PROFESSOR: I now want to do a classic problem of hitting a billiard ball and then see what's going to happen.

This is a frictionless table, which is a little bit artificial. And here is the billiard ball. The billiard ball has a mass M. The mass is uniformly distributed throughout. And it has a radius r. Let this be the center of the billiard ball C, so this radius in r. And there is no fiction.

I hit the ball. I give it an impulse. I hit it here with an impulse, which is given. Let's say j. And the impulse is given at a distance h above that point C. We will also evaluate what happens when we change h. We can even make h negative. If I call this positive, I could call this negative h. And h0 would be an impulse right through the point C.

You expect-- and your intuition is correct-- that if you give it this impulse that the ball will go to the right, of course. And that it will start to rotate, at least it will rotate in this direction if h is positive. If h is negative it will rotate counterclockwise. And we will evaluate that in great detail.

The hit is parallel to this surface, and this-- picture what you see here is a cross section through the center of this billiard ball. The impulse is very short and very brief.

The question always, what is the angular velocity? If you hit it above point C it will start to rotate in this direction. The angular velocity omega about the center of mass, c is the center of mass. What is that? And, you also want to know what is the velocity of the center of mass itself? Before the impulse, the ball is at rest. And after the impulse, the center of mass will have a velocity v center of mass. And there will be a clockwise rotation or there will be a counterclockwise rotation depending upon where you apply this impulse.

Now before you watch this segment, I would advise you to first visit a problem in which I have a rod on a frictionless horizontal table. And I apply an impulse to that rod. The problem is quite similar, and all the tools that I'm developing there, I'm going to use here. So I would strongly advise you to first look at that segment. In fact, there are even two segments about how I solve the problem of hitting the rod on that frictionless horizontal table. The rod will also start to move. The center of mass will have a velocity. And it will also start to rotate about center of mass. So there are lots of similarities.

All right, let's first start with Newton's first law. F equals M times a of the center of mass. That equals M times dv dt. It's a vector of the center of mass.

I take the integral dt. I take the integral dt. So I do a time integral this part and of that part.

So what do I get? I get that the integral from an initial time to a final time-- this is the time interval that the force lasts. The force doesn't have to be constant in time. F dt equals the integral over that same time interval, t initial to t final of M times dv center of mass. This is the definition of impulse. As I said earlier, the force doesn't have to be constant. It could change in time, and so the acceleration could change in time of the center of mass. But the integral over that time interval t initial to t final of F dt, that's the definition of impulse.

So this is j. And this equals M times the final velocity minus the initial velocity of the center of mass. The initial velocity is 0 because your object is not moving before I give the impulse. So you see, you get a very simple result and that is that v of the center of mass equals the impulse divided by M. So if you know the impulse and you know the mass of the ball, then you know the velocity of the center of mass. Notice the velocity of the center of mass is exactly in the same direction of j. That is sort of intuitive, isn't it?

Well, it could probably be in this direction, could it? Because it wouldn't jump off the table. Maybe if you hit it like this, yeah, it could jump off the table. All right, this is very intuitive. And so the magnitude of v center of mass equals j divided by M. Keep in mind that they are vectors in this direction and j is in this direction. Notice that the velocity of the center of mass is completely independent of h. That may not be so intuitive. In other words, it doesn't matter where you hit this billiard ball. Whether h is the maximum value, which could be r or whether h is smaller than r or whether h is even 0. You could hit it right here at the center. Or whether h is below the center of mass. In which case, it will start to spin counterclockwise. The center of mass velocity is always the same for a given value of the impulse. And that is not so intuitive. At least not for me.

Let's now calculate the angular velocity omega about the center of mass. I want to remind you of an important theorem, which you can prove if you want to, which is the following-- and I will write it out. The total angular momentum relative to a point, and let me call that point P Remember angular momentum, in general, is defined as relative to a point. You can choose different points P and you may get different angular momentums depending upon where you choose the point P. So the total angular momentum relative to a chosen point P is the sum of two components. Component number one is the angular momentum of the center of mass relative to P. And the second component is the angular momentum of rotation about the center of mass. So this portion is always taken about the center of mass. So about point C. But here you can choose P anywhere you want to. You can even choose P to be point C if you want that.

