# Continuous Mass Flow

# Strategy : Continuous Mass Flow

- Goal: Determine speed of object as function of time or mass as the mass of the object is continuously changing.

- Choose system consisting of all elements that undergo momentum change in an interval [t, t+$\Delta$t]

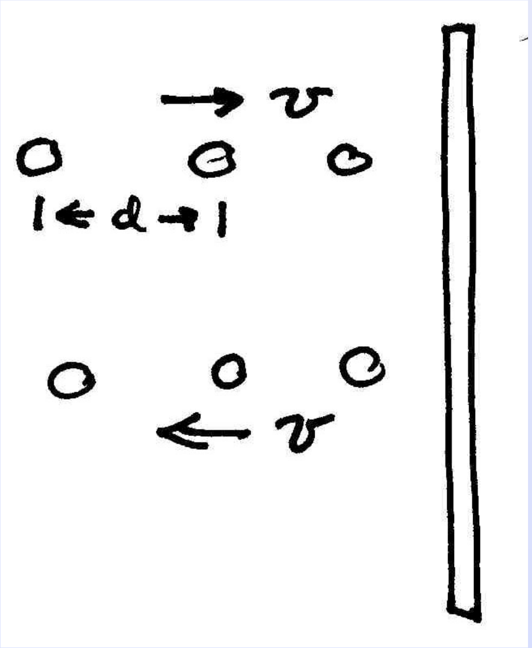
- Use momentum flow diagrams, to apply

$$\vec{\mathbf{F}}_{\text{ext}} = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{P}}^{\text{total}}(t+\Delta t) - \vec{\mathbf{P}}^{\text{total}}(t)}{\Delta t}$$

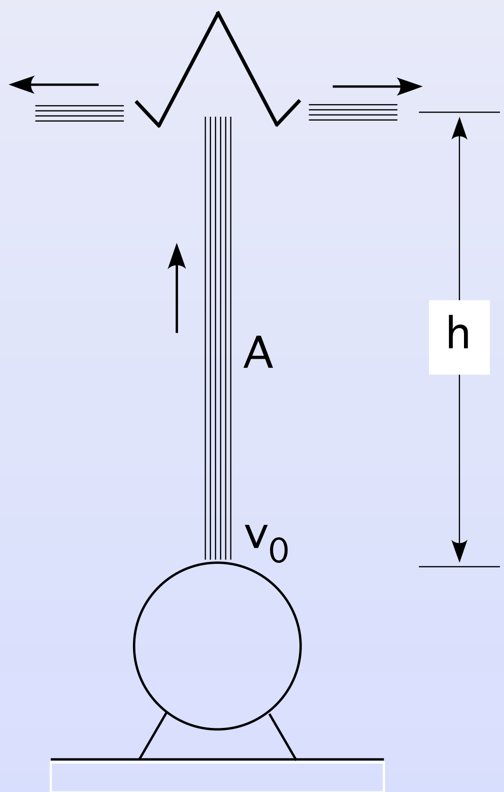
to find a differential equation that describes motion of object.

# Checkpoint Problem: stream bouncing off wall

A stream of particles of mass m and separation d hits a perpendicular surface with speed v. The stream rebounds along the original line of motion with the same speed. The mass per unit length of the incident stream is $\lambda = m/d$. What is the magnitude of the force on the surface?



# Checkpoint Problem: Clown-Hat Fountain



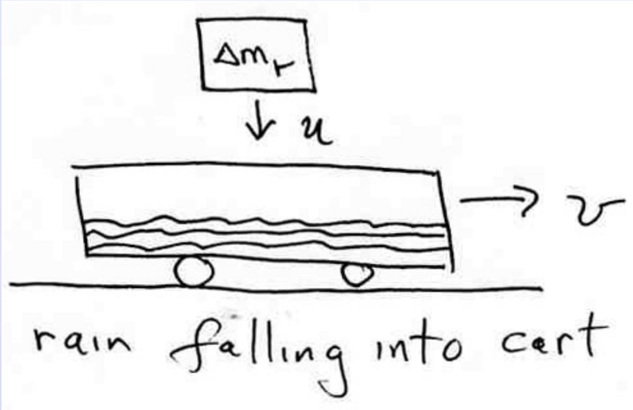

A stream of water of density $\rho$ and cross sectional area A leaves a water toy vertically at an initial speed $v_0$. After hitting a plastic hat of mass M the water is dispersed horizontally.

