Module 18: Collision Theory

18.1 Introduction

In the previous module we considered examples in which two objects collide and stick together, and either there were no external forces acting in some direction (or the collision was nearly instantaneous) so the component of the momentum of the system along that direction is constant. We shall now study collisions between objects in more detail. In particular we shall consider cases in which the objects do not stick together. The momentum along a certain direction may still be constant but the mechanical energy of the system may change. We will begin our analysis by considering two-particle collision. We introduce the concept of the relative velocity between two particles and show that it is independent of the choice of reference frame. We then show that the change in kinetic energy only depends on the change of the square of the relative velocity and therefore is also independent of the choice of reference frame. We will then study one- and two-dimensional collisions with zero change in potential energy. On particular we will characterize the types of collisions by the change in kinetic energy and analyze the possible outcomes of the collisions.

18.2 Reference Frames Relative and Velocities

We shall recall our definition of relative inertial reference frames (**add link).** Let $\vec{\mathbf{R}}$ be the vector from the origin of frame S to the origin of reference frame S'. Denote the position vector of particle i with respect to origin of reference frame S by $\vec{\mathbf{r}}_i$ and similarly, denote the position vector of particle i with respect to origin of reference frame S' by $\vec{\mathbf{r}}'_i$ (Figure 18.1).

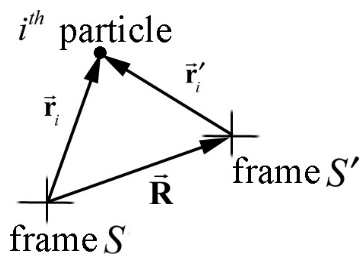


Figure 18.1 Position vector of i^{th} particle in two reference frames.

The position vectors are related by

$$\vec{\mathbf{r}}_i = \vec{\mathbf{r}}'_i + \vec{\mathbf{R}} \, . \tag{18.2.1}$$

The relative velocity (call this the *boost velocity*) between the two reference frames is given by

$$\vec{\mathbf{V}} = \frac{d\vec{\mathbf{R}}}{dt}. \tag{18.2.2}$$

Assume the boost velocity between the two reference frames is constant. Then, the relative acceleration between the two reference frames is zero,

$$\vec{\mathbf{A}} = \frac{d\vec{\mathbf{V}}}{dt} = \vec{\mathbf{0}}, \tag{18.2.3}$$

When Eq. (18.2.3) is satisfied, the reference frames S and S' are called *relatively inertial reference frames.*

Suppose the i^{th} particle in Figure 18.1 is moving; then observers in different reference frames will measure different velocities. Denote the velocity of i^{th} particle in frame S by $\vec{\mathbf{v}}_i = d\vec{\mathbf{r}}_i / dt$, and the velocity of the same particle in frame S' by $\vec{\mathbf{v}}'_i = d\vec{\mathbf{r}}' / dt$. Since the derivative of the position is velocity, the velocities of the particles in two different reference frames are related according to

$$\vec{\mathbf{v}}_i = \vec{\mathbf{v}}'_i + \vec{\mathbf{V}}. \tag{18.2.4}$$

Center of Mass Reference Frame

Let $\vec{\mathbf{R}}_{cm}$ be the vector from the origin of frame S to the center of mass of the system of particles, a point that we will choose as the origin of reference frame S_{cm}, called the *center of mass reference frame*. Denote the position vector of particle i with respect to origin of reference frame S by $\vec{\mathbf{r}}_i$ and similarly, denote the position vector of particle i with respect to origin of reference frame S_{cm} by $\vec{\mathbf{r}}_{cm,i}$ (Figure 18.1A).

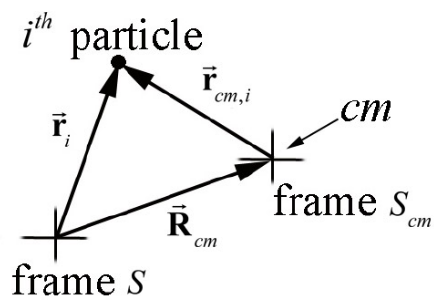


Figure 18.1A Position vector of i^{th} particle in the center of mass reference frame.

The position vector of particle i in the center of mass frame is then given by

$$\vec{\mathbf{r}}_{cm,i} = \vec{\mathbf{r}}_i - \vec{\mathbf{R}}_{cm}. \tag{18.2.5}$$

The velocity of particle i in the center of mass reference frame is then given by

$$\vec{\mathbf{v}}_{cm,i} = \vec{\mathbf{v}}_i - \vec{\mathbf{V}}_{cm}. \tag{18.2.6}$$

There are many collision problems in which the center of mass reference frame is the most convenient reference frame to analyze the collision.

Relative Velocities

Consider two particles of masses m_1 and m_2 interacting via some force.

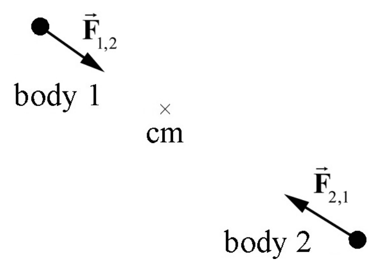


Figure 18.2 Two interacting particles

Choose a coordinate system (Figure 18.3) in which the position vector of body 1 is given by $\vec{\mathbf{r}}_1$ and the position vector of body 2 is given by $\vec{\mathbf{r}}_2$. The *relative position* of body 1 with respect to body 2 is given by $\vec{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$.

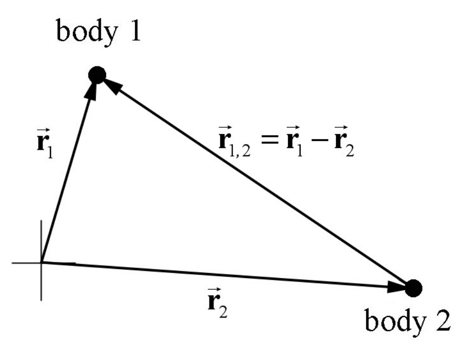


Figure 18.3 Coordinate system for two bodies.

