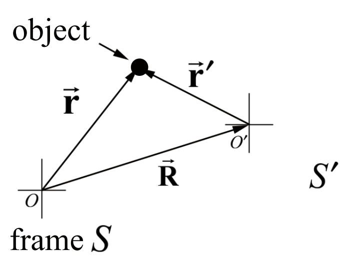


Figure 6.1 Two reference frames.

The position vectors are related by

$$\vec{\mathbf{r}}' = \vec{\mathbf{r}} - \vec{\mathbf{R}} \tag{6.2.1}$$

These coordinate transformations are called the *Galilean Coordinate Transformations*. They enable the observer in frame S to predict the position vector in frame S', based only on the position vector in frame S and the relative position of the origins of the two frames.

The relative velocity between the two reference frames is given by the time derivative of the vector $\vec{\mathbf{R}}$, defined as the limit as of the displacement of the two origins divided by an interval of time, as the interval of time becomes infinitesimally small,

$$\vec{\mathbf{V}} = \frac{d\vec{\mathbf{R}}}{dt}. \tag{6.2.2}$$

Relatively Inertial Reference Frames

If the relative velocity between the two reference frames is constant, then the relative acceleration between the two reference frames is zero,

$$\vec{\mathbf{A}} = \frac{d\vec{\mathbf{V}}}{dt} = \vec{\mathbf{0}}. \tag{6.2.3}$$

When two reference frames are moving with a constant velocity relative to each other as above, the reference frames are considered to be *relatively inertial reference frames*.

Law of Addition of Velocities: Newtonian Mechanics

Suppose the object in Figure 6.1 is moving; then observers in different reference frames will in general measure different velocities. Denote the velocity of the object in frame S by $\vec{\mathbf{v}} = d\vec{\mathbf{r}}/dt$, and the velocity of the object in frame S' by $\vec{\mathbf{v}}' = d\vec{\mathbf{r}}'/dt'$. Since the derivative of the position is velocity, the velocities of the object in two different reference frames are related according to

$$\frac{d\vec{\mathbf{r}}'}{dt'} = \frac{d\vec{\mathbf{r}}}{dt} - \frac{d\vec{\mathbf{R}}}{dt} \tag{6.2.4}$$

or

$$\vec{\mathbf{v}}' = \vec{\mathbf{v}} - \vec{\mathbf{V}} \tag{6.2.5}$$

This is called the *Law of Addition of Velocities*.

6.2.1 Example: Relative Velocities of Two Moving Planes

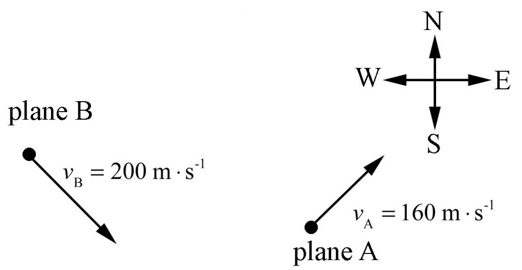
An airplane A is traveling northeast with a speed of $v_{\mathrm{A}} = 160\ \mathrm{m \cdot s^{-1}}$. A second airplane B is traveling southeast with a speed of $v_{\mathrm{B}} = 200\ \mathrm{m \cdot s^{-1}}$.

a) Choose a coordinate system and write down an expression for the velocity of each airplane as vectors, $\vec{\mathbf{v}}_{\mathrm{A}}$ and $\vec{\mathbf{v}}_{B}$. Carefully use unit vectors to express your answer.

b) Sketch the vectors $\vec{\mathbf{v}}_{\mathrm{A}}$ and $\vec{\mathbf{v}}_{B}$ on your coordinate system.

c) Find a vector expression that expresses the velocity of aircraft A as seen from an observer flying in aircraft B. Calculate this vector. What is its magnitude and direction? Sketch it on your coordinate system.

From the information given in the problem we draw the velocity vectors of the airplanes as shown in Figure 6.2A.



