Module 7: The Concept of Force

Where was the chap I saw in the picture somewhere? Ah yes, in the dead sea floating on his back, reading a book with a parasol open. Couldn't sink if you tried: so thick with salt. Because the weight of the water, no, the weight of the body in the water is equal to the weight of the what? Or is it the volume equal to the weight? It's a law something like that. Vance in High school cracking his fingerjoints, teaching. The college curriculum. Cracking curriculum. What is weight really when you say weight? Thirtytwo feet per second per second. Law of falling bodies: per second per second. They all fall to the ground. The earth. It's the force of gravity of the earth is the weight.

James Joyce, Ullysses[1]

7.1 Introduction

In our daily experience, we can cause a body to move by either pushing or pulling that body. Ordinary language use describes this action as the effect of a person's strength or *force*. However, bodies placed on inclined planes, or when released at rest and undergo free fall, will move without any push or pull. Galileo still referred to a force acting on these bodies, a description of which he published in 1623 in his *Mechanics*. In 1687, Isaac Newton published his three laws of motion in the *Philosophiae Naturalis Principia Mathematica* ("Mathematical Principles of Natural Philosophy"), which extended Galileo's observations. The First Law expresses the idea that when a no force acts on a body, it will remain at rest or maintain uniform motion; when a force is applied to a body, it will change its state of motion.

> *Law 1: Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.*

> *Projectiles continue in their motions, so far as they are not retarded by the resistance of air, or impelled downwards by the force of gravity. A top, whose parts by their cohesion are continually drawn aside from rectilinear motions, does not cease its rotation, otherwise than as it is retarded by air. The greater bodies of planets and comets, meeting with less resistance in freer spaces, preserve their motions both progressive and circular for a much longer time.*

The idea that force produces motion was recognized before Newton by many scientists, especially Galileo, but Newton extended the concept of force to any circumstance that produces acceleration. When the body is initially at rest, the direction of our push or pull corresponds to the direction of motion of the body. If the body is moving, the direction of the applied force may change both the direction of motion of the body and how fast it is moving. This enables us to precisely define force in terms of

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[1] James Joyce, *Ulysses*, The Corrected Text edited by Hans Walter Gabler with Wolfhard Steppe and Claus Melchior, Random House, New York.

acceleration.

We shall define force first in terms of its effect on the standard body we introduced in Section 1.4, which by definition has a mass $m_s = 1\,\text{kg}$. We apply an action to the standard body that will induce the body to accelerate with a magnitude $|\vec{\mathbf{a}}|$ that can be measured by an accelerometer (any device that measures acceleration).

Definition: Force

Force is a vector quantity. The magnitude of the total force $|\vec{\mathbf{F}}|$ acting on the object is the product of the mass m_s with the magnitude of the acceleration $|\vec{\mathbf{a}}|$. The direction of the total force on the standard body is defined to be the direction of the acceleration of the body. Thus

$$\vec{\mathbf{F}} \equiv m_s \vec{\mathbf{a}} \tag{7.1.1}$$

The SI units for force are $[\text{kg} \cdot \text{m} \cdot \text{s}^{-2}]$. This unit has been named the newton [N] and $1\,\text{N} = 1\,\text{kg} \cdot \text{m} \cdot \text{s}^{-2}$.

In order to justify the statement that force is a vector quantity, we need to apply two forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ simultaneously to our standard body and show that the resultant force $\vec{\mathbf{F}}^T$ is the vector sum of the two forces when they are applied one at a time.
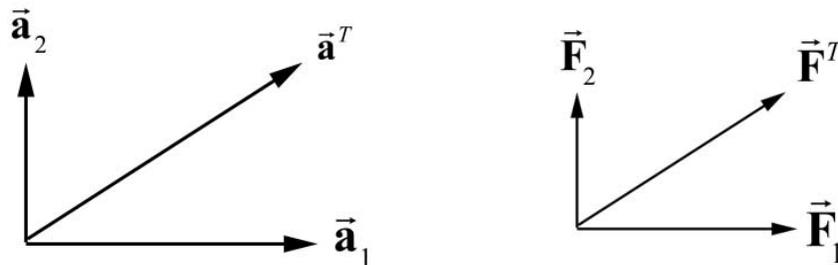

