Module 17: Systems, Conservation of Momentum and Center of Mass

17.1 External and Internal Forces and the Change in Momentum of a System

So far we have restricted ourselves to considering how the momentum of an object changes under the action of a force. For example, if we analyze in detail the forces acting on the cart rolling down the inclined plane (Figure 17.1), we determine that there are three forces acting on the cart: the force $\vec{\mathbf{F}}_{\text{cart, spring}}$ the spring applies to the cart; the gravitational interaction $\vec{\mathbf{F}}_{\text{cart, earth}}$ between the cart and the earth; and the contact force $\vec{\mathbf{F}}_{\text{cart, plane}}$ between the inclined plane and the cart. If we define the cart as our *system*, then everything else acts as the *surroundings*. We illustrate this division of system and surroundings in Figure 17.1.

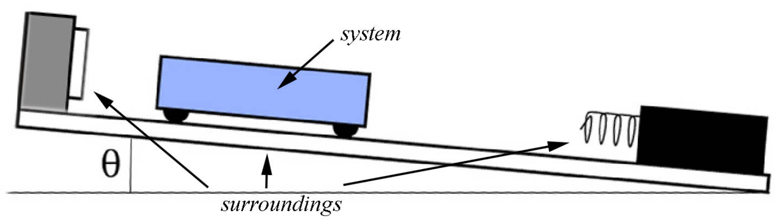


Figure 17.1 A diagram of a cart as a system and its surroundings

The forces acting on the cart are *external* forces. We refer to the vector sum of these external forces that are applied to the system (the cart) as the total external force,

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \vec{\mathbf{F}}_{\text{cart, spring}} + \vec{\mathbf{F}}_{\text{cart, earth}} + \vec{\mathbf{F}}_{\text{cart, plane}}. \tag{17.1.1}$$

Then Newton's Second Law applied to the cart, in terms of impulse, is

$$\Delta\vec{\mathbf{p}}_{\text{system}} = \int_{t_0}^{t_f} \vec{\mathbf{F}}_{\text{ext}}^{\text{total}}\, dt \equiv \vec{\mathbf{I}}_{\text{system}}. \tag{17.1.2}$$

Let's extend our system to two interacting objects, for example the cart and the spring. The forces between the spring and cart are now *internal* forces, $\vec{\mathbf{F}}_{\text{int}}$. Both objects, the cart and the spring, experience these internal forces, which by Newton's Third Law are equal in magnitude and applied in opposite directions. So when we sum up the internal forces for the whole system, they cancel. Thus the total internal force is always zero,

$$\vec{\mathbf{F}}_{\text{int}}^{\text{total}} = \vec{\mathbf{0}}. \tag{17.1.3}$$

External forces are still acting on our system; the gravitational force, the contact force between the inclined plane and the cart, and also a new external force, the force between the spring and the force sensor. The total force acting on the system is the sum of the internal and the external forces. However, as we have shown, the internal forces cancel, so we have that

$$\vec{\mathbf{F}}^{\text{total}} = \vec{\mathbf{F}}^{\text{total}}_{\text{ext}} + \vec{\mathbf{F}}^{\text{total}}_{\text{int}} = \vec{\mathbf{F}}^{\text{total}}_{\text{ext}}. \tag{17.1.4}$$

System of Particles

Suppose we have a system of N particles labeled by the index $i = 1, 2, 3, \ldots, N$. The total force on the i^{th} particle is

$$\vec{\mathbf{F}}_i^{\text{total}} = \vec{\mathbf{F}}^{\text{total}}_{\text{ext},i} + \sum_{j=1,\, j\neq i}^{j=N} \vec{\mathbf{F}}_{i,j}. \tag{17.1.5}$$

In this expression $\vec{\mathbf{F}}_{i,j}$ is the force on the i^{th} particle due to the interaction between the i^{th} and j^{th} particles. We sum over all j particles with $j \neq i$ since a particle cannot exert a force on itself (equivalently, we could define $\vec{\mathbf{F}}_{i,i} = \vec{\mathbf{0}}$), yielding the total internal force acting on the i^{th} particle,

$$\vec{\mathbf{F}}^{\text{total}}_{\text{int},i} = \sum_{j=1,\, j\neq i}^{j=N} \vec{\mathbf{F}}_{i,j}. \tag{17.1.6}$$

The total force acting on the system is the sum over all i particles of the total force acting on each particle,

$$\vec{\mathbf{F}}^{\text{total}} = \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i^{\text{total}} = \sum_{i=1}^{i=N} \vec{\mathbf{F}}^{\text{total}}_{\text{ext},i} + \sum_{i=1}^{i=N} \sum_{j=1,\, j\neq i}^{j=N} \vec{\mathbf{F}}_{i,j} = \vec{\mathbf{F}}^{\text{total}}_{\text{ext}}. \tag{17.1.7}$$