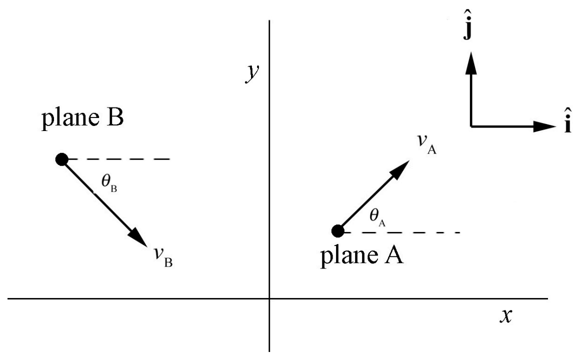


Figure 6.2A: Motion of two planes **Figure 6.2B: Coordinate System**

An observer at rest with respect to the ground defines a reference frame S. Choose a coordinate system shown in Figure 6.2B. According to this observer, airplane A is moving with velocity $\vec{\mathbf{v}}_A = v_A \cos\theta_A \hat{\mathbf{i}} + v_A \sin\theta_A \hat{\mathbf{j}}$, and airplane B is moving with velocity $\vec{\mathbf{v}}_B = v_B \cos\theta_B \hat{\mathbf{i}} + v_B \sin\theta_B \hat{\mathbf{j}}$. According to the information given in the problem airplane A flies northeast so $\theta_A = \pi/4$ and airplane B flies southeast east so $\theta_B = -\pi/4$. Thus $\vec{\mathbf{v}}_A = (80\sqrt{2}\ \text{m}\cdot\text{s}^{-1})\hat{\mathbf{i}} + (80\sqrt{2}\ \text{m}\cdot\text{s}^{-1})\hat{\mathbf{j}}$ and $\vec{\mathbf{v}}_B = (100\sqrt{2}\ \text{m}\cdot\text{s}^{-1})\hat{\mathbf{i}} - (100\sqrt{2}\ \text{m}\cdot\text{s}^{-1})\hat{\mathbf{j}}$

Consider a second observer moving along with airplane B, defining reference frame S'. What is the velocity of airplane A according to this observer moving in airplane B? The velocity of the observer moving along in airplane B with respect to an observer at rest on the ground is just the velocity of airplane B and is given by $\vec{\mathbf{V}} = \vec{\mathbf{v}}_B = v_B \cos\theta_B \hat{\mathbf{i}} + v_B \sin\theta_B \hat{\mathbf{j}}$. Using the Law of Addition of Velocities, Equation (6.2.5), the velocity of airplane A with respect to an observer moving along with Airplane B is given by

$$
\begin{aligned}
\vec{\mathbf{v}}'_A &= \vec{\mathbf{v}}_A - \vec{\mathbf{V}} = (v_A \cos\theta_A \hat{\mathbf{i}} + v_A \sin\theta_A \hat{\mathbf{j}}) - (v_B \cos\theta_B \hat{\mathbf{i}} + v_B \sin\theta_B \hat{\mathbf{j}}) \\
&= (v_A \cos\theta_A - v_B \cos\theta_B)\hat{\mathbf{i}} + (v_A \sin\theta_A - v_B \sin\theta_B)\hat{\mathbf{j}} \\
&= ((80\sqrt{2}\ \text{m}\cdot\text{s}^{-1}) - (100\sqrt{2}\ \text{m}\cdot\text{s}^{-1}))\hat{\mathbf{i}} + ((80\sqrt{2}\ \text{m}\cdot\text{s}^{-1}) + (100\sqrt{2}\ \text{m}\cdot\text{s}^{-1}))\hat{\mathbf{j}}. \quad (6.2.6) \\
&= -(20\sqrt{2}\ \text{m}\cdot\text{s}^{-1})\hat{\mathbf{i}} + (180\sqrt{2}\ \text{m}\cdot\text{s}^{-1})\hat{\mathbf{j}} \\
&= v'_{Ax}\hat{\mathbf{i}} + v'_{Ay}\hat{\mathbf{j}}
\end{aligned}
$$

Figure 6.2C shows the velocity of airplane A with respect to airplane B in reference frame S'.

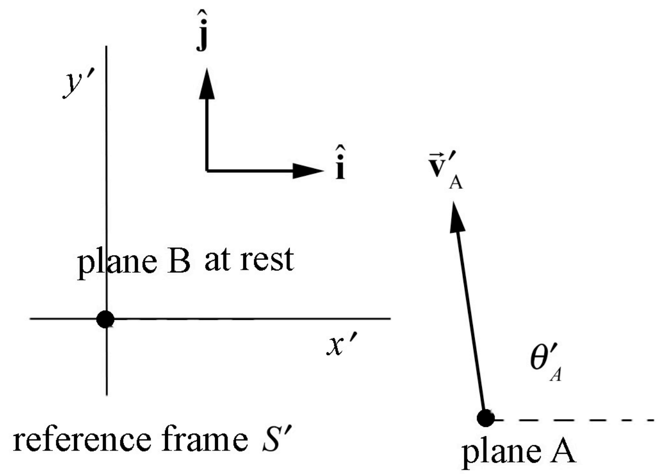


Figure 6.2C Airplane A as seen from observer in airplane B

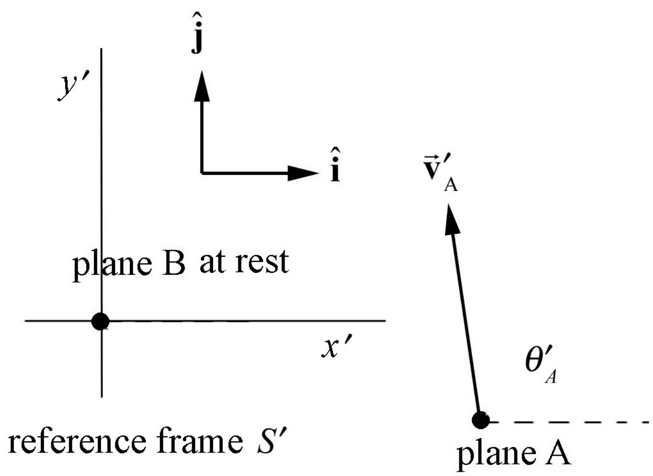
We can now use Equation (2.3.5) to find the magnitude of velocity of airplane A as seen by an observer moving with airplane B,

$$\left|\vec{v}'_{\mathrm{A}}\right| = (v'^{\,2}_{Ax} + v'^{\,2}_{Ay})^{1/2} = ((-20\sqrt{2}\ \mathrm{m\cdot s^{-1}})^2 + (180\sqrt{2}\ \mathrm{m\cdot s^{-1}})^2)^{1/2} = 256\ \mathrm{m\cdot s^{-1}}. \quad (6.2.7)$$