Find the equilibrium height of the hat,

$$h = h(A, \rho, v_0, g, M)$$

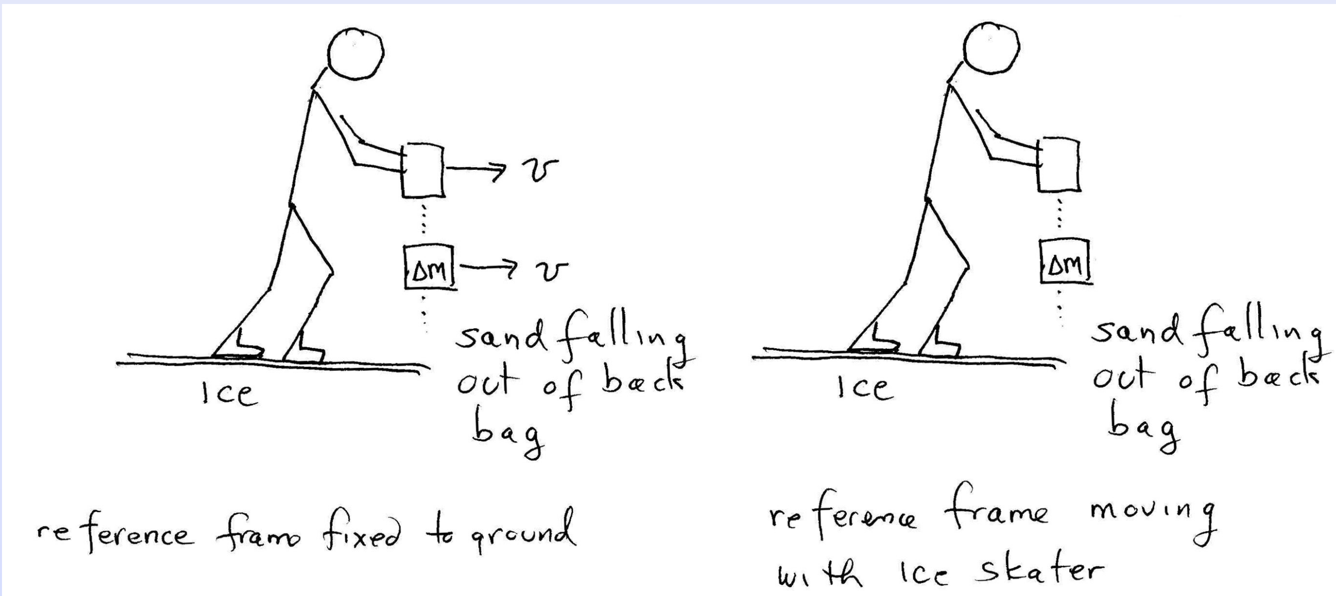
# Category 1: Adding Rain

There is a transfer of material into the object but no transfer of momentum in the direction of motion of the object. Consider for example rain falling vertically downward into a moving cart. A small amount of rain has no component  of momentum in the direction of motion of the cart.