During the course of the interaction, body 1 is displaced by $d\vec{\mathbf{r}}_1$ and body 2 is displaced by $d\vec{\mathbf{r}}_2$, so the *relative displacement* of the two bodies during the interaction is given by $d\vec{\mathbf{r}}_{1,2} = d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2$. The *relative velocity* between the particles is

$$\vec{\mathbf{v}}_{1,2} = \frac{d\vec{\mathbf{r}}_{1,2}}{dt} = \frac{d\vec{\mathbf{r}}_1}{dt} - \frac{d\vec{\mathbf{r}}_2}{dt} = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2 \,. \tag{18.2.7}$$

We shall now show that the relative velocity between the two particles is independent of the choice of reference frame providing that the reference frames are relatively inertial. The relative velocity $\vec{\mathbf{v}}'_{12}$ in reference frame S' can be determined from using Eq. (18.2.4) to express Eq. (18.2.7) in terms of the velocities in the in reference frame S',

$$\vec{\mathbf{v}}_{1,2} = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2 = (\vec{\mathbf{v}}'_1 + \vec{\mathbf{V}}) - (\vec{\mathbf{v}}'_2 + \vec{\mathbf{V}}) = \vec{\mathbf{v}}'_1 - \vec{\mathbf{v}}'_2 = \vec{\mathbf{v}}'_{1,2} \tag{18.2.8}$$

and is equal to the relative velocity in frame S.

For a two-particle interaction, the relative velocity between the two vectors is independent of the choice of reference frame.

In **Appendix 12.A: Work Done on a System of Two Particles** (**add link**), we showed that when two particles of masses m_1 and m_2 interact, the change of kinetic energy between the final state B and the initial state A due to the interaction force only is equal to

$$\Delta K = \frac{1}{2}\mu(v_B^2 - v_A^2) \tag{18.2.9}$$

where $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass of the two-particle system. (If Eq. (18.2.3) did not hold, Eq. (18.2.9) would not be valid in all frames.)

In Eq. (18.2.9), the square of the final relative velocity $(\vec{\mathbf{v}}_1)_B - (\vec{\mathbf{v}}_2)_B$ is given by

$$(v_{1,2}^2)_B = ((\vec{\mathbf{v}}_1)_B - (\vec{\mathbf{v}}_2)_B)\cdot((\vec{\mathbf{v}}_1)_B - (\vec{\mathbf{v}}_2)_B) \tag{18.2.10}$$

and the square of the initial relative velocity $(\vec{\mathbf{v}}_1)_A - (\vec{\mathbf{v}}_2)_A$ is given by

$$(v_{1,2}^2)_A = ((\vec{\mathbf{v}}_1)_A - (\vec{\mathbf{v}}_2)_A)\cdot((\vec{\mathbf{v}}_1)_A - (\vec{\mathbf{v}}_2)_A)\,. \tag{18.2.11}$$

By expressing the change of kinetic energy in terms of the relative velocity, a quantity that is independent of the reference frame,

the change in kinetic energy is therefore independent of the choice of reference frame.

18.3 Characterizing Collisions

In a collision, the ratio of the magnitudes of the initial and final relative velocities is called the coefficient of restitution and denoted by the symbol e,

$$e = \frac{v_B}{v_A}. \tag{18.3.1}$$

If the magnitude of the relative velocity does not change during a collision, $e = 1$, then the change in kinetic energy is zero, (Eq. (18.2.9)). Collisions in which there is no change in kinetic energy are called *elastic collisions*,

$$\Delta K = 0, \quad \textit{elastic collision}. \tag{18.3.2}$$

If the magnitude of the final relative velocity is less than the magnitude of the initial relative velocity, $e < 1$, then the change in kinetic energy is negative. Collisions in which the kinetic energy decreases are called inelastic collisions,

$$\Delta K < 0, \ \textit{inelastic collision}. \tag{18.3.3}$$

If the two objects stick together after the collision, then the relative final velocity is zero, $e = 0$. Such collisions are called *totally inelastic*. The change in kinetic energy can be found from Eq. (18.2.9),

$$\Delta K = -\frac{1}{2}\mu v_A^2 = -\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2} v_A^2, \quad \textit{totally inelastic collision}. \tag{18.3.4}$$

If the magnitude of the final relative velocity is greater than the magnitude of the initial relative velocity, $e > 1$, then the change in kinetic energy is positive. Collisions in which the kinetic energy increases are called *superelastic collisions*,

$$\Delta K > 0, \ \textit{superelastic collision}. \tag{18.3.5}$$

18.4 One-Dimensional Elastic Collision Between Two Objects

Consider a one-dimensional completely elastic collision between two objects moving in the x-direction. One object, with mass m_1 and initial x-component of the velocity $v_{x1,0}$, collides with an object of mass m_2 and initial x-component of the velocity $v_{x2,0}$. The scalar components $v_{x1,0}$ and $v_{x2,0}$ can be positive, negative or zero. No forces other than the interaction force between the objects act during the collision. After the collision, the velocity components are $v_{x1,f}$ and $v_{x2,f}$.

Because there are no external forces x-direction, momentum is constant in the x-direction. Equating the momentum components before and after the collision gives the relation

$$m_1 v_{x1,0} + m_2 v_{x2,0} = m_1 v_{x1,f} + m_2 v_{x2,f} \,. \tag{18.4.1}$$

Because the collision is elastic, kinetic energy is constant. Equating the kinetic energy before and after the collision gives the relation

$$m_1 v_{x1,0}^2 + m_2 v_{x2,0}^2 = m_1 v_{x1,f}^2 + m_2 v_{x2,f}^2 \tag{18.4.2}$$

where the factor of $1/2$ before each term in Eq. (18.4.2) has been divided out. Rewrite these Eq.s as

$$m_1(v_{x1,0} - v_{x1,f}) = m_2(v_{x2,f} - v_{x2,0}) \tag{18.4.3}$$

$$m_1(v_{x1,0}^2 - v_{x1,f}^2) = m_2(v_{x2,f}^2 - v_{x2,0}^2) \,. \tag{18.4.4}$$

The second Eq. above (18.4.4)can be written as

$$m_1(v_{x1,0} - v_{x1,f})(v_{x1,0} + v_{x1,f}) = m_2(v_{x2,f} - v_{x2,0})(v_{x2,f} + v_{x2,0}) \,. \tag{18.4.5}$$