We can now use Equation (2.3.5) to find the angle of velocity of airplane A as seen by an observer moving with airplane B,

$$\begin{aligned}\theta'_A &= \tan^{-1}(v'_{Ay} / v'_{Ax}) = \tan^{-1}((180\sqrt{2}\ \mathrm{m\cdot s^{-1}}) / (-20\sqrt{2}\ \mathrm{m\cdot s^{-1}})) \\ &= \tan^{-1}(-9) = 180^\circ - 83.7^\circ = 96.3^\circ\end{aligned}. \quad (6.2.8)$$

Figure 6.2C shows the velocity of airplane A with respect to airplane B in reference frame S'.



6.3 Projectile Motion

A special case of two-dimensional motion occurs when the vertical component of the acceleration is constant and the horizontal component is zero. Then the complete set of equations for position and velocity for each independent direction of motion are given by

$$\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} = \left(x_0 + v_{x,0}t\right)\hat{\mathbf{i}} + \left(y_0 + v_{y,0}\,t + \frac{1}{2}a_y\,t^2\right)\hat{\mathbf{j}}, \tag{6.3.1}$$

$$\vec{\mathbf{v}}(t) = v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}} = v_{x,0}\,\hat{\mathbf{i}} + \left(v_{y,0} + a_y\,t\right)\hat{\mathbf{j}}, \tag{6.3.2}$$

$$\vec{\mathbf{a}}(t) = a_x(t)\,\hat{\mathbf{i}} + a_y(t)\,\hat{\mathbf{j}} = a_y\,\hat{\mathbf{j}}. \tag{6.3.3}$$

Consider the motion of a body that is released with an initial velocity $\vec{v}_0$ at a height h above the ground. Two paths are shown in Figure 6.14.

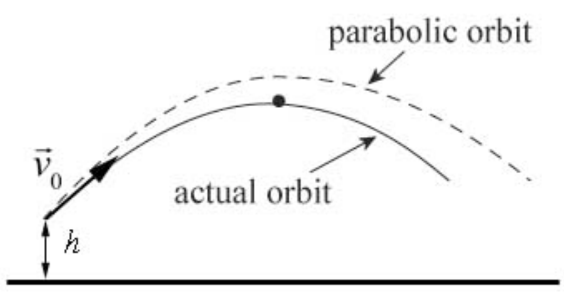


Figure 6.14 Actual orbit and parabolic orbit of a projectile

The dotted path represents a *parabolic* trajectory and the solid path represents the actual orbit. The difference between the paths is due to air resistance. There are other factors that can influence the path of motion; a rotating body or a special shape can alter the flow of air around the body, which may induce a curved motion or lift like the flight of a baseball or golf ball. We shall begin our analysis by neglecting all influences on the body except for the influence of gravity.

We shall choose coordinates with our y-axis in the vertical direction with $\hat{\mathbf{j}}$ directed upwards and our x-axis in the horizontal direction with $\hat{\mathbf{i}}$ directed in the direction that the body is moving horizontally. We choose our origin to be the place where the body is released at time $t = 0$. Figure 6.15 shows our coordinate system with the position of the body at time t and the coordinate functions $x(t)$ and $y(t)$.

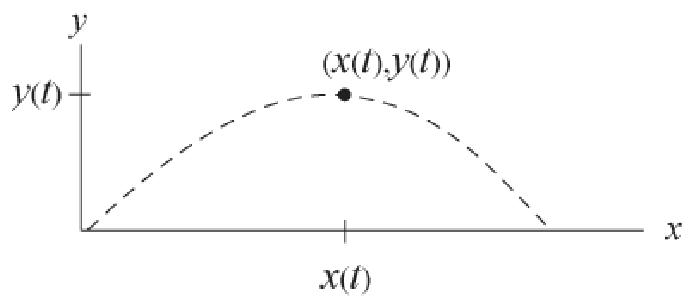


Figure 6.15 A coordinate sketch for parabolic motion.

The coordinate function $y(t)$ represents the distance from the body to the origin along the y-axis at time t, and the coordinate function $x(t)$ represents the distance from the body to the origin along the x-axis at time t.

The y-component of the acceleration,

$$a_y = -g, \tag{6.3.4}$$

is a constant and is independent of the mass of the body. Notice that $a_y < 0$; this is because we chose our positive y-direction to point upwards.

Since we are ignoring the effects of any horizontal forces, the acceleration in the horizontal direction is zero,

$$a_x = 0; \tag{6.3.5}$$

therefore the x-component of the velocity remains unchanged throughout the flight.