Figure 7.1: Force is a vector concept

When we apply the two forces simultaneously, we measure the acceleration $\vec{\mathbf{a}}$, and define

$$\vec{\mathbf{F}}^T \equiv m_s \vec{\mathbf{a}} \,. \tag{7.1.2}$$

We then apply each force separately and measure the accelerations $\vec{\mathbf{a}}_1$ and $\vec{\mathbf{a}}_2$, noting that

$$\vec{\mathbf{F}}_1 = m_s \vec{\mathbf{a}}_1 \qquad\qquad (7.1.3)$$

$$\vec{\mathbf{F}}_2 = m_s \vec{\mathbf{a}}_2 . \qquad\qquad (7.1.4)$$

We then compare the accelerations. The results of these three measurements, and for that matter any similar experiment, confirms that the accelerations add as vectors

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}_1 + \vec{\mathbf{a}}_2 . \qquad\qquad (7.1.5)$$

Therefore the forces add as vectors as well,

$$\vec{\mathbf{F}}^T = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 . \qquad\qquad (7.1.6)$$

This last statement is not a definition but a consequence of the experimental result described by Equation (7.1.5) and our definition of force.

Free Body Force Diagram

Since force is a vector concept, the total force may be the vector sum of individual forces acting on the body

$$\vec{\mathbf{F}}^T = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots \qquad\qquad (7.1.7)$$

A *free body force diagram* is a representation of the sum of all the forces that act on the isolated body. For each body, represent each force acting on the body by an arrow that indicates the direction of the force. For example, the forces that regularly appear in free body diagram are contact forces, tension, gravitation, friction, pressure forces, spring forces, electric and magnetic forces which we shall introduce below.

Suppose we choose a Cartesian coordinate system, then we can resolve the total force into its component vectors

$$\vec{\mathbf{F}}^T = F_x^T \, \hat{\mathbf{i}} + F_y^T \, \hat{\mathbf{j}} + F_z^T \, \hat{\mathbf{k}} \qquad\qquad (7.1.8)$$

Each one of the component vectors is itself a vector sum of the individual component vectors from each contributing force. We can use the free body force diagram to make these vector decompositions of the individual forces. For example, the x - component of the total force is

$$F_x^T = F_{1,x}^T + F_{2,x}^T + \cdots . \qquad\qquad (7.1.9)$$

Example 1 Two horizontal ropes are attached to a post that is stuck in the ground. The ropes pull the post producing the vector forces $\vec{F}_1 = 70 \text{ N } \hat{i} + 20 \text{ N } \hat{j}$ and $\vec{F}_2 = -30 \text{ N } \hat{i} + 40 \text{ N } \hat{j}$ as shown in the figure. Find the direction and magnitude of the sum of the forces exerted by the ropes on the post.
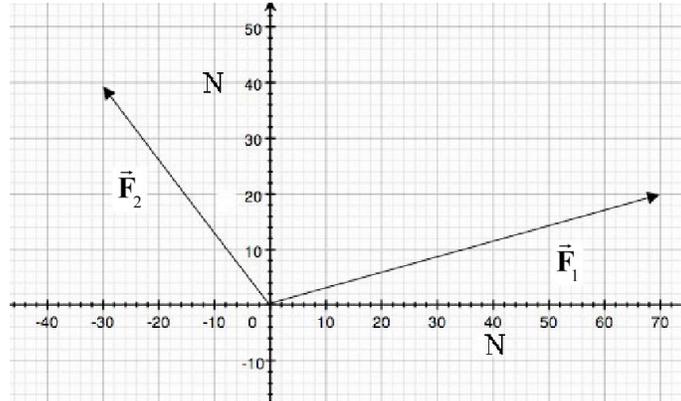
Solution: Let $\vec{F} \equiv \vec{F}_1 + \vec{F}_2$ denote the sum of the forces due to the ropes. Then we can write the sum as