Divide the kinetic energy Eq. (18.4.4) by the momentum Eq. (18.4.3), yielding

$$v_{x1,0} + v_{x1,f} = v_{x2,0} + v_{x2,f} \,. \tag{18.4.6}$$

Eq. (18.4.6) may be expressed as

$$v_{x1,0} - v_{x2,0} = v_{x2,f} - v_{x1,f} \,. \tag{18.4.7}$$

The relative velocity between the two objects is defined to be

$$\vec{\mathbf{v}}_{rel} = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2 \,. \tag{18.4.8}$$

The quantity $v_{xrel,0} = v_{x1,0} - v_{x2,0}$ is the initial component of the relative velocity, and $v_{xrel,f} = -(v_{x2,f} - v_{x1,f})$ is the final component of the relative velocity. So we find that

$$v_{xrel,0} = -v_{xrel,f} \,. \tag{18.4.9}$$

or by taking absolute values that the initial and final relative speeds are equal. Eq. (18.4.7) may be rewritten as

$$v_{x1,f} = v_{x2,f} - v_{x1,0} + v_{x2,0} \tag{18.4.10}$$

Now substitute Eq. (18.4.10) into Eq. (18.4.1) yields

$$m_1 v_{x1,0} + m_2 v_{x2,0} = m_1 v_{x1,f} + m_2 (v_{x1,0} + v_{x1,f} - v_{x2,0}). \tag{18.4.11}$$

Solving Eq. (18.4.11) for $v_{x1,f}$ involves some algebra and yields

$$v_{x1,f} = v_{x1,0} \frac{m_1 - m_2}{m_1 + m_2} + v_{x2,0} \frac{2m_2}{m_1 + m_2} \tag{18.4.12}$$

and a similar calculation yields

$$v_{x2,f} = v_{x2,0} \frac{m_2 - m_1}{m_2 + m_1} + v_{x1,0} \frac{2m_1}{m_2 + m_1}. \tag{18.4.13}$$

Consider what happens in the limits $m_1 \gg m_2$ in Eq. (18.4.12). Then

$$v_{x1,f} \to v_{x1,0} + v_{x2,0}(2m_2 / m_1); \tag{18.4.14}$$

the more massive object's velocity component is only slightly changed by an amount proportional to the less massive object's momentum component.

$$v_{x2,f} \to -v_{x2,0} + 2v_{x1,0} = v_{x1,0} + v_{x1,0} - v_{x2,0}. \tag{18.4.15}$$

We can rewrite this as

$$v_{x2,f} - v_{x1,0} = v_{x1,0} - v_{x2,0}. \tag{18.4.16}$$

i.e. the less massive object "rebounds" with the same speed relative to the more massive object.

If the objects are identical, or have the same mass, Eqs. (18.4.12) and (18.4.13) become

$$v_{x1,f} = v_{x2,0}, \quad v_{x2,f} = v_{x1,0}; \tag{18.4.17}$$

the objects have exchanged velocities, and unless we could somehow distinguish the objects, we might not be able to tell if there was a collision at all.

One-Dimensional Collision Between Two Objects – Center of Mass Reference Frame

Consider the one-dimensional elastic collision described above. Now let's view the collision from the center of mass (CoM) frame. The velocity component of the center of mass is

$$v_{\mathrm{x,cm}} = \frac{m_1\, v_{x1,0} + m_2\, v_{x2,0}}{m_1 + m_2}. \tag{18.4.18}$$

With respect to the center of mass, the velocity components of the objects are

$$\begin{aligned} v'_{x1,0} &= v_{x1,0} - v_{\mathrm{x,cm}} = (v_{x1,0} - v_{x2,0})\frac{m_2}{m_1 + m_2} \\ v'_{x2,0} &= v_{x2,0} - v_{\mathrm{x,cm}} = (v_{x,2,0} - v_{x,1,0})\frac{m_1}{m_1 + m_2}. \end{aligned} \tag{18.4.19}$$

In the CoM frame there is no total momentum before the collision and hence no total momentum after the collision. For an elastic collision, the only way for both momentum and kinetic energy to be the same before and after the collision is for either the objects have the same velocity (a miss) or to reverse the direction of the velocities. Symbolically, in the CoM frame, the final velocity components are

$$\begin{aligned} v'_{x1,f} &= -v'_{x1,0} = (v_{x2,0} - v_{x1,0})\frac{m_2}{m_1 + m_2} \\ v'_{x2,f} &= -v'_{x2,0} = (v_{x2,0} - v_{x1,0})\frac{m_1}{m_1 + m_2}. \end{aligned} \tag{18.4.20}$$

The final velocity components are then given by

$$\begin{aligned} v_{x1,f} &= v'_{x1,f} + v_{\mathrm{x,cm}} \\ &= (v_{x2,0} - v_{x1,0})\frac{m_2}{m_1 + m_2} + \frac{m_1\, v_{1,0} + m_2\, v_{2,0}}{m_1 + m_2} \\ &= v_{x1,0}\frac{m_1 - m_2}{m_1 + m_2} + v_{x2,0}\frac{2\,m_2}{m_1 + m_2} \end{aligned} \tag{18.4.21}$$

as in Eq. (18.4.12) and a similar calculation that reproduces Eq. (18.4.13).

18.5 Worked Examples

18.5.1 Example Elastic One-Dimensional Collision Between Two Objects Consider the elastic collision of two carts along a track; the incident cart 1 has mass m_1 and moves with initial speed $v_{1,0}$. The target cart has mass $m_2 = 2m_1$ and is initially at rest, $v_{2,0} = 0$. Immediately after the collision, the incident cart has final speed $v_{1,f}$ and the target cart has final speed $v_{2,f}$. Calculate the final velocities of the carts as a function of the initial speed $v_{1,0}$.

Solution

Draw a "momentum flow" diagram for the objects before (initial state) and after (final state) the collision (Figure 18.5, with a greatly simplified rendering of a "cart").