Kinematic Equations of Motion

The kinematic equations of motion for the position and velocity components of the object are

$$x(t) = x_0 + v_{x,0}\, t\,, \tag{6.3.6}$$

$$v_x(t) = v_{x,0}\,, \tag{6.3.7}$$

$$y(t) = y_0 + v_{y,0}\, t - \frac{1}{2} g t^2\,, \tag{6.3.8}$$

$$v_y(t) = v_{y,0} - g t\,. \tag{6.3.9}$$

Initial Conditions

In these equations, the initial velocity vector is

$$\vec{\mathbf{v}}_0(t) = v_{x,0}\, \hat{\mathbf{i}} + v_{y,0}\, \hat{\mathbf{j}}\,. \tag{6.3.10}$$

Often the description of the flight of a projectile includes the statement, "a body is projected with an initial speed v_0 at an angle θ_0 with respect to the horizontal." The vector decomposition diagram for the initial velocity is shown in Figure 6.16. The components of the initial velocity are given by

$$v_{x,0} = v_0 \cos\theta_0\,, \tag{6.3.11}$$

$$v_{y,0} = v_0 \sin\theta_0\,. \tag{6.3.12}$$

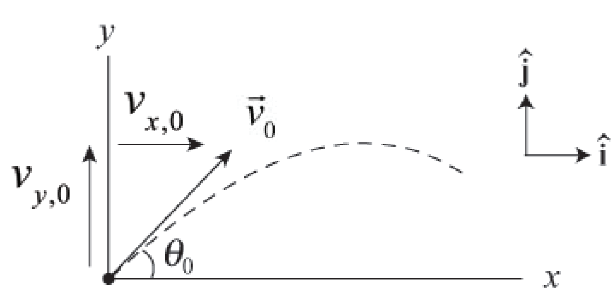


Figure 6.16 A vector decomposition of the initial velocity

Since the initial speed is the magnitude of the initial velocity, we have

$$v_0 = \left(v_{x,0}^2 + v_{y,0}^2\right)^{1/2}. \tag{6.3.13}$$

The angle θ_0 is related to the components of the initial velocity by

$$\theta_0 = \tan^{-1}\left(\frac{v_{y,0}}{v_{x,0}}\right). \tag{6.3.14}$$

The initial position vector appears with components

$$\vec{\mathbf{r}}_0 = x_0\,\hat{\mathbf{i}} + y_0\,\hat{\mathbf{j}}. \tag{6.3.15}$$

Note that the trajectory in Figure 6.16 has $x_0 = y_0 = 0$, but this will not always be the case, as in the example below.

Example: Time of Flight and Maximum Height of a Projectile

A person throws a stone at an initial angle $\theta_0 = 45^\circ$ from the horizontal with an initial speed of $v_0 = 20\ \mathrm{m\cdot s^{-1}}$. The point of release of the stone is at a height $d = 2\ \mathrm{m}$ above the ground. You may neglect air resistance. a) How long does it take the stone to reach the highest point of its trajectory? b) What was the maximum vertical displacement of the stone? Ignore air resistance.

Solution:

Choose the origin on the ground directly underneath the point where the stone is released. We choose upwards for the positive y-direction and along the projection of the path of the stone along the ground for the positive x-direction. Set $t = 0$ the instant the stone is released. At $t = 0$ the initial conditions are then $x_0 = 0$ and $y_0 = d$. The initial x- and y-components of the velocity are given by Eq. (6.3.11) and Eq. (6.3.12)

The y-component of the position of the stone $y(t)$ is plotted as a function of time in Figure 3 with $d = 2\ \mathrm{m}$, $v_0 = 20\ \mathrm{m\cdot s^{-1}}$, and $\theta_0 = 45^\circ$. At time t the stone has coordinates $(x(t), y(t))$. These coordinate functions are shown in Figure 6.17.

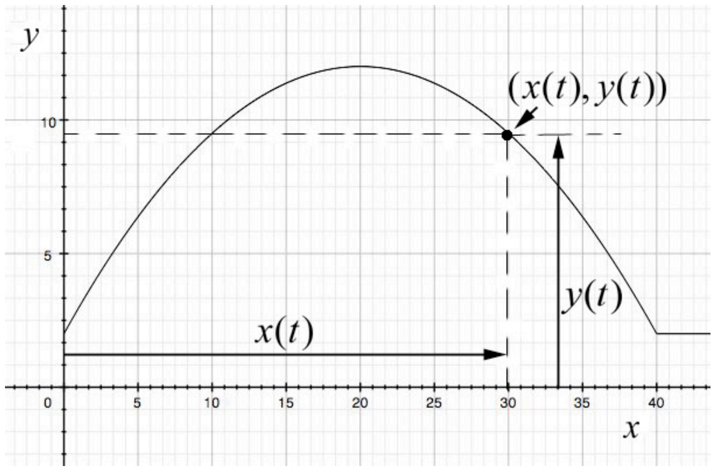


Figure 6.17: Coordinate functions for stone

The y-component of the position of the stone, $y(t)$, is plotted as a function of time in Figure 6.18. The slope of this graph at any time t yields the instantaneous y-component of the velocity $v_y(t)$ at that time t.

