Momentum and Impulse
Challenge Problem Solutions

Problem 1: Estimation

Estimate the magnitude of the momentum of the following moving objects: a) a person walking; b) a car moving along a highway; c) a truck moving along a highway; d) a passenger jet at cruising speed; e) a freight train transporting coal.
Indicate your choice of reference frame for each case.

Problem 1 Solution:

These must be "answers" instead of "solutions," as a wide range of values for the estimated quantities could be used. Note that for common experience or everyday examples, it's sometimes convenient to make estimates in English units and convert to metric. For these answers, scientific notation is not always helpful.

All of the following estimates are with respect to a stationary observer on the ground. Using a frame moving with the walker or the respective vehicle is not really in the spirit of the problem.

a) Time yourself while walking. For some of us, $1\,\mathrm{m\cdot s^{-1}}$ is more of a stroll, while $2\,\mathrm{m\cdot s^{-1}}$ is a fairly fast clip. So, using a mass of $100\,\mathrm{kg}$ and a walking speed of $1.5\,\mathrm{m\cdot s^{-1}}$, the magnitude of the momentum is $150\,\mathrm{kg\cdot m\cdot s^{-1}}$. Larger walkers (such as the writer of these answers, who could stand to lose a few pounds) have in general a larger mass and a longer stride and a faster walking pace.

b) For a small car, use a mass of $1000\,\mathrm{kg}$. For highway speeds, the exact conversion in English units (still used in this country) is $60\,\mathrm{mph} = 88\,\mathrm{ft/sec}$, or about 25 - $30\,\mathrm{m\cdot s^{-1}}$, for an approximate momentum magnitude of $\sim 2500\,\mathrm{kg\cdot m\cdot s^{-1}}$.

c) Trucks vary greatly in mass, from lightweight pickups to the tandem eighteen-wheelers. I'm used to a ton-and-a-half flatbed (that's the capacity, not the weight), so five tons (the weight limit of a small bridge) would mean a mass of roughly $5000\,\mathrm{kg}$. Using the same speed from part b) of $25\,\mathrm{m\cdot s^{-1}}$ gives an order of magnitude momentum magnitude of $10^5\,\mathrm{kg\cdot m\cdot s^{-1}}$.

d) For the mass of the plane, let's use the value of $5\times10^4\,\mathrm{kg}$ cited in Problem Set 3 and a cruising speed of $200\,\mathrm{m\cdot s^{-1}}$ (roughly two-thirds the speed of sound) for a momentum magnitude of $10^6\,\mathrm{kg\cdot m\cdot s^{-1}}$.

e) Lots of crude estimates are needed here. We're not given the number of cars, whether the coal is anthracite or bituminous, and some of us would have to guess at the size of a car and the density of coal. Suppose instead that each car has the same magnitude of momentum as the truck in part c) and that we have a train of ten cars (seems low, but this train may be dealing with mountainous terrain). In this case, the magnitude of the momentum is $10^6\,\mathrm{kg \cdot m \cdot s^{-1}}$. If anything, this estimate is low, in that it's a short train at slow speeds, and we haven't included the engine.

Problem 2: Impulse and Center of Mass Velocity: *Elevator Ride*

A person is riding the Building 16 elevator, while standing on the force platform that was used in the lecture demonstration on Monday Oct 16. The force platform measures the force of the person's feet on the platform. The data in Figure 17.1a shows a plot of the force of platform on the person (normal force) vs. time for the person in the elevator.