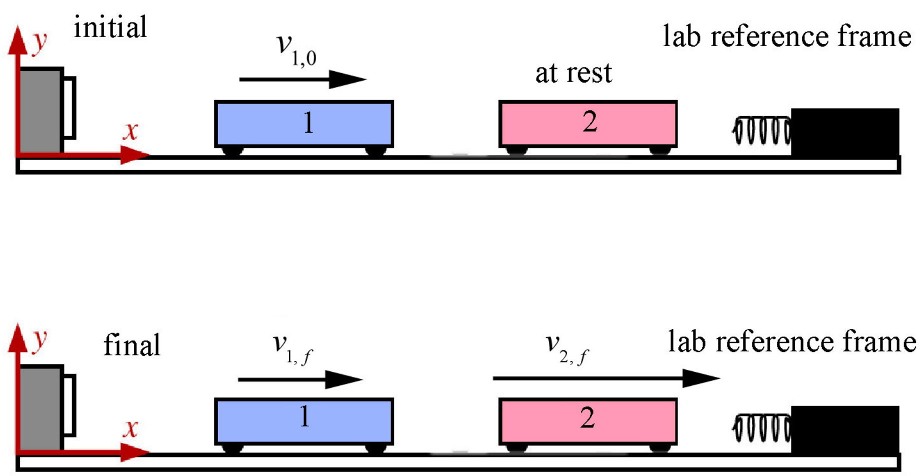


Figure 18.5 Momentum flow diagram for elastic one-dimensional collision

We can immediately use our results above with $m_2 = 2m_1$ and $v_{2,0} = 0$. The final x-component of velocity of cart 1 is given by Eq. (18.4.12)

$$v_{x1,f} = -v_{x1,0}\frac{1}{3}. \tag{18.5.1}$$

The final x-component of velocity of cart 2 is given by Eq. (18.4.13)

$$v_{x2,f} = v_{x1,0}\frac{2}{3}. \tag{18.5.2}$$

Example 18.5.2 The Dissipation of Kinetic Energy in a Completely Inelastic Collision Between Two Objects

For the general case, an incident object of mass m_1 and initial speed v_0 collides completely inelastically with a target object of mass m_2 that is initially at rest. There are no external forces acting on the objects in the direction of the collision. Find $\Delta K / K_{\text{initial}} = (K_{\text{final}} - K_{\text{initial}}) / K_{\text{initial}}$.

In the absence of any net force on the system consisting of the two objects, the momentum after the collision will be the same as before the collision. After the collision the objects will move in the direction of the initial velocity of the incident object with a common speed v_f found from

$$
\begin{aligned}
(m_1 + m_2)v_f &= m_1 v_0 \\
v_f &= v_0 \frac{m_1}{m_1 + m_2}.
\end{aligned}
\tag{18.5.3}
$$

The change in kinetic energy to the initial kinetic energy $\Delta K = (K_{\text{final}} - K_{\text{initial}})$ is

$$
\begin{aligned}
\Delta K = (K_{\text{final}} - K_{\text{initial}}) &= \frac{1}{2}(m_1 + m_2)v_f^2 - \frac{1}{2}v_0^2 \\
&= \frac{1}{2}(m_1 + m_2)v_0^2\left(\frac{m_1}{m_1 + m_2}\right)^2 - \frac{1}{2}v_0^2 \\
&= \frac{1}{2}m_1 v_0^2\left(\frac{m_1}{m_1 + m_2} - 1\right) \\
&= -\frac{1}{2}m_1 v_0^2 \frac{m_2}{m_1 + m_2}.
\end{aligned}
\tag{18.5.4}
$$

The ratio of the change in kinetic energy to the initial kinetic energy is then

$$
\Delta K / K_{\text{initial}} = -\frac{m_2}{m_1 + m_2}. \tag{18.5.5}
$$

18.5.3 Example: Elastic Two-Dimensional Collision

Object 1 with mass m_1 is initially moving with a speed $v_{1,0} = 3.0\,\text{m}\cdot\text{s}^{-1}$ and collides elastically with object 2 that has the same mass, $m_2 = m_1$, and is initially at rest. After the collision, object 1 moves with an unknown speed $v_{1,f}$ at an angle $\theta_{1,f} = 30^\circ$ with respect to its initial direction of motion and object 2 moves with an unknown speed $v_{2,f}$, at an unknown angle $\theta_{2,f}$ (as shown in the Figure 18.6). Find the final speeds of each of the objects and the angle $\theta_{2,f}$.

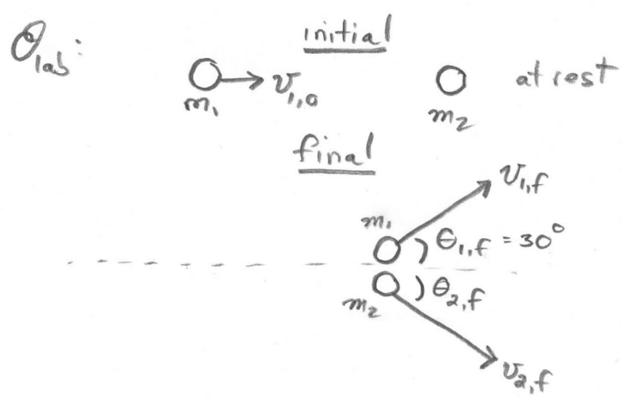


Figure 18.6 Momentum flow diagram for two-dimensional elastic collision

Solution:

Choose a set of positive unit vectors for the initial and final states as shown in Figure 18.7. We designate the respective speeds of each of the particles on the momentum flow diagrams.