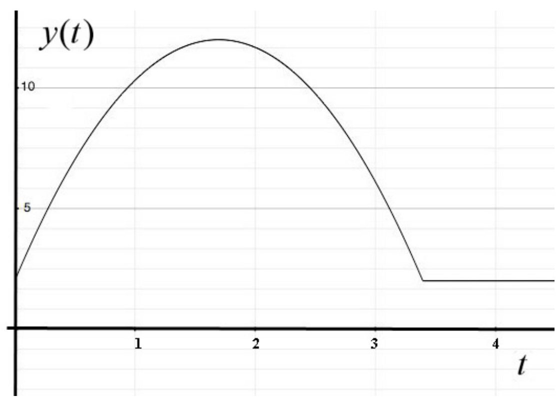


Figure 6.18: Plot of the y-component of the position as a function of time

There are several important things to notice about Figures 6.17 and 6.18. The first point is that the abscissa axes are different in both figures, Figure 6.17 is a plot of y vs. x and Figure 6.18 is a plot of y vs. t. The second thing to notice is that at $t = 0$, the slope of the graph in Figure 6.18 is equal to Let $t = t_{top}$ correspond to the instant the stone is at its maximal vertical position. or the highest point in the flight. The final thing to notice about Figure 6.18 is that at $t = t_{top}$ the slope is zero or $v_y(t = t_{top}) = 0$. Therefore

$$v_y(t_{top}) = v_0 \sin\theta_0 - gt_{top} = 0.$$

We can solve this equation for t_{top},

$$t_{top} = \frac{v_0 \sin\theta_0}{g} = \frac{(20\ \text{m}\cdot\text{s}^{-1})\sin(45^\circ)}{9.8\ \ \text{m}\cdot\text{s}^{-2}} = 1.44\ \text{s}\,.$$

The y-component of the velocity as a function of time is graphed in Figure 6.19.

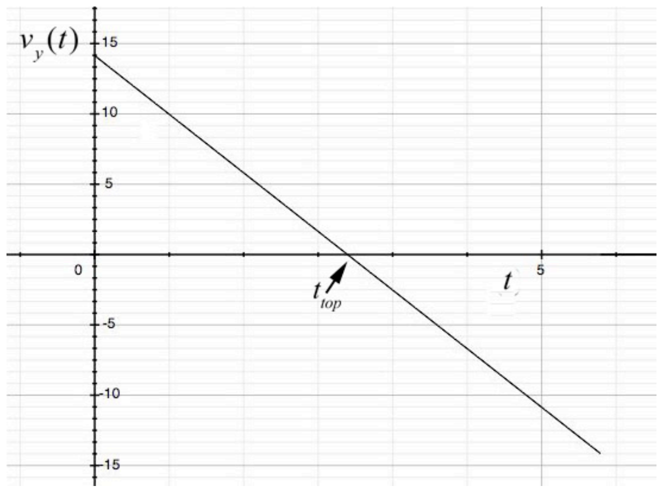


Figure 6.19: y-component of the velocity as a function of time

Notice that at $t = 0$ the intercept is positive indicting the initial y-component of the velocity is positive which means that the stone was thrown upwards. The y-component of the velocity changes sign at $t = t_{top}$ indicating that it is reversing its direction and starting to move downwards.

We can now use Eq. (6.3.8) to find the maximal height of the stone above the ground

$$\begin{aligned} y(t = t_{top}) &= d + v_0 \sin\theta_0 \frac{v_0 \sin\theta_0}{g} - \frac{1}{2} g \left(\frac{v_0 \sin\theta_0}{g} \right)^2 \\ &= d + \frac{{v_0}^2 \sin^2\theta_0}{2g} = 2\ \text{m} + \frac{(20\ \text{m}\cdot\text{s}^{-1})^2 \sin^2(45^\circ)}{2(9.8\ \ \text{m}\cdot\text{s}^{-2})} = 12.2\ \text{m} \end{aligned}, \qquad (6.3.16)$$

Orbit equation

So far our description of the motion has emphasized the independence of the spatial dimensions, treating all of the kinematic quantities as functions of time. We shall now eliminate time from our equation and find the *orbit equation* of the body. We begin with Equation (6.3.6) for the x-component of the position,

$$x(t) = x_0 + v_{x,0}\, t \qquad (6.3.17)$$

and solve Equation (6.3.17) for time t as a function of $x(t)$,

$$t = \frac{x(t) - x_0}{v_{x,0}}. \qquad (6.3.18)$$

The vertical position of the body is given by Equation (6.3.8),

$$y(t) = y_0 + v_{y,0}\, t - \frac{1}{2} g t^2. \qquad (6.3.19)$$

We then substitute the above expression, Equation (6.3.18) for time t into our equation for the y-component of the position yielding

$$y(t) = y_0 + v_{y,0}\left(\frac{x(t) - x_0}{v_{x,0}}\right) - \frac{1}{2} g \left(\frac{x(t) - x_0}{v_{x,0}}\right)^2. \qquad (6.3.20)$$