Note that the double sum vanishes,

$$\sum_{i=1}^{i=N} \sum_{j=1,\, j\neq i}^{j=N} \vec{\mathbf{F}}_{i,j} = \vec{\mathbf{0}}, \tag{17.1.8}$$

because all internal forces cancel in pairs,

$$\vec{\mathbf{F}}_{i,j} = -\vec{\mathbf{F}}_{j,i}. \tag{17.1.9}$$

With the assumption that the mass of the i^{th} particle does not change, the total force on the i^{th} particle is equal to the rate of change in momentum of the i^{th} particle,

$$\vec{\mathbf{F}}_i^{\text{total}} = m_i \frac{d\,\vec{\mathbf{v}}_i}{dt} = \frac{d\,\vec{\mathbf{p}}_i}{dt}. \qquad (17.1.10)$$

When can now substitute Equation (17.1.10) into Equation (17.1.7) and determine that that the total external force is equal to the sum over all particles of the momentum change of each particle,

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \sum_{i=1}^{i=N} \frac{d\,\vec{\mathbf{p}}_i}{dt}. \qquad (17.1.11)$$

The total momentum of the system is given by the sum

$$\vec{\mathbf{p}}_{\text{system}} = \sum_{i=1}^{i=N} \vec{\mathbf{p}}_i\,; \qquad (17.1.12)$$

momenta add as vectors.

We conclude that the total external force causes the total momentum of the system to change, and we thus restate and generalize Newton's Second Law for a system of objects as

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \frac{d\,\vec{\mathbf{p}}_{\text{system}}}{dt}. \qquad (17.1.13)$$

In terms of impulse, this becomes the statement

$$\Delta\vec{\mathbf{p}}_{\text{system}} = \int_{t_0}^{t_f} \vec{\mathbf{F}}_{\text{ext}}^{\text{total}} dt \equiv \vec{\mathbf{I}}. \qquad (17.1.14)$$

17.2 Center of Mass

Consider two point-like particles with masses m_1 and m_2. Choose a coordinate system with a choice of origin such that body 1 has position $\vec{\mathbf{r}}_1$ and body 2 has position $\vec{\mathbf{r}}_2$ (Figure 17.2).

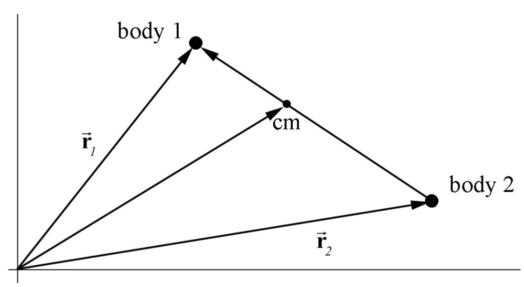


Figure 17.2 Center of mass coordinate system.

The center of mass vector, $\vec{\mathbf{R}}_{\text{cm}}$, of the two-body system is defined as

$$\vec{\mathbf{R}}_{\text{cm}} = \frac{m_1 \vec{\mathbf{r}}_1 + m_2 \vec{\mathbf{r}}_2}{m_1 + m_2}. \tag{17.2.1}$$

We shall now extend the concept of the center of mass to more general systems. Suppose we have a system of N particles labeled by the index $i = 1, 2, 3, \ldots, N$. Choose a coordinate system and denote the position of the i^{th} particle as $\vec{\mathbf{r}}_i$. The total mass of the system is given by the sum

$$m^{\text{total}} = \sum_{i=1}^{i=N} m_i \tag{17.2.2}$$

and the position of the center of mass of the system of particles is given by

$$\vec{\mathbf{R}}_{\text{cm}} = \frac{1}{m^{\text{total}}} \sum_{i=1}^{i=N} m_i \vec{\mathbf{r}}_i . \tag{17.2.3}$$

(For a continuous rigid body, each point-like particle has mass dm and is located at the position $\vec{\mathbf{r}}'$. The center of mass is then defined as an integral over the body,

$$\vec{\mathbf{R}}_{\text{cm}} = \frac{\int_{\text{body}} dm\, \vec{\mathbf{r}}'}{\int_{\text{body}} dm}. \tag{17.2.4}$$

In Chapter 13 we will learn how to specifically calculate the above integrals. For our discussion in this chapter, we will only consider finite sums, or extended objects where the center of mass is easily determined.)

Translational Motion of the Center of Mass

The velocity of the center of mass is found by differentiation,

$$\vec{\mathbf{V}}_{\text{cm}} = \frac{1}{m^{\text{total}}} \sum_{i=1}^{i=N} m_i \vec{\mathbf{v}}_i = \frac{\vec{\mathbf{p}}^{\text{total}}}{m^{\text{total}}} . \tag{17.2.5}$$

The total momentum is then expressed in terms of the velocity of the center of mass by

$$\vec{\mathbf{p}}^{\text{total}} = m^{\text{total}} \vec{\mathbf{V}}_{\text{cm}}. \tag{17.2.6}$$

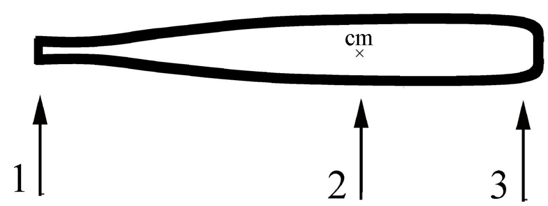
We have already determined that the total external force is equal to the change of the total momentum of the system (Equation (17.1.13)). If we now substitute Equation (17.2.6) into Equation (17.1.13), and continue with our assumption of constant masses m_i, we have that

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \frac{d\,\vec{\mathbf{p}}^{\text{total}}}{dt} = m^{\text{total}} \frac{d\,\vec{\mathbf{V}}_{\text{cm}}}{dt} = m^{\text{total}} \vec{\mathbf{A}}_{\text{cm}}, \tag{17.2.7}$$

where $\vec{\mathbf{A}}_{\text{cm}}$, the derivative with respect to time of $\vec{\mathbf{V}}_{\text{cm}}$, is the acceleration of the center of mass.