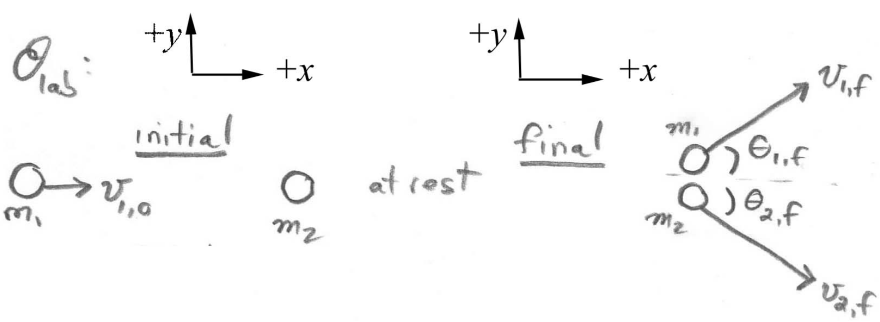


Figure 18.7 Choice of unit vectors for momentum flow diagram

Initial State: The components of the total momentum $\vec{\mathbf{p}}_0^{\text{total}} = m_1\vec{\mathbf{v}}_{1,0} + m_2\vec{\mathbf{v}}_{2,0}$ in the initial state are given by

$$
\begin{aligned}
p_{x,0}^{\text{total}} &= m_1 v_{1,0} \\
p_{y,0}^{\text{total}} &= 0.
\end{aligned}
\tag{18.5.6}
$$

Final State: The components of the momentum $\vec{\mathbf{p}}_f^{\text{total}} = m_1\vec{\mathbf{v}}_{1,f} + m_2\vec{\mathbf{v}}_{2,f}$ in the final state are given by

$$\begin{aligned} p_{x,f}^{\text{total}} &= m_1 v_{1,f} \cos\theta_{1,f} + m_1 v_{2,f} \cos\theta_{2,f} \\ p_{y,f}^{\text{total}} &= m_1 v_{1,f} \sin\theta_{1,f} - m_1 v_{2,f} \sin\theta_{2,f}. \end{aligned} \tag{18.5.7}$$

There are no any external forces acting on the system, so each component of the total momentum remains constant during the collision,

$$p_{x,0}^{\text{total}} = p_{x,f}^{\text{total}} \tag{18.5.8}$$

$$p_{y,0}^{\text{total}} = p_{y,f}^{\text{total}}. \tag{18.5.9}$$

These two Eq.s become

$$\begin{aligned} m_1 v_{1,0} &= m_1 v_{1,f} \cos\theta_{1,f} + m_1 v_{2,f} \cos\theta_{2,f} \\ 0 &= m_1 v_{1,f} \sin\theta_{1,f} - m_1 v_{2,f} \sin\theta_{2,f}. \end{aligned} \tag{18.5.10}$$

The collision is elastic; the kinetic energy is the same before and after the collision,

$$K_0^{\text{total}} = K_f^{\text{total}}, \tag{18.5.11}$$

or

$$\frac{1}{2} m_1 v_{1,0}^2 = \frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_1 v_{2,f}^2. \tag{18.5.12}$$

We have three Eq.s, two momentum Eq.s and one energy Eq., with three unknown quantities, $v_{1,f}$, $v_{2,f}$ and $\theta_{2,f}$ since we are already given that $v_{1,0} = 3.0\ \text{m}\cdot\text{s}^{-1}$ and $\theta_{1,f} = 30^o$. We first rewrite the expressions in Eq. (18.5.10), canceling the factors of m_1, as

$$\begin{aligned} v_{2,f} \cos\theta_{2,f} &= v_{1,0} - v_{1,f} \cos\theta_{1,f} \\ v_{2,f} \sin\theta_{2,f} &= v_{1,f} \sin\theta_{1,f}. \end{aligned} \tag{18.5.13}$$

Add the squares of the expressions in Eq. (18.5.13), yielding

$$v_{2,f}^2 \cos^2\theta_{2,f} + v_{2,f}^2 \sin^2\theta_{2,f} = (v_{1,0} - v_{1,f} \cos\theta_{1,f})^2 + v_{1,f}^2 \sin^2\theta_{1,f}. \tag{18.5.14}$$

We can use the identities $\cos^2\theta_{2,f} + \sin^2\theta_{2,f} = 1$ and $\cos^2\theta_{1,f} + \sin^2\theta_{1,f} = 1$ to simplify Eq. (18.5.14), yielding

$$v_{2,f}^2 = v_{1,0}^2 - 2 v_{1,0} v_{1,f} \cos\theta_{1,f} + v_{1,f}^2. \tag{18.5.15}$$

Substituting Eq. (18.5.15) into Eq. (18.5.12) yields

$$\frac{1}{2}m_1 v_{1,0}^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_1(v_{1,0}^2 - 2v_{1,0}\, v_{1,f}\cos\theta_{1,f} + v_{1,f}^2). \qquad (18.5.16)$$

Eq. (18.5.16) simplifies to

$$0 = 2v_{1,f}^2 - 2v_{1,0}v_{1,f}\cos\theta_{1,f}, \qquad (18.5.17)$$

which may be solved for the final speed of object 1,

$$v_{1,f} = v_{1,0}\cos\theta_{1,f} = (3.0\ \mathrm{m\cdot s^{-1}})\cos 30^\circ = 2.6\ \mathrm{m\cdot s^{-1}}. \qquad (18.5.18)$$

Divide the expressions in Eq. (18.5.13), yielding

$$\frac{v_{2,f}\sin\theta_{2,f}}{v_{2,f}\cos\theta_{2,f}} = \frac{v_{1,f}\sin\theta_{1,f}}{v_{1,0} - v_{1,f}\cos\theta_{1,f}}. \qquad (18.5.19)$$

Eq. (18.5.19) simplifies to

$$\tan\theta_{2,f} = \frac{v_{1,f}\sin\theta_{1,f}}{v_{1,0} - v_{1,f}\cos\theta_{1,f}}. \qquad (18.5.20)$$

Thus object 2 moves at an angle

$$\begin{aligned}
\theta_{2,f} &= \tan^{-1}\left(\frac{v_{1,f}\sin\theta_{1,f}}{v_{1,0} - v_{1,f}\cos\theta_{1,f}}\right) \\
\theta_{2,f} &= \tan^{-1}\left(\frac{(2.6\ \mathrm{m\cdot s^{-1}})\sin 30^\circ}{3.0\ \mathrm{m\cdot s^{-1}} - (2.6\ \mathrm{m\cdot s^{-1}})\cos 30^\circ}\right) \\
&= 60^\circ.
\end{aligned} \qquad (18.5.21)$$

The above results for $v_{1,f}$ and $\theta_{2,f}$ may be substituted into either of the expressions in Eq. (18.5.13), or Eq. (18.5.12), to find $v_{2,f} = 1.5\,\mathrm{m\cdot s^{-1}}$.