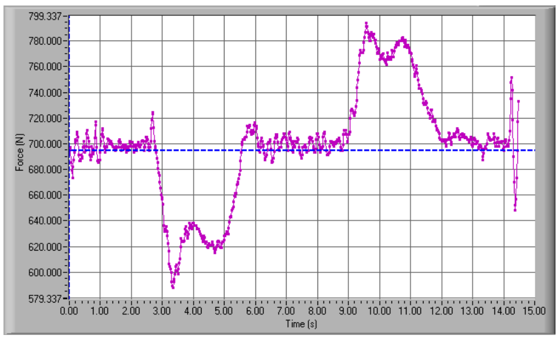


Figure 17.1a

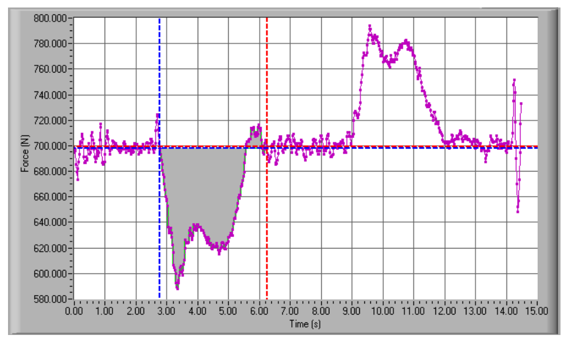
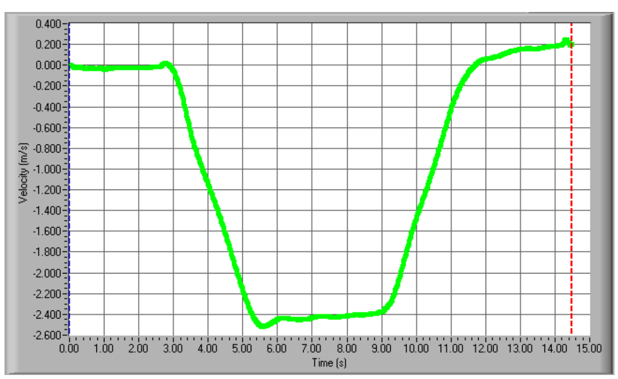


Figure 17.1b

Figure 17.1b highlights the area for the time interval $[t_1 = 2.75\text{ s}, t_2 = 6.35\text{ s}]$. The impulse for this interval is:

$$\vec{\mathbf{I}}[t_1,t_2] = \int_{t_1=2.75\text{ s}}^{t_2=6.35\text{ s}} \vec{\mathbf{F}}^{\text{total}}(t)\,dt = -174\text{ N}\cdot\text{s} \qquad (2.1)$$

Figure 17.2 show a plot of the vertical component of the center of mass velocity of the person vs. time.



a) What does the negative sign signify in Eq.(2.1) in terms of the choice of positive direction for a relevant coordinate axis? Is the choice of positive direction up or down? Based on the plot in Figure 17.1a, describe the motion of the elevator during the interval $[t_1 = 2.75\text{ s}, t_f = 12.0\text{ s}]$. In particular, identify any distinct stages. For each distinct stage describe whether the elevator is moving up or down, whether or not the acceleration is constant (or zero), or non-constant. Explain your reasoning.

b) Explain why the shaded area in Figure 17.1b is with respect to a baseline force of $700\ \mathrm{N}$ and not $0\ \mathrm{N}$.

c) Explain how you can use the data in Figure 17.1a to calculate the change in the vertical component of the velocity of the center of mass of the person? In particular, what is the vertical component of the velocity of the center of mass of the person at (i) $t_1 = 2.75\ \mathrm{s}$, (ii) $t_2 = 6.35\ \mathrm{s}$. How does your calculation compare with the results for the vertical component of the velocity of the center of mass of the person shown in the graph in Figure 17.2?

d) What do you expect the impulse to equal for the time interval $[t_1 = 2.75\ \mathrm{s},\ t_f = 12.0\ \mathrm{s}]$?

Problem 2 Solutions:

a) The downward force that the person exerts on the scale has decreased in magnitude, and hence the magnitude of the upward force that the scale exerts on the person has decreased. The force that the scale exerts on the person is less than the person's weight, and hence the minus sign in expression for impulse. The positive direction must be upward; scales as a rule do not measure forces tending to move the top part of the scale, where the person is standing, upward. The plot in Figure 3.1a is for the elevator (and the scale, and the person) descending by accelerating downward, moving at more or less constant speed (zero acceleration) for about three seconds and then accelerating upward while still moving down until the elevator comes to a stop.

b) There are two forces on the person, the downward gravitational force (the person's weight) and the upward force that the scale exerts on the person's feet. The net impulse is found from the difference between these two forces, and hence the person's weight, approximately $700\ \mathrm{N}$ as seen from the graph before the elevator began its descent and after it stopped, is subtracted from the scale reading. To restate one of the answers to part a), for the period when the elevator is accelerating downward, the net impulse is negative if the positive direction is upward.