From Equation (17.2.7) we can conclude that in considering the linear motion of the center of mass, the sum of the external forces may be regarded as acting at the center of mass.

Concept Question 17.2.1: Suppose you push a baseball bat lying on a nearly frictionless table at the center of mass (position 2) with a force $\vec{\mathbf{F}}_{\text{ext}}$. Will the acceleration of the center of mass be *greater than, equal to, or less than* if you push the bat with the same force at either end (position 2 and 3)?



Answer: The acceleration of the center of mass will be equal in the three cases. From our previous discussion, (Equation (17.2.7)), the acceleration of the center of mass is independent of where the force is applied. However, the bat undergoes a very different motion if we apply the force at one end or at the center of mass. When we apply the force at the center of mass all the particles in the baseball bat will undergo linear motion. When we push the bat at one end, the particles that make up the baseball bat will no longer undergo a linear motion even though the center of mass undergoes linear motion. In fact, each particle will rotate about the center of mass of the bat while the center of mass of the bat accelerates in the direction of the applied force.

17.3 Constancy of Momentum and Isolated Systems

Suppose we now completely isolate our system from the surroundings. This means that the total external force acting on the system is zero,

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \vec{\mathbf{0}}. \tag{17.3.1}$$

Then the change in the total momentum of the system is zero,

$$\Delta\vec{\mathbf{p}}_{\text{system}} = \vec{\mathbf{0}}; \tag{17.3.2}$$

the total momentum of the closed (isolated) system is constant. The initial momentum of our system is the sum of the initial momentum of the individual particles,

$$\vec{\mathbf{p}}_0^{\text{total}} = m_1\vec{\mathbf{v}}_{1,0} + m_2\vec{\mathbf{v}}_{2,0} + \cdots. \tag{17.3.3}$$

The final total momentum is the sum of the final momentum of the individual particles,

$$\vec{\mathbf{p}}_f^{\text{total}} = m_1\vec{\mathbf{v}}_{1,f} + m_2\vec{\mathbf{v}}_{2,f} + \cdots. \tag{17.3.4}$$

Note that the right-hand-sides of Equations. (17.3.3) and (17.3.4) are *vector* sums.

This section may be summarized as:

> *When the total external force on a system is zero, then the total initial momentum of the system equals the total final momentum of the system,*

$$\vec{\mathbf{p}}_0^{\text{total}} = \vec{\mathbf{p}}_f^{\text{total}}. \tag{17.3.5}$$

17.4 Momentum Changes and Non-isolated Systems

Suppose the total force acting on the system is not zero,

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} \neq \vec{\mathbf{0}}. \tag{17.4.1}$$

By Newton's Third Law, the sum of the total force on the surroundings is equal in magnitude but opposite in direction to the total external force acting on the system, so the

$$\vec{\mathbf{F}}_{\text{surr}}^{\text{total}} = -\vec{\mathbf{F}}_{\text{ext}}^{\text{total}}. \tag{17.4.2}$$

It's important to note that in Equation (17.4.2), all internal forces in the surroundings sum to zero.

Thus the sum of the total external force acting on the system and the total force acting on the surroundings is zero,

$$\vec{\mathbf{F}}_{\text{surr}}^{\text{total}} + \vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \vec{\mathbf{0}}. \tag{17.4.3}$$

We have already found (Equation (17.1.13)) that the total external force $\vec{\mathbf{F}}_{\text{ext}}^{\text{total}}$ on the system is the rate of change of the momentum of the system. Similarly, the total force on the surrounding is the rate of change of the momentum of the surroundings. Therefore the total momentum of both the system and surroundings is always conserved.

Definition: Conservation of Momentum

> *For a system and all of the surroundings that undergo any change of state, the total change in the momentum of the system and its surroundings is zero,*

$$\Delta\vec{\mathbf{p}}_{\text{system}} + \Delta\vec{\mathbf{p}}_{\text{surroundings}} = \vec{\mathbf{0}}. \tag{17.4.4}$$

17.5 Worked Examples

Problem Solving Strategies

When solving problems involving changing momentum in a system, we shall employ our general problem solving strategy involving four basic steps:

I. Understand – get a conceptual grasp of the problem.
II. Devise a Plan - set up a procedure to obtain the desired solution.
III. Carry our your plan – solve the problem!
IV. Look Back – check your solution and method of solution.

We shall develop a set of guiding ideas for the first two steps.