$$\vec{F} \equiv \vec{F}_1 + \vec{F}_2 = (F_{x1} + F_{x2})\,\hat{i} + (F_{y1} + F_{y2})\,\hat{j} = (70 \text{ N} + -30 \text{ N})\,\hat{i} + (20 \text{ N} + 40 \text{ N})\,\hat{j}$$
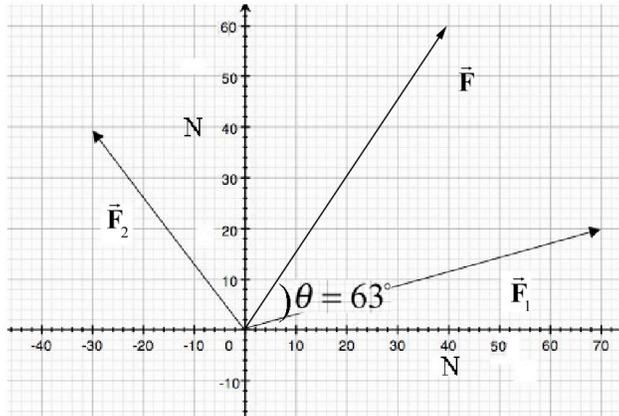$$= (40 \text{ N})\,\hat{i} + (60 \text{ N})\,\hat{j}$$
.(7.1.10)

The magnitude is

$$\left|\vec{F}\right| = \sqrt{(40 \text{ N})^2 + (60 \text{ N})^2} = 72 \text{ N}.$$
(7.1.11)

The angle the force $\vec{F}$ makes with the positive x-axis given by

$$\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\left[\frac{60 \text{ N}}{40 \text{ N}}\right]\right) = 63°.$$
(7.1.12)

as shown in the figure below.

7.2 Mass

Our definition of force is based on proportionality between force and acceleration,

$$\vec{F} \propto \vec{a} .$$
(7.2.1)

In order to define the magnitude of the force, we introduced a constant of proportionality, the *inertial mass*, which Newton called a "quantity of matter" and he was the first to clearly understand that inertial mass was a property of a body different from "weight" (see section 3.5). So far, we have only used the standard body to measure force. Instead of performing experiments on the standard body, we can calibrate the masses of all other bodies in terms of the standard mass by the following experimental procedure.

We apply a force of magnitude F to the standard body and measure the magnitude of the acceleration a_s. Then we apply the same force to a second body of unknown mass m_u. We measure the magnitude of the unknown acceleration a_u. Since the same force is applied to both bodies,

$$F = m_u \, a_u = m_s \, a_s ,$$
(7.2.2)

Therefore the ratio of the unknown mass to the standard mass is equal to the inverse ratio of the magnitudes of the accelerations,

$$\frac{m_u}{m_s} = \frac{a_s}{a_u} .$$
(7.2.3)

Therefore the second body has mass equal to

$$m_u \equiv m_s \frac{a_s}{a_u} .$$
(7.2.4)

This method is justified by the fact that we can repeat the experiment using a different force and still find that the ratios of the acceleration are the same.

7.3 Newton's Second Law, Force Laws, and Predicting Motion

Newton's Second Law expresses the idea that when a body undergoes acceleration, a force is acting on it:

> *Law II: The change of motion is proportional to the motive force impresses, and is made in the direction of the right line in which that force is impressed.*
>
> *If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force is impressed altogether and at once or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added or subtracted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both.*

Since we defined force in terms of the change in motion, the Second Law appears to be a restatement of this definition, and devoid of predictive power since force is only determined by measuring acceleration. What transforms the Second Law from a definition into the Equations of Motion for a physical system that can, in principle, predict the future positions and velocities of all bodies is the additional input that comes from *Force Laws* that are based on experimental observations on the interactions between bodies.

There are forces that don't change appreciably from one instant to another, which we refer to as constant in time, and forces that don't change appreciably from one point to another, which we refer to as constant in space. The gravitation force on a body near the surface of the earth is an example of a force that is constant in space.

There are forces that increase as you move away. When a mass is attached to one end of a spring and the spring is stretched a distance $|x|$, the spring force increases in strength proportional to the stretch.