c) The product of the person's mass and velocity is the integral of the net force with respect to time, which is the area between the horizontal line representing the person's weight and the curve representing the force on the scale. For calculation purposes, this area can be calculated numerically by a computer. From an eyeball examination of the graph in Figure 3.2, the elevator is not moving at $t_1 = 2.75\ \mathrm{s}$. We are given (*via* a numerical integration) that the vertical component of the net impulse in the interval $[2.75\ \mathrm{s},\ 6.35\ \mathrm{s}]$ was $-174\ \mathrm{N \cdot s}$. The person's mass is $700\ \mathrm{N}/g$, giving a vertical component of velocity of $-2.44\ \mathrm{m \cdot s^{-1}}$ (which is likely more significant figures than we need or want).

d) Between these two times, the elevator has descended and ideally has come to a stop. The impulse, the net change in momentum, should be zero. From Figure 3.2, this is almost the case, and if the elevator and person are not at rest at the end of the trip, either the elevator is dangerous or there was some error in the computer's numerical integration process.

Problem 3: ***Compressive Strength of Bones***

The compressive force per area necessary to break the tibia in the lower leg is about $F / A = 1.6 \times 10^8\ \mathrm{N} \cdot \mathrm{m}^{-2}$. The smallest cross sectional area of the tibia, about $3.2\ \mathrm{cm}^2$, is slightly above the ankle. Suppose a person of mass $m = 60\ \mathrm{kg}$ jumps to the ground from a height $h_0 = 2.0\ \mathrm{m}$ and absorbs the shock of hitting the ground by bending the knees. **Assume that there is constant deceleration during the collision**. During the collision, the person lowers his center of mass by an amount $\Delta d = 1.0\ \mathrm{cm}$.

a) What is the collision time Δt_{col}?

b) Find the average force of the ground on the person during the collision.

c) What is the ratio of average force of the ground on the person to the gravitational force on the person? Can we effectively ignore the gravitational force during the collision?

d) Will the person break his ankle?

Problem 3 Solutions:

a) What is the collision time Δt_{col}?

> Let the person be the system. Choose a coordinate system with the positive y-direction pointing up, and the origin on the ground.

The *y*-component of the velocity just as contact is with the ground is

$$v_{y,o} = -\sqrt{2gh_0} \qquad (3.1)$$

The displacement $\Delta y = y(t) - y_0 = -\Delta d$ of the person while in contact with the ground is given by

$$\Delta y = v_{y,o} \Delta t_{\mathrm{col}} + \frac{1}{2} a_{cm} \Delta t_{\mathrm{col}}{}^2 \qquad (3.2)$$

The *y*-component of the velocity is given by

$$v_y(t) = v_{y,o} + a_{cm} \Delta t \qquad (3.3)$$

The y-component of the velocity is zero $v_y(t = \Delta t_{\mathrm{col}}) = 0$ when the person's displacement is $\Delta y = -\Delta d$.

Solving Eq.(3.3) at $t = \Delta t_{\text{col}}$ yields

$$\Delta t_{\text{col}} = -\frac{v_{y,o}}{a_{cm}} \tag{3.4}$$

Substitute Eq. (3.4) into Eq. (3.2) and solve for the acceleration giving

$$a_{cm} = -\frac{1}{2}\frac{(v_{y,o})^2}{\Delta y} \tag{3.5}$$

Substitute Eq. (3.5) into Eq. (3.4) and solve for the collision time

$$\Delta t_{col} = \frac{2\Delta y}{v_{y,o}} \tag{3.6}$$

The initial y-velocity component from Eq. (3.1) is $v_{y,o} = -\sqrt{2gh_0}$, the displacement is $\Delta y = -\Delta d$, so with these values the collision time is

$$\Delta t_{\text{col}} = \frac{2\Delta d}{\sqrt{2gh_0}} = \frac{2(1\times 10^{-2}\text{ m})}{\sqrt{2(9.8\text{ m}\cdot\text{s}^{-2})(2.0\text{ m})}} = 3.2\times 10^{-3}\text{ s} \tag{3.7}$$

b) Find the average force of the ground on the person during the collision.