I. Understand – get a conceptual grasp of the problem

The first question you should ask is whether or not momentum is constant in some system that is changing its state after undergoing an interaction. First you must identify the objects that compose the system and how they are changing their state due to the interaction. As a guide, try to determine which objects change their momentum in the course of interaction. You must keep track of the momentum of these objects before and after any interaction. Second, momentum is a vector quantity so the question of whether momentum is constant or not must be answered in each relevant direction. In order to determine this, there are two important considerations. You should identify any external forces acting on the system. Remember that a non-zero total external force will cause the total momentum of the system to change, (Equation (17.1.13) above),

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \frac{d\,\vec{\mathbf{p}}_{\text{system}}}{dt}. \tag{17.5.1}$$

Equation (17.5.1) is a vector equation; if the total external force in some direction is zero, then the change of momentum in that direction is zero. In some cases, external forces may act but the time interval during which the interaction takes place is so small that the impulse is small in magnitude compared to the total momentum and might be neglible. Recall that the total average external impulse changes the momentum of the system

$$\vec{\mathbf{I}} = \vec{\mathbf{F}}_{\text{ext}}^{\text{total}} \Delta t_{\text{interaction}} = \Delta \vec{\mathbf{p}}_{\text{system}} . \tag{17.5.2}$$

If the interaction time is small enough, the momentum of the system is constant, $\Delta \vec{\mathbf{p}} \to \vec{\mathbf{0}}$. If the momentum is not constant then you must apply either Equation (17.5.1) or Equation (17.5.2). If the total momentum of the system is constant, then you can apply Equation (17.3.5),

$$\vec{\mathbf{p}}_0^{\text{total}} = \vec{\mathbf{p}}_f^{\text{total}} . \tag{17.5.3}$$

If there is no net external force in some direction, for example the x-direction, the component of momentum is constant in that direction, and you must apply

$$p_{x,0}^{\text{total}} = p_{x,f}^{\text{total}} . \tag{17.5.4}$$

II. Devise a Plan - set up a procedure to obtain the desired solution

Draw diagrams of all the elements of your system for the two states immediately before and after the system changes its state. Choose symbols to identify each mass and velocity in the system. Identify a set of positive directions and unit vectors for each state. Choose your symbols to correspond to the state and motion (this facilitates an easy interpretation, for example $(v_{x,0})_1$ represents the x-component of the velocity of object 1 in the initial state and $(v_{x,f})_1$ represents the x-component of the velocity of object 1 in the final state). Decide whether you are using components or magnitudes for your velocity symbols. Since momentum is a vector quantity, identify the initial and final vector components of the total momentum. We shall refer to these diagrams as *momentum flow diagrams*. Based on your model you can now write expressions for the initial and final momentum of your system. As an example in which two objects are moving only in the x-direction, the initial x-component of the momentum is

$$p_{x,0}^{\text{total}} = m_1 (v_{x,0})_1 + m_2 (v_{x,0})_2 . \tag{17.5.5}$$

The final x-component of the momentum is

$$p_{x,f}^{\text{total}} = m_1 (v_{x,f})_1 + m_2 (v_{x,f})_2 . \tag{17.5.6}$$

If the x-component of the momentum is constant then

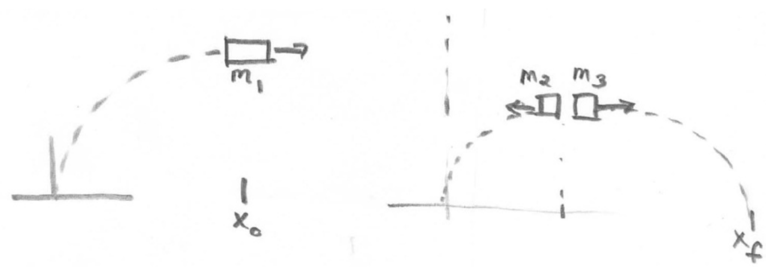
$$p_{x,0}^{\text{total}} = p_{x,f}^{\text{total}} . \tag{17.5.7}$$

We can now substitute Equations (17.5.5) and (17.5.6) into Equation (17.5.7), yielding

$$m_1(v_{x,0})_1 + m_2(v_{x,0})_2 = m_1(v_{x,f})_1 + m_2(v_{x,f})_2 . \qquad (17.5.8)$$

Equation (17.5.8) can now be used for any further analysis required by a particular problem. For example, you may have enough information to calculate the final velocities of the objects after the interaction. If so then carry out your plan and check your solution, especially dimensions or units and any relevant vector directions.

Example 17.5.1 Exploding Projectile An instrument-carrying projectile of mass m_1 accidentally explodes at the top of its trajectory. The horizontal distance between launch point and the explosion is x_0. The projectile breaks into two pieces that fly apart horizontally. The larger piece, m_3, has three times the mass of the smaller piece, m_2. To the surprise of the scientist in charge, the smaller piece returns to earth at the launching station. Neglect air resistance and effects due to the earth's curvature.



How far away, x_{3f}, from the original launching point does the larger piece land?

Solution: We can solve this problem two different ways. The easiest approach is to use the concept that the center of mass of the system follows a parabolic trajectory. From the information given in the problem $m_2 = m_1 / 4$ and $m_3 = 3m_1 / 4$. Thus when the two objects return to the ground the center of mass of the system has traveled a distance $R_{cm} = 2x_0$.