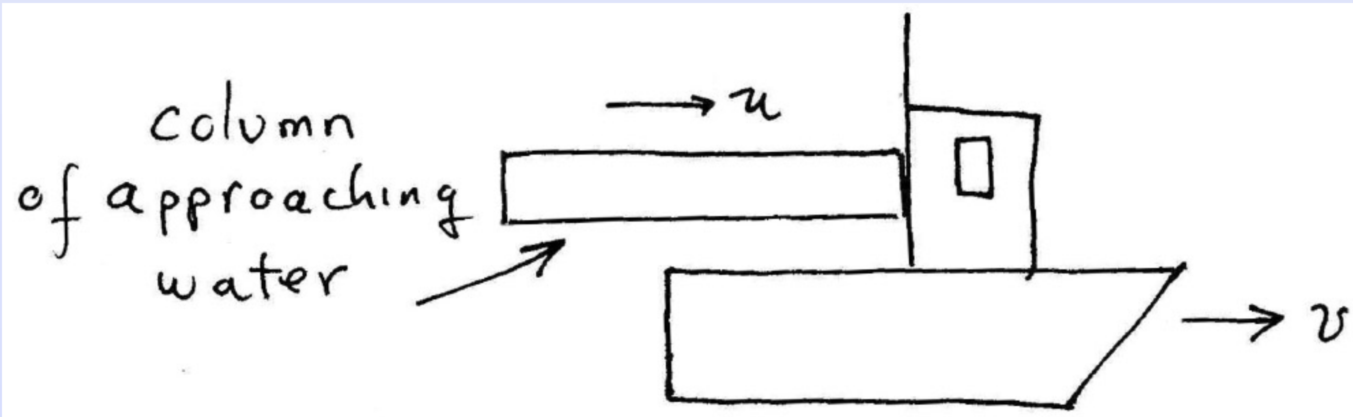
# Category 2: Leaking Sand

The material continually leaves the object but it does not transport any momentum away from the object in the direction of motion of the object. For example, consider an ice skater gliding on ice holding a bag of sand that is leaking straight down with respect to the moving skater.



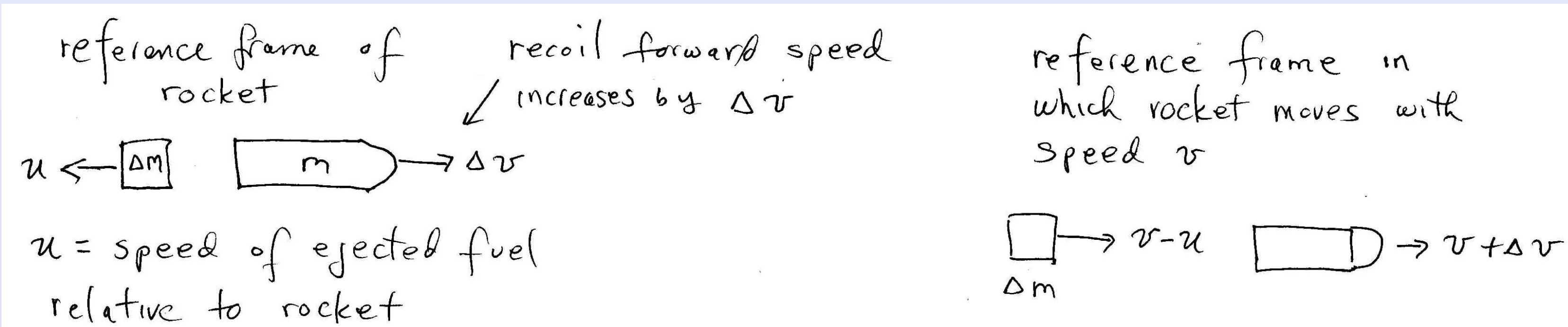
# Category 3: Impulse

The material continually hits the object providing an impulse resulting in a transfer of momentum to the object in the direction of motion. For example, suppose a fire hose is used to put out a fire on a boat. The incoming water continually hits the boat impulsing it forward.



# Category 4: Recoil

The material continually is ejected from the object, resulting in a recoil of the object. For example when fuel is ejected from the back of a rocket, the rocket recoils forward.