Before going on, it must be noted that the fact that $\theta_{1,f} + \theta_{2,f} = 90^\circ$, that is, the objects move away from the collision point at right angles, is not a coincidence. A vector derivation is presented below. We can see this result algebraically from the above result. Using the result of Eq. (18.5.18), $v_{1,f} = v_{1,0}\cos\theta_{1,f}$, in Eq. (18.5.20) yields

$$\tan\theta_{2,f} = \frac{\cos\theta_{1,f}\sin\theta_{1,f}}{1-\cos\theta_{1,f}^2} = \cot\theta_{1,f}\,; \tag{18.5.22}$$

the angles $\theta_{1,f}$ and $\theta_{2,f}$ are complements.

It should be noted that Eq. (18.5.17) also has the solution $v_{2,f} = 0$, which would correspond to the incident particle missing the target completely.

18.5.4 Example Equal Mass Particles in a Two-Dimensional Elastic Collision Emerge at Right Angles

We can prove that the equal mass particles emerge from the collision at right angles by making explicit use of the fact that momentum is a vector quantity.

Since there are no external forces acting on the two objects during the collision (the collision forces are all internal), momentum is constant. Therefore

$$\vec{\mathbf{p}}_0^{\text{total}} = \vec{\mathbf{p}}_f^{\text{total}} \tag{18.5.23}$$

which becomes

$$m_1\vec{\mathbf{v}}_{1,0} = m_1\vec{\mathbf{v}}_{1,f} + m_1\vec{\mathbf{v}}_{2,f} \tag{18.5.24}$$

Eq. (18.5.24) simplifies to

$$\vec{\mathbf{v}}_{1,0} = \vec{\mathbf{v}}_{1,f} + \vec{\mathbf{v}}_{2,f}\ . \tag{18.5.25}$$

Recall the vector identity that the square of the speed is given by the dot product

$$\vec{\mathbf{v}}\cdot\vec{\mathbf{v}} = v^2\ . \tag{18.5.26}$$

With this identity in mind, we take the dot product of each side of Eq. (18.5.25) with itself,

$$\begin{aligned}\vec{\mathbf{v}}_{1,0}\cdot\vec{\mathbf{v}}_{1,0} &= (\vec{\mathbf{v}}_{1,f} + \vec{\mathbf{v}}_{2,f})\cdot(\vec{\mathbf{v}}_{1,f} + \vec{\mathbf{v}}_{2,f}) \\ &= \vec{\mathbf{v}}_{1,f}\cdot\vec{\mathbf{v}}_{1,f} + 2\vec{\mathbf{v}}_{1,f}\cdot\vec{\mathbf{v}}_{2,f} + \vec{\mathbf{v}}_{2,f}\cdot\vec{\mathbf{v}}_{2,f}.\end{aligned} \tag{18.5.27}$$

This becomes

$$v_{1,0}^2 = v_{1,f}^2 + 2\vec{\mathbf{v}}_{1,f}\cdot\vec{\mathbf{v}}_{2,f} + v_{2,f}^2\ . \tag{18.5.28}$$

Recall that kinetic energy is the same before and after an elastic collision, and the masses of the two objects are equal, so Eq. (18.5.12) simplifies to

$$v_{1,0}^2 = v_{1,f}^2 + v_{2,f}^2 \,. \tag{18.5.29}$$

Comparing Eq. (18.5.28) with Eq. (18.5.29), we see that

$$\vec{\mathbf{v}}_{1,f} \cdot \vec{\mathbf{v}}_{2,f} = 0 \quad . \tag{18.5.30}$$

The dot product of two nonzero vectors is zero when the two vectors are at right angles to each other.

18.5.5 Example: Two-Dimensional Collision

A particle of mass m_{inc} (for "mass of the incident particle") with initial speed v_0 collides with a particle of mass m_{tar} (for "mass of the target particle"), initially at rest. After the collision the directions of the incident particle and the target particle are observed to be perpendicular. The speeds of the incident and target particles particle after the collision are determined to be

$$\begin{aligned} v_{f,\,\text{inc}} &= \frac{3}{5} v_0 \\ v_{f,\,\text{tar}} &= \frac{1}{5} v_0 . \end{aligned} \tag{18.5.31}$$

What is the ratio $m_{\text{tar}} / m_{\text{inc}}$? Forces other than the interaction between the particles should be neglected.

Solutions:

Two methods of solution will be presented here. The first is more straightforward, and the second involves a bit of a trick.

First, take the positive x-direction to be the initial direction of motion of the incident particle, and denote the direction of motion of the incident particle after the collision as making an angle α with respect to the x-direction and the direction of motion of the target particle after the collision as making an angle β with respect to the x-direction, as shown in the diagram. (The magnitudes of the velocity vectors in the diagram are not to scale.)

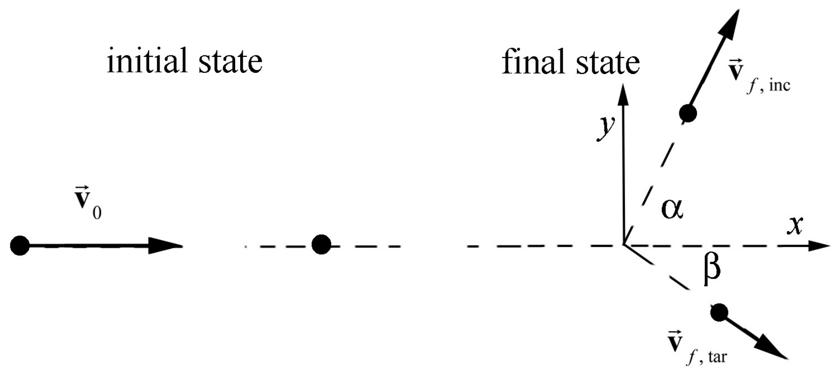


Figure 18.9 Momentum flow diagram for two-particle collision

From the conditions given in the problem, α and β are complementary angles, $\alpha + \beta = \pi/2$. The linear momentum is a constant,

$$m_{\mathrm{inc}}\vec{\mathbf{v}}_0 = m_{\mathrm{inc}}\vec{\mathbf{v}}_{f,\,\mathrm{inc}} + m_{\mathrm{tar}}\vec{\mathbf{v}}_{f,\,\mathrm{tar}}\,. \tag{18.5.32}$$