Now, I am therefore, free to choose any point that I want to choose in space. I could choose this point, I could choose this point, I could choose this point or this point. I will choose now a point, any point on this line through C parallel to j, and I call that the line DCE. So that's where I will choose a point.

I could've picked any point and whatever follows will then be almost the same. I will choose point D. So I have to evaluate now the component number one, which is the angular momentum of the center of mass relative to point P.

The angular momentum relative to point D-- D is my choice-- equals the position vector relative to point D cross product with the momentum of the center of mass because it's the angular momentum of the center of mass relative to that point, and my point is D. And that I claim is 0.

If you choose point D, the position vector goes from D to the center of mass. That would be r of D. That is this r of D. That is the angle that it makes with v center of mass is 0. So this cross product equals 0. Because P and the velocity of the center of mass are in the same direction. This is M, capital M, times v center of mass.

If I had chosen point E, then the angle between the position vector and the momentum of the center of mass would have been 180 degrees. So this cross product would also have been 0. So it's clear that this component is 0 if I choose the point that I've chosen. If I choose any point on this line, the angular momentum of the center of mass relative to any point on this line is 0. So that's why it may be beneficial to choose such a point. Because that contribution 1 will then completely disappear. And I think it would probably be the fastest way to solve this problem. But you don't have to choose a point on the line DCE. You don't have to.

All right, so this component is not there. So now we have to go to the second component, which is the angular momentum of rotation about the center of mass. So we give the impulse before we give the the angular momentum about the center of mass is 0. And after the impulse there is an angular momentum about the center of mass C. At any moment in time, there will be a force acting in this direction during the time that this force lasts, which is from the initial time to the final time, and the time integral of F dt will be the impulse. And at any moment in time I have here the position vector, which I will call r of C. And this distance here we have called h. We will call h positive in this direction and we'll call h negative in this direction.

So, recall that dL dt, and I now have to choose point C because I have to calculate the angular momentum of rotation about the center of mass. That equals the torque relative to point C. And the torque relative to point C is the cross product between the position vector and the force. So that is r of C cross F.

I now do a time integral. Integral of this dt, integral of this dt. So I'm going to get t initial to t final of dL relative to point C equals the integral t initial to t final of rC cross F dt.

What is this integral? Well, this integral is L after the impact. That means that time t final minus L before the impact that was at time tI, t initial, and this is 0. So that equals now this integral. You can take r of C outside the bracket if you want to think of it that way. The integral t initial t final of F dt, that integral pe definition is J. That's the impulse. So this integral would be r of C cross J. So we now have a very simple result that this vector angular momentum about the center of mass-- going to write a C here-- equals rC cross J. The magnitude of the angular momentum about the center of mass is the moment of inertia about the center of mass times the angular velocity after the impulse has been given. And the magnitude of this quantity-- and I want you to think of that yourself. r of C cross J. J is also in this direction. That is going to be h times J. And you work that out for yourself.

So I get h times J. This is the magnitude of this vector. And this is the magnitude of this vector. The direction of this vector would be in the paper if h were larger than 0. The ball would start to rotate clockwise and therefore, the angular momentum and also this cross product is in the paper. That vector would be out of the paper if h were less than 0.

If I hit the ball here, it would start to rotate in this direction. And if it rotates in this direction, the angular momentum vector about point C is up. So these are the magnitudes and here you have the directions depending upon h.

The moment of inertia about point C of a solid sphere, a uniform mass distribution, equals 2/5 M R squared.

If I substitute that in here, I find immediately that omega equals hJ divided by I center of mass. And that becomes 5 times h times J divided by 2M R squared. And this is my answer for the angular velocity after the impulse. Notice that omega does depends on h, whereas the center of mass velocity did not depend on h. It's linearly proportional with h. There's a maximum value that omega can have, which is when h becomes R. Notice also that when h equals 0 that omega equals 0. That is kind of intuitive, isn't it? Because if I hit that ball right through the center, you wouldn't expect any rotation. That's quite intuitive. So if I hit it right here through C, you wouldn't expect this to rotate. You would only expect the ball to translate. The center of mass would have a translation, but there would be no rotation. If h is positive we get a clockwise rotation. Omega becomes positive.

If h is negative, if I'm below C, then think of omega as being negative, which would give me a counterclockwise rotation.