There are forces that stay constant in magnitude but always point towards the center of a circle; for example when a ball is attached to a rope and spun in a horizontal circle with constant speed, the tension force acting on the ball is directed towards the center of the circle. This type of attractive central force is called a *centripetal force*.

There are forces that spread out in space such that their influence becomes less with distance. Common examples are the gravitation and electric forces. The gravitational force between two bodies falls off as the inverse square of the distance separating the bodies provided the bodies are of a small dimension compared to the

distance between them. More complicated arrangements of attracting and repelling things give rise to forces that fall off with other powers of r : constant, $1/r$, $1/r^2$, $1/r^3$, etc.,

How do we determine if there is any mathematical relationship, a *force law* that describes the relationship between the force and some measurable property of the bodies involved?

Hooke's Law

We shall illustrate this procedure by considering the force that compressed or stretched springs exert on bodies. In order to stretch or compress a spring from its equilibrium length, a force must be exerted on the spring. Attach a body of mass m to one end of a spring and fix the other end of the spring to a wall (Figure 7.2). Assume that the contact surface is smooth and hence frictionless in order to consider only the effect of the spring force.
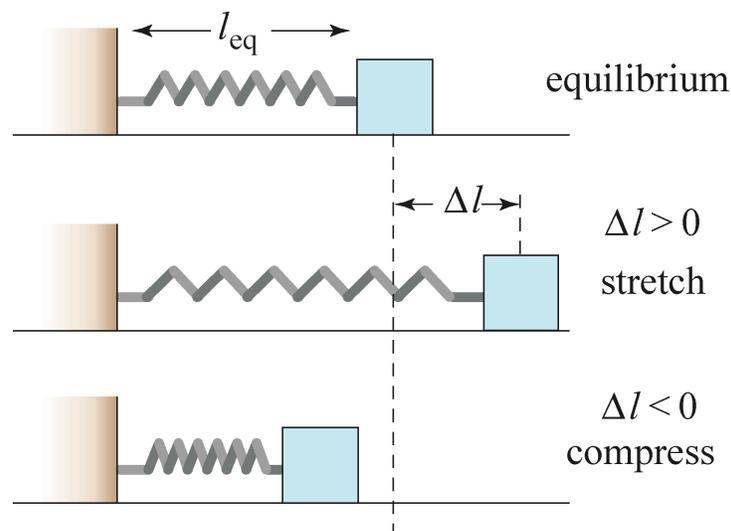

Figure 7.2: A sketch of a spring attached to a wall and an object.

Initially stretch the spring a distance $\Delta l > 0$ (or compress the spring by $\Delta l < 0$), release the body, measure the acceleration, and then calculate the magnitude of the force of the spring acting on the body using the definition of force $\left| \vec{F} \right| = m \left| \vec{a} \right|$. Now repeat the experiment for a range of stretches (or compressions). Experiments will shown that for some range of lengths, $\Delta l_0 < \Delta l < \Delta l_1$, the magnitude of the measured force is proportional to the stretched length and is given by the formula

$$\left| \vec{F} \right| \propto \left| \Delta l \right|. \qquad (7.3.1)$$

In addition, the direction of the acceleration is always towards the equilibrium position when the spring is neither stretched nor compressed. This type of force is called

a *restoring force.* From the experimental data, the constant of proportionality, the *spring constant k*, can be determined.
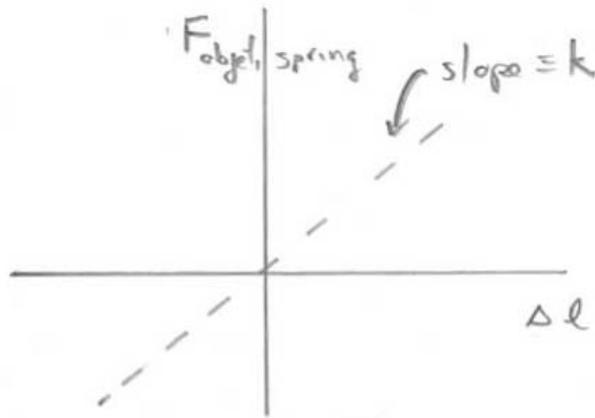