# Continuous Mass Flow: Strategy

Choose a system by considering all the mass elements that undergo momentum change during a time interval [ t , t+$\Delta$t ]

Identify any infinitesimal mass element $\Delta$m that undergoes a momentum change during this time interval along with its velocities at time t and time t+$\Delta$t

Momentum flow diagram: Draw the system at time t, and the system at time t+$\Delta$t with labels for the velocities of all the mass elements other than infinitesimal mass element that undergoes a momentum change $\Delta$m. Distinguish velocities at time t from velocities at time t +$\Delta$t by using:

$$\vec{\mathbf{v}}(t) = \text{velocity at time t}$$

$$\vec{\mathbf{v}}(t+\Delta t) = \vec{\mathbf{v}}(t) + \Delta\vec{\mathbf{v}} = \text{velocity at time t} + \Delta \text{t}$$

# Continuous Mass Flow: Strategy

Calculate the change in momentum for the system

Are there any external forces at time *t*?

Apply Momentum Principle to find a differential equation for the velocity of the mass elements that describes how the system evolves

$$\vec{\mathbf{F}}_{\text{ext}} = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{P}}^{\text{total}}(t+\Delta t) - \vec{\mathbf{P}}^{\text{total}}(t)}{\Delta t}$$

In taking limits, you can ignore contributions of terms that are proportional to $(\Delta m)(\Delta v)/\Delta t$

Conservation of Mass: Check to see if there is any relationship between the rate of change of one mass element with respect to some other mass element.

Differential Equation: Explore whether or not you can exactly solve the equation to find the velocity as a function of time.

# Strategy : Rocket Problem

- Goal: Determine velocity of rocket as function of time as mass is continuously ejected at rate d*m*/d*t* with speed ***u*** relative to rocket.

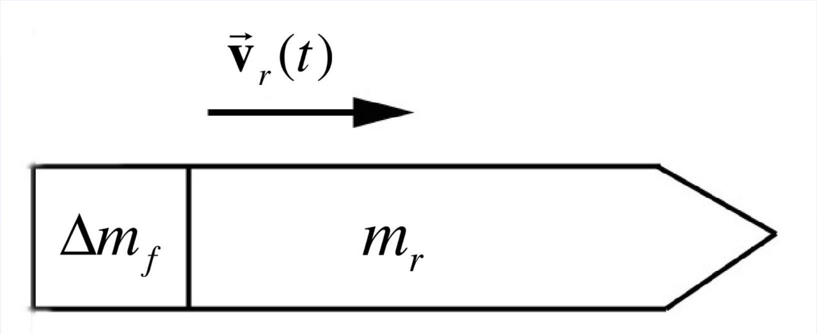
- System: consider all elements that undergo momentum change: rocket and fuel

- Using Momentum flow diagram, apply

$$\vec{\mathbf{F}}_{\text{ext}} = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{P}}^{\text{total}}(t + \Delta t) - \vec{\mathbf{P}}^{\text{total}}(t)}{\Delta t}$$

to find differential equation that describes motion.

# Worked Example: Rocket



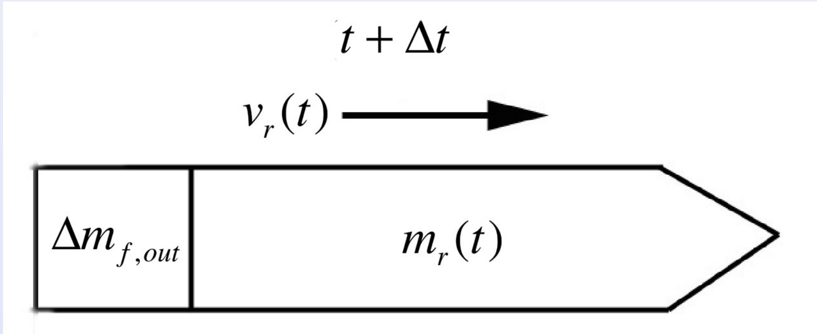

A rocket at time t is moving with speed $v_{r,0}$ in the positive x-direction in empty space. The rocket burns the fuel at a rate $dm_{f,out}/dt = b > 0$. The fuel is ejected backward with speed u relative to the rocket.

a) What is the relationship between the time rate of change of exhaust mass $dm_f/dt$, and the time rate of change of rocket mass $dm_r/dt$?

b) Find an equation for the rate of change of the speed of the rocket in terms $m_r(t)$, u, and $dm_r/dt$ and solve for v.