If we treat the person as the system then there are two external forces acting on the person, the gravitational force, $\vec{\mathbf{F}}_{grav} = -mg\,\hat{\mathbf{j}}$, and a normal force between the ground and the person, $\vec{\mathbf{F}}_{ground} = N\,\hat{\mathbf{j}}$. This normal force varies with time but we shall consider the time average, $(\vec{\mathbf{F}}_{ground})_{ave} = N_{ave}\,\hat{\mathbf{j}}$.

The impulse is equal to the change in momentum so

$$F_y^{\text{total}}\Delta t_{\text{col}} = -mv_{y,0} \tag{3.8}$$

Using our expressions for the total force, and the collision time and, we have

$$\left(N_{ave} - mg\right)\left(\frac{2\Delta y}{v_{y,o}}\right) = -mv_{y,0} \tag{3.9}$$

Solving for the average normal force

$$N_{ave} = mg - \frac{m(v_{y,0})^2}{2\Delta y} \qquad (3.10)$$

Finally substituting in the initial y–component of the velocity and the displacement. the average normal force is

$$N_{ave} = mg(1 + \frac{h_0}{\Delta d}) = (60\text{ kg})(9.8\text{ m}\cdot\text{s}^{-2})(1 + \frac{(2.0\text{ m})}{(1\times 10^{-2}\text{ m})}) = 1.2\times 10^5\text{ N} \qquad (3.11)$$

c) What is the ratio of the average (normal) force of the ground on the person to the gravitational force on the person? Can we effectively ignore the gravitational force during the collision?

$$\frac{N_{ave}}{mg} = (1 + \frac{h_0}{\Delta d}) = \left(1 + \frac{(2.0\text{ m})}{(1\times 10^{-2}\text{ m})}\right) = 2.01\times 10^2\text{ N} \qquad (3.12)$$

Notice that the factor $(1 + \frac{h_0}{\Delta d}) \simeq \frac{h_0}{\Delta d} = 200$ so we can effectively ignore the external gravitational force during the collision.

d) Will the person break his ankle?

We approximate the compressional force per area by the average normal force divided by the smallest cross sectional area of the tibia

$$P = \frac{N_{ave}}{A} = \frac{mg}{A}(1 + \frac{h_0}{\Delta d}) = (1.2\times 10^5\text{ N})/(3.2\times 10^{-4}\text{m}^2) = 3.7\times 10^8\text{N}\cdot\text{m}^{-2} \qquad (3.13)$$

The compressive force per area necessary to break the tibia in the lower leg is about $F / A = 1.6\times 10^8\text{ N}\cdot\text{m}^{-2}$, so this fall is enough to break the tibia.

Remark: In order to find the minimum displacement to avoid breaking the bone, we could set

$$\frac{mg}{A}(1 + \frac{h_0}{\Delta d_{\min}}) \simeq \frac{mg}{A}(\frac{h_0}{\Delta d_{\min}}) = 1.6\times 10^8\text{N}\cdot\text{m}^{-2} \qquad (3.14)$$

Thus

$$\Delta d_{\min} \simeq (60\text{ kg})(9.8\text{ m}\cdot\text{s}^{-2})\left(\frac{(2.0\text{ m})}{(1.6\times10^{8}\text{ N}\cdot\text{m}^{-2})(3.2\times10^{-4}\text{m}^{2})}\right) = 2.3\times10^{-2}\text{ cm} \quad \textit{(3.15)}$$

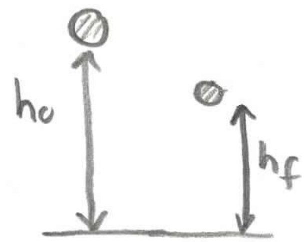
The ratio

$$\frac{h_0}{\Delta d_{\min}} \simeq \frac{(A)(1.6\times10^{8}\text{ N}\cdot\text{m}^{-2})}{mg} = 87 \qquad \textit{(3.16)}$$

This depends on the compressive strength of the bone, the cross sectional area, and inversely on the weight of the person. Note that the maximum normal force is anywhere from two to ten times the average normal force. So a safe distance to lower the center of mass would be about 20 cm.

Problem 4: ***Momentum and Impulse***

A superball of m_1, starting at rest, is dropped from a height h_0 above the ground and bounces back up to a height of h_f. The collision with the ground occurs over a time interval Δt_c.



a) What is the momentum of the ball immediately before the collision?

b) What is the momentum of the ball immediately after the collision?

c) What is the average force of the ground on the ball?

d) What impulse is imparted to the ball?

e) What is the change in the kinetic energy during the collision?