In component form,

$$\begin{aligned} m_{\mathrm{inc}}v_0 &= m_{\mathrm{inc}}v_{f,\,\mathrm{inc}}\cos\alpha + m_{\mathrm{tar}}v_{f,\,\mathrm{tar}}\cos\beta \\ 0 &= m_{\mathrm{inc}}v_{f,\,\mathrm{inc}}\sin\alpha - m_{\mathrm{tar}}v_{f,\,\mathrm{tar}}\sin\beta. \end{aligned} \tag{18.5.33}$$

At this point, it's best to use the information given in the problem, that $\alpha + \beta = \pi/2$, to express the trigonometric functions of β in terms of α, $\cos\beta = \sin\alpha$, $\sin\beta = \cos\alpha$, so that the expressions in (18.5.33) reduce to

$$\begin{aligned} m_{\mathrm{inc}}v_0 &= m_{\mathrm{inc}}v_{f,\,\mathrm{inc}}\cos\alpha + m_{\mathrm{tar}}v_{f,\,\mathrm{tar}}\sin\alpha \\ 0 &= m_{\mathrm{inc}}v_{f,\,\mathrm{inc}}\sin\alpha - m_{\mathrm{tar}}v_{f,\,\mathrm{tar}}\cos\alpha. \end{aligned} \tag{18.5.34}$$

There are many valid ways to proceed from Eq.s (18.5.34). Any algebraic manipulation will be equivalent to multiplying the first expression in (18.5.34) by $\cos\alpha$ and the second by $\sin\alpha$ and adding, canceling the $m_{\mathrm{tar}}v_{f,\,\mathrm{tar}}$ terms, with the result

$$m_{\mathrm{inc}}v_0\cos\alpha = m_{\mathrm{inc}}v_{f,\,\mathrm{inc}}(\cos^2\alpha + \sin^2\alpha) = m_{\mathrm{inc}}v_{f,\,\mathrm{inc}}\,. \tag{18.5.35}$$

From the observed speeds as given in (18.5.31), $\cos\alpha = 3/5$ and so $\sin\alpha = 4/5$ (in degrees, $\alpha \approx 37^\circ$). From either expression in (18.5.34),

$$\begin{aligned}\frac{m_{\text{tar}} v_{f,\,\text{tar}}}{m_{\text{inc}} v_{f,\,\text{inc}}} &= \frac{m_{\text{tar}}}{m_{\text{inc}}} \frac{v_0/5}{3v_0/5} = \frac{4/5}{3/5} \\ \frac{m_{\text{tar}}}{m_{\text{inc}}} &= 4.\end{aligned} \tag{18.5.36}$$

Alternate Solution:

For this idealized situation, we can use the algebraic expression for momentum to obtain the same result. That is, denote the initial momentum by $\vec{\mathbf{p}}_0$ and the final momenta by $\vec{\mathbf{p}}_{f,\,\text{inc}}$ and $\vec{\mathbf{p}}_{f,\,\text{tar}}$. Equating initial momentum and total final momentum,

$$\vec{\mathbf{p}}_0 = \vec{\mathbf{p}}_{f,\,\text{inc}} + \vec{\mathbf{p}}_{f,\,\text{tar}}. \tag{18.5.37}$$

The square of the magnitude of each side of Eq. (18.5.37) is found by taking the dot product of each side with itself,

$$\begin{aligned}\vec{\mathbf{p}}_0 \cdot \vec{\mathbf{p}}_0 &= (\vec{\mathbf{p}}_{f,\,\text{inc}} + \vec{\mathbf{p}}_{f,\,\text{tar}}) \cdot (\vec{\mathbf{p}}_{f,\,\text{inc}} + \vec{\mathbf{p}}_{f,\,\text{tar}}) \\ p_0^{\,2} &= (p_{f,\,\text{inc}})^2 + 2\,\vec{\mathbf{p}}_{f,\,\text{inc}} \cdot \vec{\mathbf{p}}_{f,\,\text{tar}} + (p_{f,\,\text{tar}})^2 \\ &= (p_{f,\,\text{inc}})^2 + (p_{f,\,\text{tar}})^2\end{aligned} \tag{18.5.38}$$

where the fact that the final velocities and hence the final momenta are perpendicular, $\vec{\mathbf{p}}_{f,\,\text{inc}} \cdot \vec{\mathbf{p}}_{f,\,\text{tar}} = 0$, has been used. The observations as given in (18.5.31), inserted into (18.5.38), give

$$m_{\text{inc}}^{\;2}\, v_0^{\;2} = m_{\text{inc}}^{\;2} \left(\frac{3}{5} v_0\right)^2 + m_{\text{tar}}^{\;2} \left(\frac{1}{5} v_0\right)^2 \tag{18.5.39}$$

from which $m_{\text{tar}} = 4\,m_{\text{inc}}$ readily.

It should be noted that this problem has been rigged to allow fairly simple calculation, primarily in giving the final velocities as being perpendicular and the ratios of the final speeds to the initial speed adjusted to give the final ratio of the masses to be an integer. A four-to-one ratio would not be unusual for elementary particles, but as will be seen in the next chapter, almost half of the initial kinetic energy has been lost, and in the absence of external forces, one particle would have to be in an extremely excited state. Sometimes, such collisions are taken to be representative of, for instance, ice hockey players colliding. In such collisions, losing half of the kinetic energy is typical, but having one player four times as massive as the other is bordering on unfair.