We can now use the definition of center of mass to find where the object with the greater mass hits the ground. Choose an origin at the starting point. The center of mass of the system is given by

$$\vec{\mathbf{R}}_{cm} = \frac{m_2\vec{\mathbf{r}}_2 + m_3\vec{\mathbf{r}}_3}{m_2 + m_3} .$$

So when the objects hit the ground $\vec{\mathbf{R}}_{cm} = 2x_0\ \hat{\mathbf{i}}$, the object with the smaller mass returns to the origin, $\vec{\mathbf{r}}_2 = \vec{\mathbf{0}}$, and the position vector of the other object is $\vec{\mathbf{r}}_3 = x_{3f}\ \hat{\mathbf{i}}$. So using the definition of the center of mass,

$$2x_0\,\hat{\mathbf{i}} = \frac{(3m_1/4)x_{3f}\,\hat{\mathbf{i}}}{m_1/4 + 3m_1/4} = \frac{(3m_1/4)x_{3f}\,\hat{\mathbf{i}}}{m_1} = \frac{3}{4}x_{3f}\,\hat{\mathbf{i}} \; .$$

Therefore

$$x_{3f} = \frac{8}{3}x_0 \; .$$

Note that the vertical height above the ground nor the gravitational acceleration g did not enter into our solution.

Alternatively, we can use conservation of momentum and kinematics to find the distance traveled. Since the smaller piece returns to the starting point after the collision, it must have the same speed v_0 as the projectile before the collision. Since the collision is instantaneous, the horizontal component of the momentum is constant during the collision. We can use this to determine the speed of the larger piece after the collision. Since the larger piece takes the same amount of time to return to the ground as the projectile originally takes to reach the top of the flight. We can therefore determine how far the larger piece traveled horizontally.

We begin by identifying various states in the problem.

Initial State, time t_0: The projectile is launched.

State 1 time t_1: The projectile is at the top of its flight trajectory immediately before the explosion. The mass is m_1 and the speed of the projectile is v_1.

State 2 time t_2: Immediately after the explosion, the projectile has broken into two pieces, one of mass m_2 moving backwards (in the $-x$-direction) with speed v_2 and the other of mass m_3 moving forward with speed v_3.

State 3 time t_f: The two pieces strike the ground, one at the original launch site and the other at a distance x_f from the launch site, as indicated in the figure. The pieces take the same amount of time to reach the ground since they are falling from the same height and both have no velocity in the vertical direction immediately after the explosion.

Model: Now we can pose some questions that may help us understand how to solve the problem. What is the speed of the projectile at the top of its flight just before the collision? What is the speed of the smaller piece just after the collision? What is the speed of the larger piece just after the collision?

The momentum flow diagram with State 1 as the initial state and State 2 as the final state is shown below. In the momentum flow diagrams and analysis we shall use symbols that represent the magnitudes of the magnitudes x-components of the velocities and arrows to indicate the directions of the velocities; for example the symbol $v_1 \equiv \left|(v_x(t_1))_1\right|$ for the magnitude of the x-component of the velocity of the object before the explosion at time t_1, $v_2 \equiv \left|(v_x(t_2))_2\right|$ and $v_3 \equiv \left|(v_x(t_2))_3\right|$ for the magnitudes of the x-component of the velocity of objects 2 and 3 immediately after the collision at time $t_{2.}$.

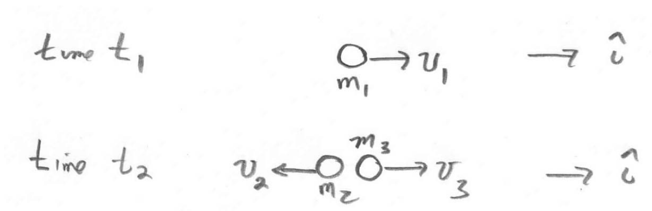


Figure 17.6 Momentum flow diagrams for the two middle states of the problem.

The initial momentum before the explosion is

$$p_{x,0}^{\text{total}} = p_x^{\text{total}}(t_1) = m_1 v_1 . \tag{17.5.9}$$

The momentum immediately after the explosion is

$$p_{x,f}^{\text{total}} = p_x^{\text{total}}(t_2) = -m_2 v_2 + m_3 v_3 \tag{17.5.10}$$

Note that in Equations (17.5.9) and (17.5.10) the signs of the terms are obtained directly from the momentum flow diagram, consistent with the use of magnitudes; we are told that the smaller piece moves in a direction opposite the original direction after the explosion.