Figure 7.3: Plot of force vs. compression and extension of spring

The spring constant has units $\mathrm{N} \cdot \mathrm{m}^{-1}$. The spring constant for each spring is determined experimentally by measuring the slope of the graph of the force vs. compression and extension stretch (Figure 7.3). Therefore for this one spring, the magnitude of the force is given by

$$\left|\vec{\mathbf{F}}\right| = k\left|\Delta l\right|. \tag{7.3.2}$$

Now perform similar experiments on other springs. For a range of stretched lengths, each spring exhibits the same proportionality between force and stretched length, although the spring constant may differ for each spring.

Since it would be extremely impractical to experimentally determine whether this proportionality holds for all springs, and since a modest sampling of springs has confirmed the relation, we shall *infer* that all springs will produce a restoring force which is linearly proportional to the stretched (or compressed) length. This experimental relation regarding force and stretched (or compressed) lengths for a finite set of springs has now been inductively generalized into the above mathematical model for springs, a model we refer to as a *force law*.

This *Newtonian induction* is the critical step that makes physics a predictive science. Suppose a spring, attached to an object of mass m, is stretched by an amount Δl. A prediction can be made, using the force law, that the magnitude of the force between the rubber band and the object is $\left|\vec{\mathbf{F}}\right| = k\left|\Delta l\right|$ *without having to experimentally measure the acceleration.*

Now we can use Newton's second Law to predict the magnitude of acceleration of the body;

$$|\vec{\mathbf{a}}| = \frac{|\vec{\mathbf{F}}|}{m} = \frac{k|\Delta l|}{m}. \tag{7.3.3}$$

Now perform the experiment, and measure the acceleration within some error bounds. If the magnitude of the predicted acceleration disagrees with the measured result, then the model for the force law needs modification. The ability to adjust, correct or even reject models based on new experimental results enables a description of forces between objects to cover larger and larger experimental domains.

Gravitational Force near the Surface of the Earth

Near the surface of the earth, the gravitational interaction between a body and the earth is mutually attractive and has a magnitude of

$$|\vec{\mathbf{F}}_{grav}| = m_{grav}\, g \tag{7.3.4}$$

where m_{grav} is the *gravitational mass* of the body and g is a positive constant.

The International Committee on Weights and Measures has adopted as a standard value for the acceleration of a body freely falling in a vacuum $g = 9.80665\ \mathrm{m}\cdot\mathrm{s}^{-2}$. The actual value of g varies as a function of elevation and latitude. If ϕ is the latitude and h the elevation in meters then the acceleration of gravity in SI units is

$$g = \left(9.80616 - 0.025928\cos(2\phi) + 0.000069\cos^2(2\phi) - 3.086\times10^{-4}h\right)\mathrm{m}\cdot\mathrm{s}^{-2}. \tag{7.3.5}$$

This is known as Helmert's equation. The strength of the gravitational force on the standard kilogram at $42°$ latitude is $9.80345\ \mathrm{N}\cdot\mathrm{kg}^{-1}$, and the acceleration due to gravity at sea level is therefore $g = 9.80345\ \mathrm{m}\cdot\mathrm{s}^{-2}$ for all objects. At the equator, $g = 9.78\ \mathrm{m}\cdot\mathrm{s}^{-2}$ (to three significant figures), and at the poles $g = 9.83\ \mathrm{m}\cdot\mathrm{s}^{-2}$. This difference is primarily due to the earth's rotation, which introduces an apparent repulsive force that affects the determination of g as given in Equation (7.3.4) and also flattens the spherical shape earth (the distance from the center of the earth is larger at the equator than it is at the poles by about $26.5\ \mathrm{km}$). Both the magnitude and the direction of the gravitational force also show variations that depend on local features to an extent that's useful in prospecting for oil and navigating submerged nuclear submarines. Such variations in g can be measured with a sensitive spring balance. Local variations have been much studied over the past two decades in attempts to discover a proposed "fifth force" which would fall off faster than the gravitational force that falls off as the inverse square of the distance between the masses.