c) Find the differential equation describing the motion of the rocket is it is in a constant gravitational field of magnitude *g*

# Hint: State at time t




1. Rocket with total mass $m_r(t)$ moves with speed $v_r(t)$ in positive x-direction according to observer

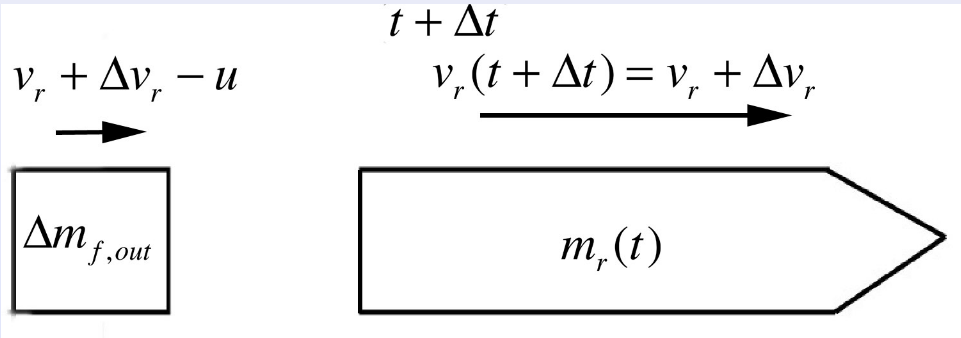
$$m_r(t) = m_{r,0} + m_{f,in}(t)$$

2. Total mass consists of mass of rocket $m_{r,0}$ and fuel $m_{f,in}(t)$

3. Fuel element with mass $\Delta m_{f,out}$ moves with speed of rocket $v_r(t)$ at time $t$, is ejected during interval $[t, t+\Delta t]$

4. x-component of momentum at time $t$ $\quad P_{x,\text{sys}}(t) = (m_r(t) + \Delta m_{f,out})v_r(t)$

# State at $t + \Delta t$




- Rocket is propelled forward by ejected fuel with new rocket speed

$$v_r(t + \Delta t) = v_r(t) + \Delta v_r$$

- Fuel is ejected backward with speed $u$ relative to rocket. Relative to observer's frame, ejected fuel element has speed

$$v_r + \Delta v_r - u$$

- X-component of system's momentum at time $t+\Delta t$

$$P_{x,sys}(t + \Delta t) = m_r(t)(v_r + \Delta v_r) + \Delta m_{f,out}(v_r + \Delta v_r - u)$$

# Rocket Equation

Are there any external forces at time *t*?

Two cases:

(1) Taking off $\vec{\mathbf{F}}_{\text{ext}}(t) = m_r(t)\vec{\mathbf{g}}$ (2) Negligible gravitational field $\vec{\mathbf{F}}_{\text{ext}} = \vec{\mathbf{0}}$

Apply Momentum Principle: $\vec{\mathbf{F}}_{\text{ext}} = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{P}}^{\text{total}}(t+\Delta t) - \vec{\mathbf{P}}^{\text{total}}(t)}{\Delta t}$

$$\vec{\mathbf{F}}_{\text{ext}} = \lim_{\Delta t \to 0} \frac{m_r(t)(\vec{\mathbf{v}}_r + \Delta\vec{\mathbf{v}}_r) + \Delta m_f \vec{\mathbf{u}} - m_r(t)\vec{\mathbf{v}}_r}{\Delta t} = \lim_{\Delta t \to 0} \frac{m_r(t)\Delta\vec{\mathbf{v}}_r + \Delta m_f \vec{\mathbf{u}}}{\Delta t}$$

$$m_r(t)\vec{\mathbf{g}} = m_r(t)\frac{d\vec{\mathbf{v}}_r}{dt} + \frac{dm_f}{dt}\vec{\mathbf{u}}$$