This explosion is a situation described above, in that during the duration of the explosion, impulse due to the external force, gravity in this case, may be neglected. The collision is considered to be instantaneous, and momentum is constant. In the horizontal direction,

$$p_x^{\text{total}}(t_1) = p_x^{\text{total}}(t_2) . \tag{17.5.11}$$

If the collision were not instantaneous, then the masses would descend during the explosion, and the action of gravity would add downward velocity to the system. Equation (17.5.11) would still be valid, but our analysis of the motion between State 2 and State 3 would be affected. Substituting Equations (17.5.9) and (17.5.10) into Equation (17.5.11) yields

$$m_1 v_1 = -m_2 v_2 + m_3 v_3 . \quad (17.5.12)$$

The mass of the projectile is equal to the sum of the masses of the ejected pieces,

$$m_1 = m_2 + m_3 . \quad (17.5.13)$$

The heavier fragment is three times the mass of the lighter piece, $m_3 = 3m_2$. Therefore

$$m_2 = (1/4)m_1, \quad m_3 = (3/4)m_1 . \quad (17.5.14)$$

There are still two unknowns to consider, v_2 and v_3. However there is an additional piece of information. We know that the lighter object returns exactly to the starting position, which implies that $v_2 = v_1$ (we have already accounted for the change in direction by considering magnitudes, as discussed above.)

Recall from our study of projectile motion that the horizontal distance is given by $x_0 = v_1 t_1$, independent of the mass. The time that it takes the lighter mass to hit the ground is the same as the time it takes the original projectile to reach the top of its flight (neglecting air resistance). Therefore the speeds must be the same since the original projectile and the smaller fragment traveled the same distance. We can use the values for the respective masses (Equation (17.5.14)) in Equation (17.5.12), which becomes

$$m_1 v_1 = -\frac{1}{4} m_1 v_1 + \frac{3}{4} m_1 v_3 . \quad (17.5.15)$$

Equation (17.5.15) can now solved for the speed of the larger piece immediately after the collision,

$$v_3 = \frac{5}{3} v_1 . \quad (17.5.16)$$

The larger piece also takes the same amount of time t_1 to hit the ground as the smaller piece. Hence the larger piece travels a distance

$$x_3 = v_3 t_1 = \frac{5}{3} v_1 t_1 = \frac{5}{3} x_0 . \quad (17.5.17)$$

Therefore the total distance the larger piece traveled from the launching station is

$$x_f = x_0 + \frac{5}{3} x_0 = \frac{8}{3} x_0 , \quad (17.5.18)$$

in agreement with our previous approach.

17.5.2 Example: Recoil in Different Frames

A person of mass m_1 is standing on a cart of mass m_2. Assume that the cart is free to move on its wheels without friction. The person throws a ball of mass m_3 at an angle of θ with respect to the horizontal as measured by the person in the cart. The ball is thrown with a speed v_0 with respect to the cart (Figure 17.7).

a) What is the final velocity of the ball as seen by an observer fixed to the ground?

b) What is the final velocity of the cart as seen by an observer fixed to the ground?

c) With what angle, with respect to the horizontal, does the fixed observer see the ball leave the cart?

Figure 17.7 Recoil of a person on cart due to thrown ball

Solution:

a), b) Our reference frame will be that fixed to the ground. We shall take as our initial state that before the ball is thrown (cart, ball, throwing person stationary) and our final state that after the ball is thrown. We are assuming that there is no friction, and so there are no external forces acting in the horizontal direction.

The initial x-component of the total momentum is zero,

$$p_{x,0}^{\text{total}} = 0\,. \tag{17.5.19}$$

After the ball is thrown, the cart and person have a final momentum

$$\vec{\mathbf{p}}_{f,\text{cart}} = -(m_2 + m_1)v_{f,\text{cart}}\,\hat{\mathbf{i}} \tag{17.5.20}$$

as measured by the person on the ground, where $v_{f,\text{cart}}$ is the speed of the person and cart. (The person's center of mass will move with respect to the cart while the ball is being thrown, but since we're interested in velocities, not positions, we need only assume that the person is at rest with respect to the cart after the ball is thrown.)

The ball is thrown with a speed v_0 and at an angle θ with respect to the horizontal as measured by the person in the cart. Therefore the person in the cart throws the ball with velocity

$$\vec{\mathbf{v}}'_{f,\text{ball}} = v_0 \cos\theta \,\hat{\mathbf{i}} + v_0 \sin\theta \,\hat{\mathbf{j}}. \qquad (17.5.21).$$

Since the cart is moving in the negative x-direction with speed $v_{f,\text{cart}}$ just as the ball leaves the person's hand, the x-component of the velocity of the ball as measured by an observer on the ground is given by

$$v_{xf,\text{ball}} = v_0 \cos\theta - v_{f,\text{cart}}. \qquad (17.5.22)$$

The ball appears to have a smaller x-component of the velocity according to the observer on the ground. The velocity of the ball as measured by an observer on the ground is

$$\vec{\mathbf{v}}_{f,\text{ball}} = (v_0 \cos\theta - v_{f,\text{cart}}) \,\hat{\mathbf{i}} + v_0 \sin\theta \,\hat{\mathbf{j}}. \qquad (17.5.23)$$

The final momentum of the ball according to an observer on the ground is

$$\vec{\mathbf{p}}_{f,\text{ball}} = m_3 \left[(v_0 \cos\theta - v_{f,\text{cart}}) \,\hat{\mathbf{i}} + v_0 \sin\theta \,\hat{\mathbf{j}} \right]. \qquad (17.5.24)$$

The momentum flow diagram is then that shown in (Figure 17.8):