The Principle of Equivalence (see section 6.4) states that the gravitational mass is identical to the inertial mass that is determined with respect to the standard kilogram.

$$m_{\text{gravitational}} = m_{\text{inertial}} . \tag{7.3.6}$$

From this point on, inertial and gravitational mass will be denoted by the symbol m.

Newton's Third Law: Action-Reaction Pairs

Newton realized that when two bodies interact via a force, then the force on one body is equal in magnitude and opposite in direction to the force acting on the other body.

> *Law III: To every action there is always opposed an equal reaction: or, the mutual action of two bodies upon each other are always equal, and directed to contrary parts.*
>
> *Whatever draws or presses another is as much drawn or pressed by that other. If you press a stone with your finger, the finger is also pressed by the stone.*

The Third Law is the most surprising of the three laws. Newton's great discovery was that when two objects interact, they each impart the same magnitude of force on each other. Any pair of forces that satisfy the third law are referred to as an *action-reaction* pair of forces. A pair of action-reaction forces can never act on the same body.

Consider two bodies engaged in a mutual interaction. Label the bodies 1 and 2 respectively. Let $\vec{F}_{1,2}$ be the force on body 1 due to the interaction with body 2. Let $\vec{F}_{2,1}$ be the force on body 2 due to the interaction with body 1. These forces are depicted in Figure 7.4.

Figure 7.4: Action-reaction pair of forces

These two vector forces are equal in magnitude and opposite in direction,

$$\vec{F}_{1,2} = -\vec{F}_{2,1} . \tag{7.3.7}$$

Models in Physics: Fundamental Laws of Nature

Force laws are mathematical models of physical processes. They arise from observation and experimentation, and they have limited ranges of applicability. Does the linear force law for the spring hold for all springs? Each spring will most likely have a different range of linear behavior. So the model for stretching springs still lacks a universal character. As

such, there should be some hesitation to generalize this observation to all springs unless some property of the spring, universal to all springs, is responsible for the force law.

Perhaps springs are made up of very small components, which when pulled apart tend to contract back together. This would suggest that there is some type of force that contracts spring molecules when they are pulled apart. What holds molecules together? Can we find some fundamental property of the interaction between atoms that will suffice to explain the macroscopic force law? This search for *fundamental forces* is a central task of physics.

In the case of springs, this could lead into an investigation of the composition and structural properties of the atoms that compose the steel in the spring. We would investigate the geometric properties of the lattice of atoms and determine whether there is some fundamental property of the atoms that create this lattice. Then we ask how stable is this lattice under deformations. This may lead to an investigation into the electron configurations associated with each atoms and how they overlap to form bonds between atoms. These particles carry charges, which obey Coulomb's Law, but also the Laws of Quantum Mechanics. So in order to arrive at a satisfactory explanation of the elastic restoring properties of the spring, we need models that describe the fundamental physics that underline Hooke's Law.

Universal Law of Gravitation

At points significantly away from the surface of the earth, the gravitation force is no longer constant with respect to the distance to the surface. *Newton's Universal Law of Gravitation* describes the gravitation force between two bodies with masses, m_1 and m_2. This force points along the line connecting the bodies, is attractive, and its magnitude is proportional to the inverse square of the distance, $r_{1,2}$, between the bodies (Figure 7.6a). The force on body 1 due to the gravitational interaction between the two bodies is given by

$$\vec{\mathbf{F}}_{1,2} = -G\frac{m_1 m_2}{r_{1,2}^2}\hat{\mathbf{r}}_{1,2}, \tag{7.3.8}$$