Conservation of mass: Rate of decrease of mass of rocket equals rate of ejection of mass

$$\frac{dm_f}{dt} = -\frac{dm_r(t)}{dt}$$

Rocket equation:

$$\vec{\mathbf{F}}_{\text{ext}} = m_r(t)\frac{d\vec{\mathbf{v}}_r}{dt} - \frac{dm_r}{dt}\vec{\mathbf{u}}$$

# Rocket Equation in Gravitational Field

$$\vec{\mathbf{F}}_{\text{ext}} + \frac{dm_r}{dt}\vec{\mathbf{u}} = m_r(t)\frac{d\vec{\mathbf{v}}_r}{dt}$$

- Fuel ejection term can be interpreted as thrust force
- Relative fuel ejection velocity $\vec{\mathbf{u}} = -u\hat{\mathbf{k}}$
- External force $\vec{\mathbf{F}}_{\text{ext}}(t) = -m_r g\hat{\mathbf{k}}$
- Rocket equation $-m_r g = m_r \frac{dv_{r,z}}{dt} + \frac{dm_r}{dt}u$

$$\frac{dv_{r,z}}{dt} = -\frac{1}{m_r}\frac{dm_r}{dt}u - g$$

- Integrate with respect to time $\int_{t_0=0}^{t_f} \frac{dv_{r,z}}{dt}dt = -\int_{t_0=0}^{t_f} \frac{1}{m_r}\frac{dm_r}{dt}u\,dt - \int_{t_0=0}^{t_f} g\,dt$
- Solution:

$$v_{r,z}(t_f) = u\ln\frac{m_r(t=0)}{m_r(t_f)} - gt_f$$

# Rocket Equation in Gravitational Field

- Fuel ejection term can be interpreted as thrust force

$$\vec{\mathbf{F}}_{\text{ext}} + \frac{dm_r}{dt}\vec{\mathbf{u}} = m_r(t)\frac{d\vec{\mathbf{v}}_r}{dt}$$

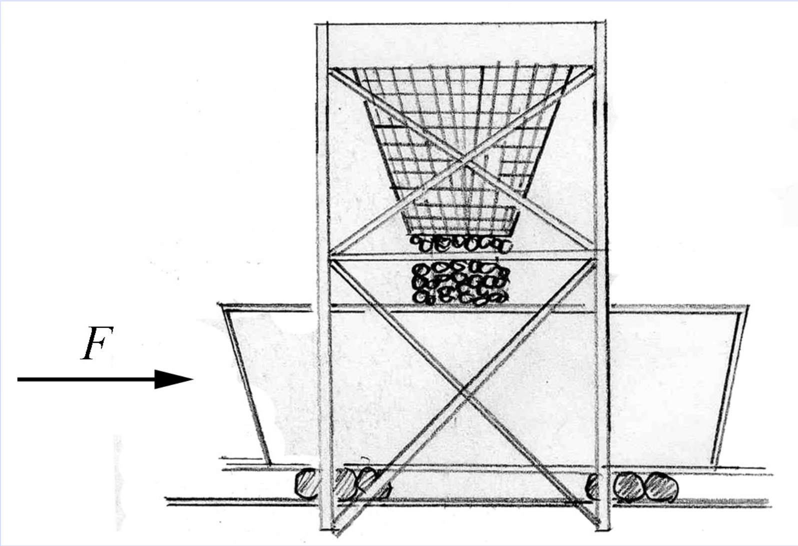
$$\vec{\mathbf{F}}_{\text{thrust}} = \frac{dm_r}{dt}\vec{\mathbf{u}}$$