Problem 4 Solutions:

For all parts, take the upward direction to be the positive $\hat{\mathbf{j}}$ direction, and use the ground to be the zero point of gravitational potential energy.

a) The initial mechanical energy is $E_0 = m_1 g h_0$; all potential, no kinetic energy. The energy when the ball first hits the ground is $E_1 = m_1 v_1^2 / 2$, all kinetic, no potential energy. Equating these energies (with the assumption that there are no nonconservative forces) and solving for the speed v_1 gives $v_1 = \sqrt{2gh_0}$, for a momentum

$$\vec{\mathbf{p}}_1 = m_1 \sqrt{2gh_0}\left(-\hat{\mathbf{j}}\right). \tag{4.1}$$

b) In order to find the momentum, we need to know the speed. The speed is found, again, through conservation of mechanical energy. Denote the upward speed after the collision as v_2, with corresponding mechanical energy $E_2 = m_1 v_2^2 / 2$. This energy must be the same as the potential energy at the top of the bounce, $E_3 = m_1 g\, h_f$. Solving for the rebound speed v_2 gives $v_2 = \sqrt{2g\, h_f}$, for a momentum

$$\vec{\mathbf{p}}_2 = m_1\sqrt{2gh_f}\,(\hat{\mathbf{j}}). \tag{4.2}$$

c) While the ball is in contact with the ground, the average net force is the change in momentum divided by the time interval,

$$\vec{\mathbf{F}}_{\text{net, ave}} = \frac{\Delta\vec{\mathbf{p}}}{\Delta t} = \frac{\vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1}{\Delta t_{\text{c}}} = m_1\sqrt{2g}\,\frac{\sqrt{h_f} + \sqrt{h_0}}{\Delta t_{\text{c}}}\,\hat{\mathbf{j}}. \tag{4.3}$$

However, there are two forces on the ball, the contact force that the ground exerts on the ball and the gravitational force. The net force is the sum of these forces, and so

$$\vec{\mathbf{F}}_{\text{net}} = \vec{\mathbf{F}}_{\text{ball, ground}} + \vec{\mathbf{F}}_{\text{grav}} = \vec{\mathbf{F}}_{\text{ball, ground}} - m_1 g\,\hat{\mathbf{j}}. \tag{4.4}$$

The gravitational force must be presumed to be constant, and hence equal to its average value. Combining the terms (watch the subscripts),

$$\vec{\mathbf{F}}_{\text{ground, ball, ave}} = \vec{\mathbf{F}}_{\text{net, ave}} + m_1 g\,\hat{\mathbf{j}} = m_1\left(\sqrt{2g}\,\frac{\sqrt{h_f} + \sqrt{h_0}}{\Delta t_{\text{c}}} + g\right)\hat{\mathbf{j}}. \tag{4.5}$$

d) The impulse is the change in momentum,

$$\Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1 = m_1\sqrt{2g}\left(\sqrt{h_f} + \sqrt{h_0}\right)\hat{\mathbf{j}}. \tag{4.6}$$

e) The change in kinetic energy during the collision is the difference between the energies found from the heights before and after the collision,

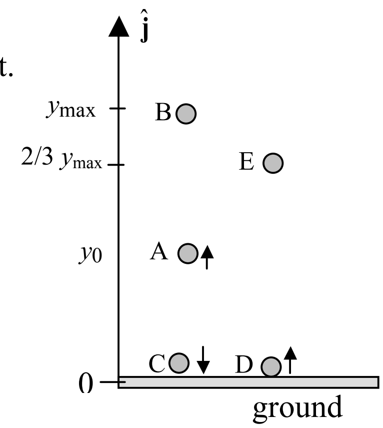
$$\Delta K = E_0 - E_2 = m_1 g h_0 - m_1 g h_f = m_1 g\left(h_0 - h_f\right). \tag{4.7}$$

Problem 5: Impulse and Newton's Second Law: ***More Bouncing Balls***

A superball of mass m is thrown straight up into the air starting from a height y_0 above the ground. The ball reaches a maximum height y_{max} above the ground. After the ball hits the ground, it bounces back up to a new maximum height equal to $(2/3)y_{max}$. Assume the collision with ground takes place over a time $\Delta t_{collision}$.