This expression can be simplified to give

$$y(t) = y_0 + \frac{v_{y,0}}{v_{x,0}}\left(x(t) - x_0\right) - \frac{1}{2}\frac{g}{v_{x,0}^2}\left(x(t)^2 - 2x(t)x_0 + x_0^2\right). \qquad (6.3.21)$$

This is seen to be an equation for a parabola by rearranging terms to find

$$y(t) = -\frac{1}{2}\frac{g}{v_{x,0}^2} x(t)^2 + \left(\frac{g x_0}{v_{x,0}^2} + \frac{v_{y,0}}{v_{x,0}}\right) x(t) - \frac{v_{y,0}}{v_{x,0}} x_0 - \frac{1}{2}\frac{g}{v_{x,0}^2} x_0^2 + y_0. \qquad (6.3.22)$$

The graph of $y(t)$ as a function of $x(t)$ is shown in Figure 6.20.

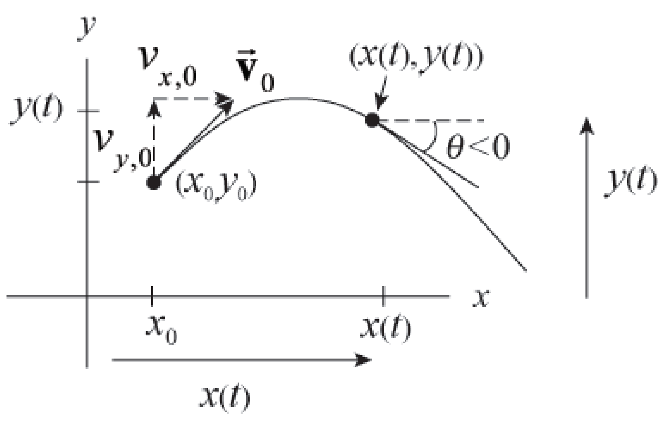


Figure 6.20 The parabolic orbit

Note that at any point $\left(x(t), y(t)\right)$ along the parabolic trajectory, the direction of the tangent line at that point makes an angle θ with the positive x-axis as shown in Figure 6.20. This angle is given by

$$\theta = \tan^{-1}\left(\frac{dy}{dx}\right), \tag{6.3.23}$$

where dy/dx is the derivative of the function $y(x) = y(x(t))$ at the point $\left(x(t), y(t)\right)$.

The velocity vector is given by

$$\vec{\mathbf{v}}(t) = \frac{dx(t)}{dt}\ \hat{\mathbf{i}} + \frac{dy(t)}{dt}\ \hat{\mathbf{j}} \equiv v_x(t)\ \hat{\mathbf{i}} + v_y(t)\ \hat{\mathbf{j}} \tag{6.3.24}$$

The direction of the velocity vector at a point $\left(x(t), y(t)\right)$ can be determined from the components. Let ϕ be the angle that the velocity vector forms with respect to the positive x-axis. Then

$$\phi = \tan^{-1}\left(\frac{v_y(t)}{v_x(t)}\right) = \tan^{-1}\left(\frac{dy/dt}{dx/dt}\right) = \tan^{-1}\left(\frac{dy}{dx}\right). \tag{6.3.25}$$

Comparing our two expressions we see that $\phi = \theta$; the slope of the graph of $y(t)$ *vs*. $x(t)$ at any point determines the direction of the velocity at that point. We cannot tell from our graph of $y(x)$ how fast the body moves along the curve; the magnitude of the velocity cannot be determined from information about the tangent line.

If, as in Figure 6.16, we choose our origin at the initial position of the body at $t = 0$, then $x_0 = 0$ and $y_0 = 0$. Our orbit equation, Equation (6.3.22) can now be simplified to

$$y(t) = -\frac{1}{2}\frac{g}{v_{x,0}^2}x(t)^2 + \frac{v_{y,0}}{v_{x,0}}x(t). \tag{6.3.26}$$

Example: Hitting the Bucket

A person is standing on a ladder holding a pail. The person releases the pail from rest at a height h_1 above the ground. A second person standing a horizontal distance s_2 from the pail aims and throws a ball the instant the pail is released in order to hit the pail. The person releases the ball at a height h_2 above the ground, with an initial speed v_0, and at an angle θ_0 with respect to the horizontal. You may ignore air resistance.

a) Find an expression for the angle θ_0 that the person aims the ball in order to hit the pail.

b) Find an expression for the height above the ground where the collision occurred as a function of the initial speed of the ball v_0, and the quantities h_1, h_2, and s_2.

I. Understand – get a conceptual grasp of the problem

There are two objects involved in this problem. Each object is undergoing free fall, so there is only one stage each. The pail is undergoing one dimensional motion. The ball is undergoing two dimensional motion. The parameters h_1, h_2, and s_2 are unspecified, so our answers will be functions of those symbolic expressions for the quantities. Figure 6.21 shows a sketch of the motion of all the bodies in this problem.