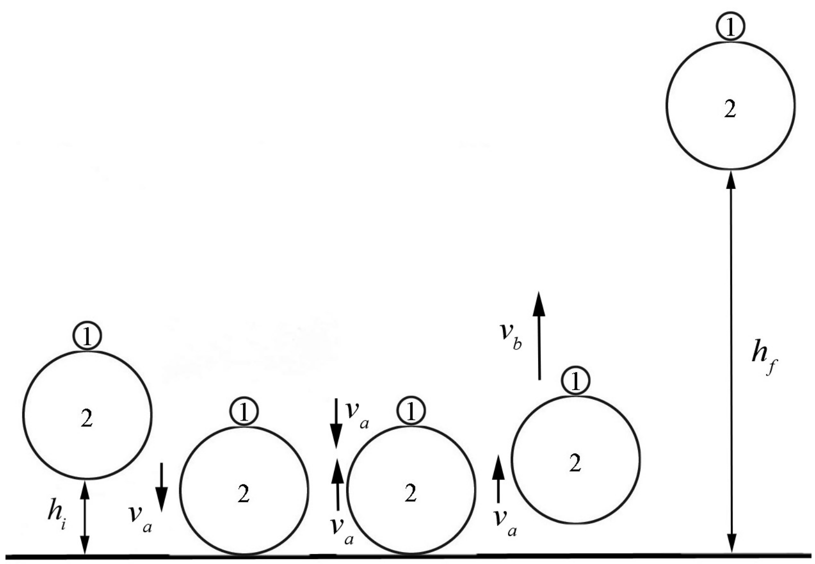
The point of this example is that for a collision that needs to be analyzed in two dimensions, there will be four quantities needed to describe the final state, either two sets of two components or two sets of components and angles. Equating initial and final momenta gives two relations, as in (18.5.33) above. The problem statement gave *three* relations involving the final velocities in terms of the initial velocities, two magnitudes and on relation between the directions. This may seem like too many relations, but consider the nature of the question; we sought the ratio of the masses, a fifth quantity. So, we had five relations between five unknowns and we were able to solve for the mass ratio.

18.5.6 Example: Bouncing Superballs

Two superballs are dropped from a height above the ground, one on top of the other. The ball on top has a mass m_1, and the ball on the bottom has a mass m_2. Assume that the when the lower ball collides with the ground there is no loss of kinetic energy. Then, as the lower ball starts to move upward, it collides with the upper ball that is still moving downwards. Assume again that the total energy of the two balls remains the same after the collision. How high will the upper ball rebound in the air? Assume $m_2 >> m_1$.

Hint: consider this collision as seen by an observer moving upward with the same speed as the ball 2 has after it collides with ground. What speed does ball 1 have in this reference frame after it collides with the ball 2?

The two balls that are dropped from a height h_i above the ground, one on top of the other. Ball 1 is on top and has mass m_1, and ball 2 is underneath and has mass m_2 with $m_2 >> m_1$. Assume that there is no loss of kinetic energy during all collisions. Ball 2 first collides with the ground and rebounds. Then, as ball 2`starts to move upward, it collides with the ball 1 which is still moving downwards. How high will ball 1 rebound in the air?



Solution:

The system consists of the two balls and the earth. There are five special states for this motion shown in the figure above.

Initial State: the balls are released from rest at a height h_i above the ground.

State A: the balls just reach the ground with speed $v_a = \sqrt{2gh_i}$.

State B: immediately before the collision of the balls collide but after ball 2 has collided with the ground and reversed direction with the same speed, v_a. Ball 1 is still moving down with speed v_a.

State C: immediately after the collision of the balls. Because we are assuming that $m_2 >> m_1$, ball 2 does not change it's speed as a result fo the collision so it is still moving upward with speed v_a. As a result of the collision, Ball 1 moves upward with speed v_b.

State D: ball 1 reaches a maximum height $h_f = v_b^2 / 2g$ above the ground.

Choice of Reference Frame:

As indicated in the hint above, this collision is best analyzed from the reference frame of an observer moving upward with speed v_a, the speed of ball 2 just after it rebounded with the ground. In this frame immediately, before the collision, ball 1 is moving downward with a speed v_b' that is twice the speed seen by an observer at rest on the ground (lab reference frame).

$$v_a' = 2v_a \tag{18.5.40}$$

The mass of ball 2 is much larger than the mass of ball 1, $m_2 >> m_1$. This enables us to consider the collision (between States B and C) to be equivalent to ball 1 bouncing off a hard wall, while ball 2 experiences virtually no recoil. Hence ball 2 remains at rest in the reference frame moving upwards with speed v_a with respect to observer at rest on ground. Before the collision, ball 1 has speed $v_a' = 2v_a$. Since there is no loss of kinetic energy during the collision, the result of the collision is that ball 1 changes direction but maintains the same speed,

$$v_b' = 2v_a . \tag{18.5.41}$$

However, according to an observer at rest on the ground, after the collision ball 1 is moving upwards with speed

$$v_b = 2v_a + v_a = 3v_a\,. \tag{18.5.42}$$

While rebounding, the mechanical energy of the smaller superball is constant (we consider the smaller superball and the Earth as a system) hence between State C and State D,

$$\Delta K + \Delta U = 0\,. \tag{18.5.43}$$

The change in kinetic energy is

$$\Delta K = -\frac{1}{2}m_1(3v_a)^2\,. \tag{18.5.44}$$

The change in potential energy is

$$\Delta U = m_1\,g\,h_f\,. \tag{18.5.45}$$

So the condition that mechanical energy is constant (Eq. (18.5.43)) is now

$$-\frac{1}{2}m_1(3v_{1a})^2 + m_1\,g\,h_f = 0\,. \tag{18.5.46}$$

We can rewrite Eq. (18.5.46) as

$$m_1\,g\,h_f = 9\frac{1}{2}m_1(v_a)^2\,. \tag{18.5.47}$$

Recall that we can also use the fact that the mechanical energy doesn't change between the Initial State and State A yielding an Eq. similar to Eq. (18.5.47),

$$m_1\,g\,h_i = \frac{1}{2}m_1(v_a)^2\,. \tag{18.5.48}$$

Now substitute the expression for the kinetic energy in Eq. (18.5.48) into Eq. (18.5.47) yielding

$$m_1\,g\,h_f = 9\,m_1\,g\,h_i\,. \tag{18.5.49}$$

Thus ball 1 reaches a maximum height

$$h_f = 9\,h_i\,. \tag{18.5.50}$$