The above process is represented in the figure to the right. The important states of the ball are labeled:

A—starting point of the ball:
B – maximum height of the ball:
C – when the ball just hits the ground (moving down):
D – when the ball just bounces from the ground (moving up):
E – new maximum height of the ball.



a) What was the magnitude of the initial velocity v_0 of the ball when it was released?

b) In terms of y_{max} and the gravitational acceleration g, what was the change in the kinetic energy of the ball due to the collision of the ball with the ground?

c) What impulse did the ground impart to the ball?

d) What was the average force of the ground on the ball during the collision?

Problem 5 Solution:

a) Choose a coordinate axis as shown in the above figure, with the positive y-direction upwards, and choose the ball and the Earth as a system for this part. Consider the states A and B (see figure above). Mechanical energy is conserved for the system between states A and B. The equations that describe the conservation of energy are

$$(U_B - U_A) + (K_E - K_E) = 0$$

$$\left(mg\,y_{max} - mg\,y_0\right) + \left(0 - \frac{1}{2}mv_A^2\right) = 0$$

$$v_0 = v_A = \sqrt{2g(y_{max} - y_0)}\ . \qquad (5.1)$$

b) Let us first calculate the kinetic energy of the ball just before it hits the ground. Consider the states B and C. The mechanical energy of the system is conserved between these states, as expressed in the equations

$$(U_C - U_B) + (K_C - K_B) = 0$$

$$K_C = mg\, y_{max}\,. \tag{5.2}$$

Next determine the kinetic energy of the ball just after it bounces up from the ground. Consider the states C and D. The mechanical energy of the system is conserved between these states, so that

$$K_D = mg\left(\frac{2}{3} y_{max}\right). \tag{5.3}$$

The change in the kinetic energy of the ball due to its collision with the ground is then

$$K_D - K_C = mg\left(\frac{2}{3} y_{max}\right) - mg\, y_{max} = -\frac{1}{3} mg\, y_{max}\,. \tag{5.4}$$

c) Choose the ball alone as the system. The impulse imparted to the system from the ground is the change in momentum of the system (ball);

$$\vec{\mathbf{I}} = \Delta\vec{\mathbf{p}} = m\,\Delta\vec{v} = m\left(v_D\,\hat{\mathbf{j}} - v_C\left(-\hat{\mathrm{j}}\right)\right). \tag{5.5}$$

Determine the velocity at points C and D from the kinetic energies.
From Equation (5.2),

$$\frac{1}{2} m v_C^2 = mg\, y_{max}\,, \tag{5.6}$$

which gives

$$v_C = \sqrt{2g\, y_{max}}\,. \tag{5.7}$$

From Equation (5.3),

$$\frac{1}{2} m\, v_D^2 = mg\left(\frac{2}{3} y_{max}\right), \tag{5.8}$$

which gives

$$v_{\mathrm{C}} = \sqrt{\frac{4}{3} g\, y_{\max}}\,. \tag{5.9}$$

Using Equations (5.7) and (5.9) in Equation (5.5) yields the expression for the velocities in Equation (5.5),

$$\vec{\mathbf{I}} = m\left(v_{\mathrm{D}} + v_{\mathrm{C}}\right)\hat{\mathbf{j}} = m\left(\sqrt{\frac{4}{3} g\, y_{\max}} + \sqrt{2 g\, y_{\max}}\right)\hat{\mathbf{j}} = \frac{2+\sqrt{6}}{\sqrt{3}} m \sqrt{g\, y_{\max}}\,\hat{\mathbf{j}}. \tag{5.10}$$

d) The impulse exerted by the ground on the ball is also $\vec{\mathbf{I}} = \vec{\mathbf{F}}_{\mathrm{ave}}\, \Delta t_{\mathrm{collision}}$ where $\Delta t_{\mathrm{collision}}$ is the time of collision. Using the expression for the impulse from Equation (5.10) to determine the average force,

$$\vec{\mathbf{F}}_{\mathrm{ave}} = \frac{\vec{\mathbf{I}}}{\Delta t} = \frac{\dfrac{2+\sqrt{6}}{\sqrt{3}} m \sqrt{g\, y_{\max}}}{\Delta t}\,\hat{\mathbf{j}}. \tag{5.11}$$