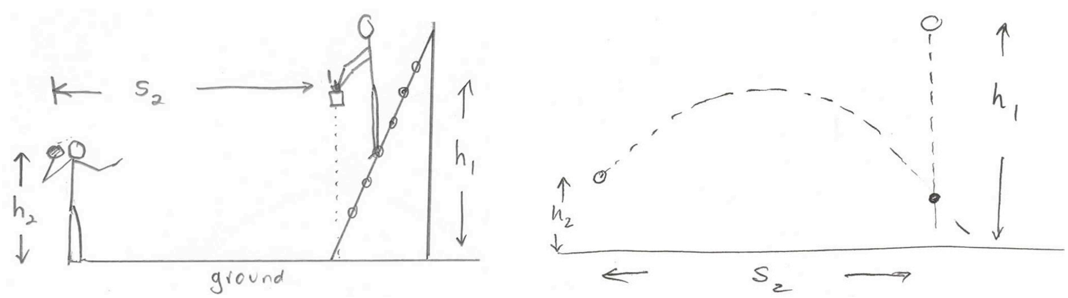


Figure 6.21: Sketch of motion.

Since the acceleration is unidirectional and constant, we will choose Cartesian coordinates, with one axis along the direction of acceleration. Choose the origin on the ground directly underneath the point where the ball is released. We choose upwards for the positive y-direction and towards the pail for the positive x-direction.

We choose position coordinates for the pail as follows. The horizontal coordinate is constant and given by $x_1 = s_2$. The vertical coordinate represents the height above the ground and is denoted by $y_1(t)$. The ball has coordinates $(x_2(t), y_2(t))$. We show these coordinates in the Figure 6.22.

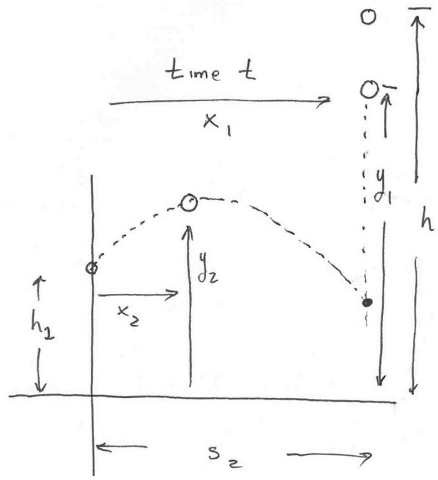


Figure 6.22: Coordinate System

II. Devise a Plan - set up a procedure to obtain the desired solution

Find an expression for the angle θ_0 that the person throws the ball as a function of h_1, h_2, and s_2.

Find an expression for the time of collision as a function of the initial speed of the ball v_0, and the quantities h_1, h_2, and s_2.

Find an expression for the height above the ground where the collision occurred as a function of the initial speed of the ball v_0, and the quantities h_1, h_2, and s_2.

Model: The pail undergoes constant acceleration $(a_y)_1 = -g$ in the vertical direction downwards and the ball undergoes uniform motion in the horizontal direction and constant acceleration downwards in the vertical direction, with $(a_x)_2 = 0$ and $(a_y)_2 = -g$.

Equations of Motion for Pail:

The initial conditions for the pail are $(v_{y,0})_1 = 0$, $x_1 = s_2$, $(y_0)_1 = h_1$. Since the pail moves vertically, the pail always satisfies the constraint condition $x_1 = s_2$ and $v_{x,1} = 0$. The equations for position and velocity of the pail simplify to

$$y_1(t) = h_1 - \frac{1}{2}gt^2 \tag{6.3.27}$$

$$v_{y,1}(t) = -gt \tag{6.3.28}$$

Equations of Motion for Ball:

The initial position is given by $(x_0)_2 = 0$, $(y_0)_2 = h_2$. The components of the initial velocity are given by $(v_{y,0})_2 = v_0 \sin(\theta_0)$ and $(v_{x,0})_2 = v_0 \cos(\theta_0)$, where v_0 is the magnitude of the initial velocity and θ_0 is the initial angle with respect to the horizontal. So the equations for position and velocity of the ball simplify to

$$x_2(t) = v_0 \cos(\theta_0) t \tag{6.3.29}$$

$$v_{x,2}(t) = v_0 \cos(\theta_0) \tag{6.3.30}$$

$$y_2(t) = h_2 + v_0 \sin(\theta_0) t - \frac{1}{2} g t^2 \tag{6.3.31}$$

$$v_{y,2}(t) = v_0 \sin(\theta_0) - gt \tag{6.3.32}$$

Note that the quantities h_1, h_2, and s_2 should be treated as known quantities although no numerical values were given, only symbolic expressions. There are six independent equations with 9 as yet unspecified quantities $y_1(t)$, t, $y_2(t)$, $x_2(t)$, $v_{y,1}(t)$, $v_{y,2}(t)$, $v_{x,2}(t)$, v_0, θ_0.