- Velocity

$$v_{r,z}(t_f) = u\ln\frac{m_r(t=0)}{m_r(t_f)} - gt_f$$

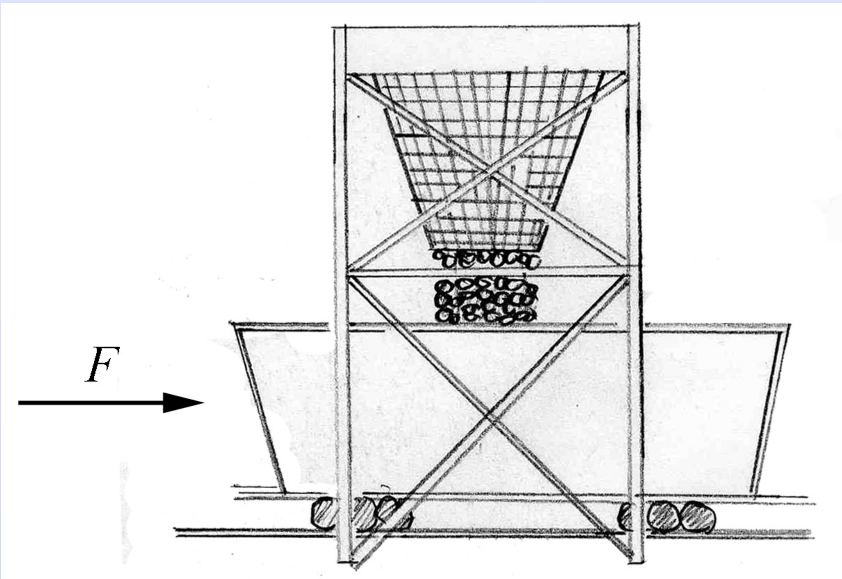
# Checkpoint Problem: Coal Car (constant force varying speed)

An empty coal car of mass m starts from rest under an applied force of magnitude F. At the same time coal begins to run into the car at a steady rate b from a coal hopper at rest along the track. Find the speed when a mass $m_c$ of coal has been transferred.



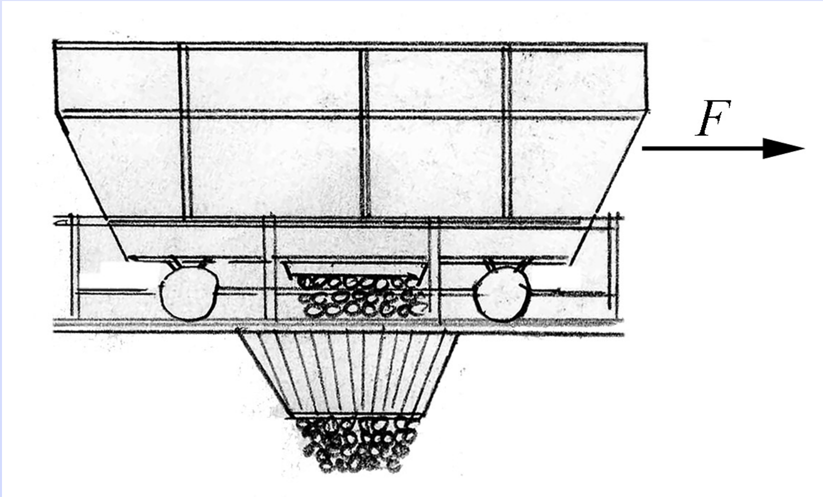
# Checkpoint Problem: Coal Car constant speed find force

An empty coal car of mass m is moving at constant speed. At the same time coal begins to run into the car at a steady rate b from a coal hopper at rest along the track. Find the applied force necessary to keep the car moving at constant speed.



# Checkpoint Problem: Emptying a Freight Car

An freight car of mass $m_c$ contains a mass of sand $m_s$ . At $t = 0$ a constant horizontal force of magnitude F is applied in the direction of rolling and at the same time a port in the bottom is opened to let the sand flow out at the constant rate $b = dm_s/dt$. Find the speed of the freight car when all the sand is gone. Assume that the freight car is at rest at $t = 0$.



# Checkpoint Problem: Sand Spraying Locomotive (relative speed of mass element)

A sand-spraying locomotive sprays sand horizontally into a freight car. The locomotive and freight car are not attached. The engineer in the locomotive maintains his speed so that the distance to the freight car is constant. The sand is transferred at a rate $dm/dt = 10$ kg/s with a velocity $u = 5$ m/s relative to the locomotive. The car starts from rest with an initial mass of 2000 kg. Find its speed after 100 s.