Now if there is no fiction the motion will continue forever. In other words, if I hit it here I would get a velocity of the center of mass. It would continue forever and ever. And the rotation counterclockwise, the omega that I've calculated here would also last forever and ever. However, in practice, if you do this on a billiard table, then of course, what you will see is very, very different. Because there is friction there. There better be fiction. That's the whole idea. And therefore, perhaps you have seen this if you hit a billiard ball on the table quite low, then the ball will first, go away from you.

You hit it here quite low. The ball will go away from you. It will go in the direction that yo hit it. It will start to spin in this direction and you shouldn't be surprised if later in time because of the friction, it starts rolling back at you. Well, we will do some separate problems on how to work them with this roll. But you can see how that happens because you give it a spin, a counterclockwise spin, and you move it away from you. And because of the friction, the ball will then, at a later time, may come back at you.

All right, there is a particular value of h for which the ball goes what me call into pure roll.

What is pure roll? I have a segment in which I discuss pure roll, so I won't go into that right now. I'll just give you the mathematical result. And that is that when the velocity of the center of mass equals the angular velocity about the center of mass times r, then and only then is the object in pure roll. These two don't have to be the same at all. Omega could be 0 and the center of mass is not 0-- the velocity of the center of mass.

If I hit the ball right in the center, omega would be 0 and v center of mass is not 0.

We'd argue that the velocity is independent of where the impulse is applied. So this would not hold in general. This only holds for one and only one point where I hit on the ball. And you can find that by substituting in here for omega, you substitute v center of mass divided by r. So when I do that I get v center of mass divided by r equals 5h J divided by 2M R squared. I lose one R and v center of mass was J divided by M. These R's are gone. Then this is correct. And so now I can lose the J and I can lose the M. And so what do I find? That when h equals 2/5 R, and only then would the ball go into pure roll. In other words, I have to hit the ball at a distance h. We have the distance h here, whereby this is 2/5 R. Then the ball will go into pure roll.

If h is larger than 2/5 R, then the velocity of the center of mass will be less than omega R. And if h is smaller than 2/5 R, but let's take for now larger than 0. So it will be above point C. Then, the velocity of the center of mass will be larger than omega R.

Now in solving this problem because I solved for the velocity of the center of mass and I solved for the angular velocity about the center of mass, I chose a point on the line DCE. You could have chosen a point on the line through the impulse. Let me get that. It's here on the floor.

I could have chosen a point anywhere on this line. The remarkable thing is, if I had chosen any point P on this line, that the total angular momentum, which before the impulse is 0, will also be 0 now after the impulse. You may say, well that's strange because in the way you solved the problem before, the angular momentum was not 0 after the impulse. That's true because I chose point C.

If I choose another point-- well, for instance, a point on the line through the impulse. Then, the angular momentum does not change. Angular momentum is now conserved. It is 0 before and it is 0 after the impulse. You will see that often in problems. That depending upon which point you choose, angular momentum may be conserved or angular momentum may not be conserved.

What is the advantage of choosing a point on the line DCE? Well, there is that contribution 1 that I mark with a 1 will be 0.

If you choose a point through the-- anywhere on the line through the impulse, then the total angular momentum is 0. But then, contribution 1 will not be 0 and contribution 2 will not be 0. But the net must be 0. So contribution 1 and contribution 2, which are both vectors, must exactly cancel each other out. One most therefore be in the paper and the other must be out of the paper.

I want you to leave with that idea and I want you to solve the problem by choosing a point P anywhere through the line through J. Through the impulse. And you should get the same answer for the velocity of the center of mass, and you should get the same answer for the angular velocity after you apply the impulse. What will help you is to look at a segment that I have taped of the rod on a frictionless horizontal surface. Which I give the hit, an impulse. It will start to rotate and there be a translation of the center of mass. And I solve that in two different ways. In one clip I solve it by choosing a point on the line through the center parallel to the impulse. And I also solve it independently by choosing a point on the line through the impulse. And that is very similar to what I'm now suggesting to you. So the total angular momentum will be conserved before the impulse at 0 and after the impulse at 0. Very interesting. I want you to think about it. Angular momentum in this case is not an intrinsic property of the motion. Angular momentum depends, in this case, on which point you choose. Good luck.