where $\hat{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_{1,2} / \left|\vec{\mathbf{r}}_{1,2}\right|$ is a unit vector directed from body 2 to body 1 (i.e. in Figure 7.6b, $\vec{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$, and $r_{1,2} = \left|\vec{\mathbf{r}}_{1,2}\right|$). The constant of proportionality in SI units is $G = 6.67 \times 10^{-11}\,\mathrm{N}\cdot\mathrm{m}^2\cdot\mathrm{kg}^{-2}$.
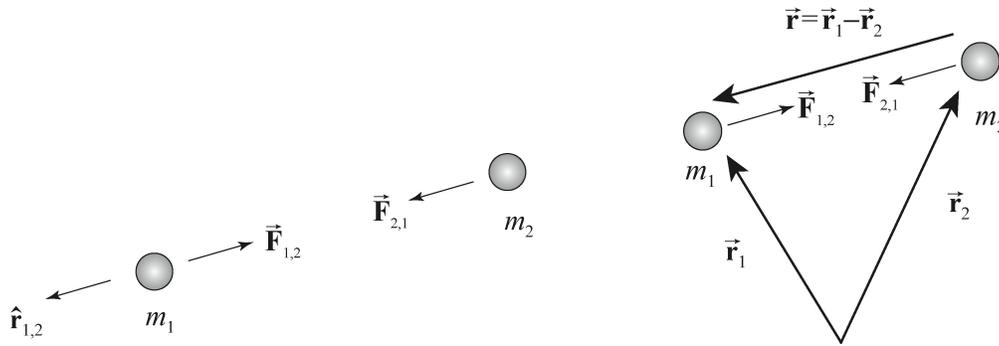
Figure 7.6a Gravitational Force between two bodies. **Figure 7.6b** Coordinate system for the two-body problem.

Electric Charge and Coulomb's Law

Matter has properties other than mass. As we have shown in the previous section, matter can also carry one of two types of observed *electric charge*, positive and negative. Like charges repel, and opposite charges attract each other. The unit of charge in the SI system of units is called the *coulomb* [C].

The smallest unit of "free" charge known in nature is the charge of an electron or proton, which has a magnitude of

$$e = 1.602 \times 10^{-19}\,\text{C}. \tag{7.3.9}$$

It has been shown experimentally that charge carried by ordinary objects is quantized in integral multiples of the magnitude of this free charge . The electron carries one unit of negative charge ($q_{\text{electron}} = -e$) and the proton carries one unit of positive charge ($q_{\text{proton}} = +e$).

In an isolated system, the total charge stays constant; in a closed system, an amount of unbalanced charge can neither be created nor destroyed. Charge can only be transferred from one body to another.

Consider two bodies with charges q_1 and q_2, separated by a distance $r_{1,2}$ in vacuum. By experimental observation, the two bodies repel each other if they are both positively or negatively charged. They attract each other if they are oppositely charged. The force exerted on q_1 due to the interaction between q_1 and q_2 is given by Coulomb's Law,

$$\vec{\mathbf{F}}_{1,2} = k_e \frac{q_1 q_2}{r_{1,2}^2} \hat{\mathbf{r}}_{1,2} \tag{7.3.10}$$

where in SI units, $k_e = 8.9875 \times 10^9 \, \text{N} \cdot \text{m}^2 \cdot \text{C}^{-2}$ and $\hat{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_{1,2} / |\vec{\mathbf{r}}_{1,2}|$ is a unit vector directed from q_2 to q_1 (*i.e.*, $\vec{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$, and $r_{1,2} = |\vec{\mathbf{r}}_{1,2}|$), as illustrated in the Figure 7.7. This law was derived empirically by Charles Augustin de Coulomb in the late 18th century by the same methods as described in previous sections.
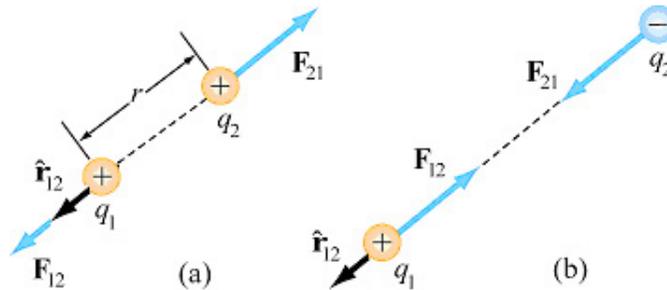
Figure 7.7 Coulomb interaction between two charges

Example 2: Coulomb's Law and the Universal Law of Gravitation

Show that Both Coulomb's Law and the Universal Law of Gravitation satisfy Newton's Third Law.