So we need two more conditions, in order to find expressions for the initial angle, θ_0, the time of collision, t_a, and the spatial location of the collision point specified by $y_1(t_a)$ or $y_2(t_a)$ in terms of the one unspecified parameter v_0. At the collision time $t = t_a$, the collision occurs when the two balls are located at the same position. Therefore

$$y_1(t_a) = y_2(t_a) \tag{6.3.33}$$

$$x_2(t_a) = x_1 = s_2 \tag{6.3.34}$$

We shall now apply these conditions that must be satisfied in order for the ball to hit the pail.

$$h_1 - \frac{1}{2} g t_a^{\ 2} = h_2 + v_0 \sin\left(\theta_0\right) t_a - \frac{1}{2} g t_a^{\ 2} \tag{6.3.35}$$

$$s_2 = v_0 \cos(\theta_0) t_a \tag{6.3.36}$$

From the first equation, the term $(1/2) g t_a^{\ 2}$ cancels from both sides. Therefore we have that

$$h_1 = h_2 + v_0 \sin(\theta_0) t_a$$

$$s_2 = v_0 \cos(\theta_0) t_a \, .$$

We can now solve these equations for $\tan(\theta_0) = \sin(\theta_0)/\cos(\theta_0)$, and thus the angle the person throws the ball in order to hit the pail.

III. Carry our your plan – solve the problem!

We rewrite these equations as

$$v_0 \sin(\theta_0) t_a = h_1 - h_2 \qquad (6.3.37)$$

$$v_0 \cos(\theta_0) t_a = s_2 \qquad (6.3.38)$$

Dividing these equations yields

$$\frac{v_0 \sin(\theta_0) t_a}{v_0 \cos(\theta_0) t_a} = \tan(\theta_0) = \frac{h_1 - h_2}{s_2} \qquad (6.3.39)$$

So the initial angle is independent of v_0, and is given by

$$\theta_0 = \tan^{-1}\left(\left(h_1 - h_2 \right) / s_2 \right) \qquad (6.3.40)$$

From the Figure 6.23 below we can see that $\tan(\theta_0) = (h_1 - h_2)/s_2$, implies that the second person aims the ball at the initial position of the pail.

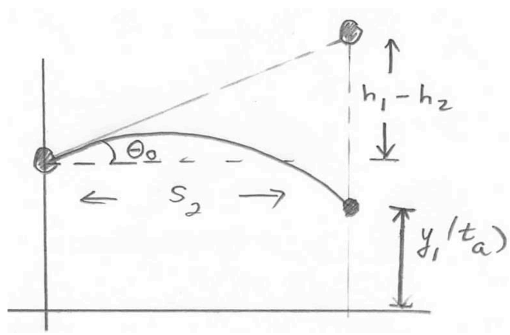


Figure 6.23: Geometry of collision

In order to find the time that the ball collides with the pail, we begin by squaring both Eqs. (6.3.37) and (6.3.38)

We square both of the equations above and utilize the trigonometric identity

$$\sin^2(\theta_0) + \cos^2(\theta_0) = 1 .$$

So our squared equations become

$$v_0{}^2 \sin^2(\theta_0) t_a{}^2 = \left(h_1 - h_2\right)^2 \tag{6.3.41}$$

$$v_0{}^2 \cos^2(\theta_0) t_a{}^2 = s_2{}^2 \tag{6.3.42}$$

Adding these equations together yields

$$v_0{}^2 (\sin^2(\theta_0) + \cos^2(\theta_0)) t_a{}^2 = s_2{}^2 + \left(h_1 - h_2\right)^2 \tag{6.3.43}$$

We now utilize the trigonometric identity

$$\sin^2(\theta_0) + \cos^2(\theta_0) = 1 .$$

So Eq. (6.3.43) becomes

$$v_0{}^2 t_a{}^2 = s_2{}^2 + \left(h_1 - h_2\right)^2 \tag{6.3.44}$$

which we can solve this for the time of collision

$$t_a = \left(\frac{s_2{}^2 + \left(h_1 - h_2\right)^2}{v_0{}^2} \right)^{1/2} \tag{6.3.45}$$

We can now use the y-coordinate function of either the ball or the pail at $t = t_a$ to find the height that the ball collides with the pail. Since it had no initial y velocity, it's easier to use the pail,

$$y_1(t_a) = h_1 - \frac{g\left(s_2{}^2 + \left(h_1 - h_2\right)^2\right)}{2v_0{}^2} \tag{6.3.46}$$

IV. Look Back – check your solution and method of solution

The person aims at the pail at the point where the pail was released. Both undergo free fall so the key result was that the vertical position obeys

$$h_1 - \frac{1}{2} g {t_a}^2 = h_2 + v_0 \sin\left(\theta_0\right) t_a - \frac{1}{2} g {t_a}^2 .$$

The distance traveled due to gravitational acceleration are the same for both so all that matters is the contribution form the initial positions and the vertical component of velocity

$$h_1 = h_2 + v_0 \sin\left(\theta_0\right) t_a .$$

Since the time is related to the horizontal distance by

$$s_2 = v_0 \cos\left(\theta_0\right) t_a$$

This is now as if both objects were moving at constant velocity.