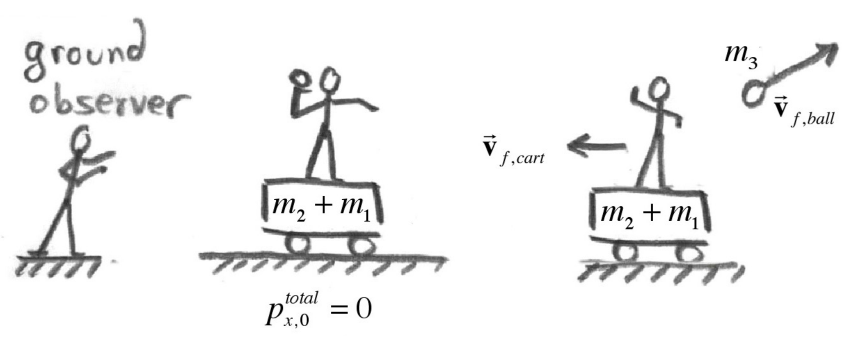


Figure 17.8 Momentum flow diagram for recoil

Since the total x-component of the momentum of the system is constant, we have that

$$
\begin{aligned}
0 &= (p_{x,f})_{\text{cart}} + (p_{x,f})_{\text{ball}} \\
&= -(m_2 + m_1)v_{f,\text{cart}} + m_3(v_0\cos\theta - v_{f,\text{cart}}).
\end{aligned}
\tag{17.5.25}
$$

We can solve Equation (17.5.25) for the final speed and velocity of the cart as measured by an observer on the ground,

$$
v_{f,\text{cart}} = \frac{m_3\, v_0 \cos\theta}{m_2 + m_1 + m_3}, \tag{17.5.26}
$$

$$
\vec{\mathbf{v}}_{f,\text{cart}} = v_{f,\text{cart}}\,\hat{\mathbf{i}} = \frac{m_3\, v_0 \cos\theta}{m_2 + m_1 + m_3}\hat{\mathbf{i}}\,. \tag{17.5.27}
$$

Note that the y-component of the momentum is not constant because as the person is throwing the ball he or she is pushing off the cart and the normal force with the ground exceeds the gravitational force so the net external force in the y-direction is non-zero.

Substituting Equation (17.5.26) into Equation (17.5.23) gives

$$
\begin{aligned}
\vec{\mathbf{v}}_{f,\text{ball}} &= (v_0\cos\theta - v_{f,\text{cart}})\,\hat{\mathbf{i}} + v_0 \sin\theta\,\hat{\mathbf{j}} \\
&= \frac{m_1 + m_2}{m_1 + m_2 + m_3}(v_0\cos\theta\,\hat{\mathbf{i}} + v_0\sin\theta)\,\hat{\mathbf{j}}.
\end{aligned}
\tag{17.5.28}
$$

As a check, note that in the limit $m_3 \ll m_1 + m_2$, $\vec{\mathbf{v}}_{f,\text{ball}}$ has speed v_0 and is directed at an angle θ above the horizontal; the fact that the much more massive person-cart combination is free to move doesn't affect the flight of the ball as seen by the fixed observer. Also note that in the unrealistic limit $m_3 \gg m_1 + m_2$ the ball is moving at a speed much smaller than v_0 as it leaves the cart.

c) The angle ϕ at which the ball is thrown as seen by the observer on the ground is given by

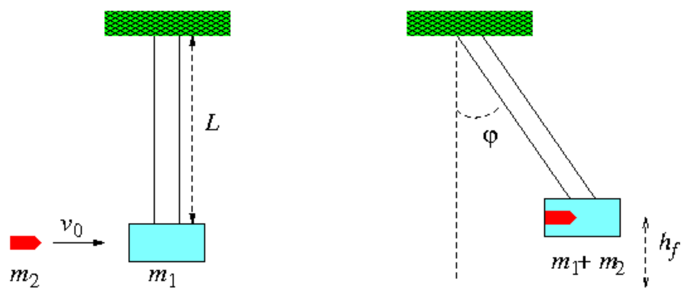
$$
\begin{aligned}
\phi &= \tan^{-1}\frac{(v_{f,\text{ball}})_y}{(v_{f,\text{ball}})_x} = \tan^{-1}\frac{v_0\sin\theta}{\big[(m_1 + m_2)/(m_1 + m_2 + m_3)\big]v_0\cos\theta} \\
&= \tan^{-1}\left[\tan\theta\frac{m_1 + m_2 + m_3}{m_1 + m_2}\right].
\end{aligned}
\tag{17.5.29}
$$

For arbitrary values for the masses, the above expression will not reduce to a simplified form. However, we can see that $\tan\phi > \tan\theta$ for arbitrary masses, and that in the limit

$m_3 \ll m_1 + m_2$, $\phi \to \theta$ and in the unrealistic limit $m_3 \gg m_1 + m_2$, $\phi \to \pi/2$. Can you explain this last odd prediction?

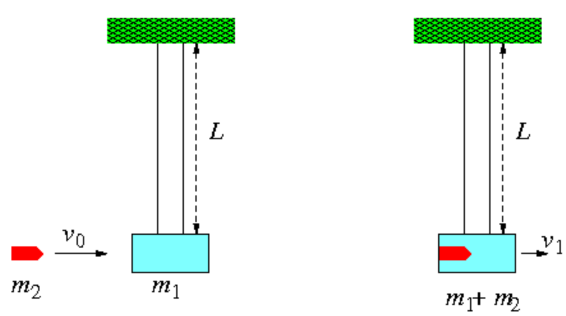
Example 17.5.3 ***Ballistic Pendulum***

A simple way to measure the speed of a bullet is with a *ballistic pendulum*, which consists of a wooden block of mass m_1 into which a bullet of mass m_2 is shot. The block is suspended from two cables, each of length L. The impact of the bullet causes the block and embedded bullet to swing through a maximum angle φ. The goal of this problem is to find a) an expression for the initial speed of the bullet v_0 as a function of m_1, m_2, L, g, and φ and then to b) determine the ratio of the lost mechanical energy (due to the collision of the bullet with the block) to the initial kinetic energy of the bullet.



Solution:

We shall use two concepts to solve this problem. We shall choose as our system the bullet and the block. We assume that the collision is nearly instantaneous and so the only external forces (gravity and tension) acting on the system are in the vertical direction and so the horizontal component of momentum is conserved. We can use conservation of momentum to determine the speed of the block immediately after the collision in terms of the speed of the bullet before the collision and the masses of the object. The momentum flow diagram is shown below.



We shall use conservation of energy to find a relation between the height that the block and bullet reached when they came to rest and the speed of the block immediately after the collision. We can then put these two pieces together to find the speed of the bullet in terms of the given quantities.

After the collision the bullet is completely embedded in the block (we shall see that this is an example of a completely inelastic collision). We can use the momentum flow diagram to analyze the collision.

Constancy of momentum in the horizontal direction is expressed as

$$m_2 v_0 = (m_1 + m_2) v_1 \tag{17.5.30}$$

where v_1 is the speed of the bullet-block combination after the collision and v_0 is the initial speed of the bullet.

The speed immediately after the collision is then

$$v_1 = \frac{m_2}{m_1 + m_2} v_0 . \tag{17.5.31}$$

Once the bullet is embedded in the block, the subsequent motion has constant energy. There is an external force acting on the system, the tension in the ropes, but that force points radially inward and since the block undergoes circular motion after being struck by the bullet, the tension does no work since the tension forces in the ropes are perpendicular to the displacement,

$$\vec{\mathbf{T}} \cdot d\vec{\mathbf{r}} = 0 . \tag{17.5.32}$$

Choose zero gravitational potential energy at the collision position. Then the mechanical energy immediately after the collision is

$$E_1 = K_1 = \frac{1}{2}\left(m_1 + m_2\right) v_1^2 \tag{17.5.33}$$

The block reaches a maximum height $h_f = L(1 - \cos\varphi)$ and the mechanical energy at that instant is then

$$E_2 = U_2 = \left(m_1 + m_2\right) g h_f = \left(m_1 + m_2\right) g L (1 - \cos\varphi) . \tag{17.5.34}$$

Conservation of mechanical energy then yields

$$
\begin{aligned}
E_1 &= \frac{1}{2}\left(m_1 + m_2\right)v_1^2 = \left(m_1 + m_2\right)gL(1-\cos\varphi) = E_2 \\
v_1^2 &= 2gL(1-\cos\varphi).
\end{aligned}
\tag{17.5.35}
$$

Substituting the result of Equation (17.5.31) for the speed v_1 immediately after the collision into Equation (17.5.35), we have that

$$
\left(\frac{m_2 v_0}{m_1 + m_2}\right)^2 = 2gL(1-\cos\varphi)\,. \tag{17.5.36}
$$

We can now solve Equation (17.5.36) for the initial speed of the bullet,

$$
v_0 = \frac{(m_1 + m_2)}{m_2}\sqrt{2gL(1-\cos\varphi)}\,. \tag{17.5.37}
$$

b) The change in mechanical energy during the collision is given by

$$
\Delta E = K_1 - K_0 = \frac{1}{2}\left(m_1 + m_2\right)v_1^2 - \frac{1}{2}m_2 v_0^2\,. \tag{17.5.38}
$$

Again substitute the result from Equation (17.5.31) for the velocity v_1 immediately after the collision into Equation (17.5.38) to obtain the "lost mechanical energy," $-\Delta E$;

$$
-\Delta E = \frac{1}{2}m_2 v_0^2\left(1 - \frac{m_2}{(m_1 + m_2)}\right) = K_0\frac{m_1}{(m_1 + m_2)}\,. \tag{17.5.39}
$$

The ratio of the lost mechanical energy to the initial kinetic energy is

$$
-\frac{\Delta E}{K_0} = \frac{m_1}{(m_1 + m_2)}\,. \tag{17.5.40}
$$

Note that this ratio only depends on the masses and is completely independent of the initial velocity or the collision forces (if the forces are so abrupt that the collision can be taken to be "instantaneous"). This lost mechanical energy has been transformed into thermal energy and also energy required to deform the internal structures of the block and bullet due to the collision.

8.01SC Physics I: Classical Mechanics