Solution: To see this, interchange 1 and 2 in the Universal Law of Gravitation to find the force on body 2 due to the interaction between the bodies. The only quantity to change sign is the unit vector

$$\hat{\mathbf{r}}_{2,1} = -\hat{\mathbf{r}}_{1,2} . \tag{7.3.11}$$

Then

$$\vec{\mathbf{F}}_{2,1} = -G \frac{m_2 \, m_1}{r_{2,1}^2} \hat{\mathbf{r}}_{2,1} = G \frac{m_1 \, m_2}{r_{1,2}^2} \hat{\mathbf{r}}_{1,2} = -\vec{\mathbf{F}}_{1,2} . \tag{7.3.12}$$

Coulomb's Law also satisfies Newton's Third Law since the only quantity to change sign is the unit vector, just as in the case of the Universal Law of Gravitation.

Modeling

One of the most central and yet most difficult tasks in analyzing a physical interaction is developing a physical model. A physical model for the interaction consists of a description of the forces acting on all the bodies. The difficulty arises in deciding which forces to include. For example in describing almost all planetary motions, the Universal Law of Gravitation was the only force law that was needed. There were anomalies, for

example the small shift in Mercury's orbit. These anomalies are interesting because they may lead to new physics. Einstein corrected Newton's Law of Gravitation by introducing General Relativity and one of the first successful predictions of the new theory was the perihelion precession of Mercury's orbit. On the other hand, the anomalies may simply be due to the complications introduced by forces that are well understood but complicated to model. When bodies are in motion there is always some type of friction present. Air friction is often neglected because the mathematical models for air resistance are fairly complicated even though the force of air resistance substantially changes the motion. Static or kinetic friction between surfaces is sometimes ignored but not always. The mathematical description of the friction between surfaces has a simple expression so it can be included without making the description mathematically intractable. A good way to start thinking about the problem is to make a simple model, excluding complications that are small order effects. Then we can check the predictions of the model. Once we are satisfied that we are on the right track, we can include more complicated effects.

7.4 Statics and Force Measurements

This procedure for experimentally measuring force in terms of acceleration is rather cumbersome and problematic in the sense that we cannot really isolate a body and only apply one force at a time. In the case of our spring, the friction between the body and the table will complicate our results. We can substitute an alternate approach to measuring forces by noting that forces can be compounded in such a manner as to leave the body in its state of rest (or in uniform motion). Thus the total force acting on the body is zero,

$$\vec{\mathbf{F}}^T = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots = \vec{\mathbf{0}} \,. \tag{7.4.1}$$

The force equation for static equilibrium is then three separate equations, one for each direction in space, and with our choice of a Cartesian coordinate system, are given by

$$F_x^{\ T} = 0 \,, \tag{7.4.2}$$

$$F_y^{\ T} = 0 \,, \tag{7.4.3}$$

$$F_z^{\ T} = 0 \,. \tag{7.4.4}$$

The science of *statics* investigates how the forces can act is such a way, and is not concerned with the motions produced by the individual forces. With this in mind, we can measure forces using a statics procedure.

We begin by using a spring scale, consisting of a spring and a hook from which we can suspend bodies (Figure 7.5). We shall choose a spring that satisfies Hooke's Law for spring forces between $0.2\,\text{N}$ and $10\,\text{N}$. If we need to measure forces in other ranges we can replace the spring.
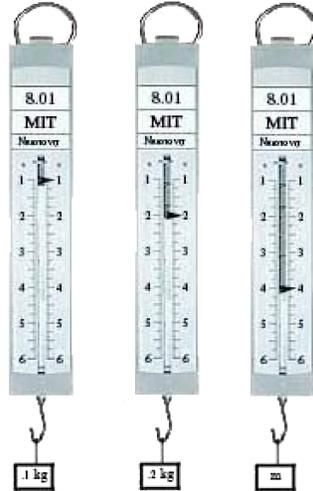
Figure 7.5: Spring Scale

In order to calibrate the scale, we need to apply a standard force that we already know the magnitude. We shall suspend a body of mass $m = 0.102\,\text{kg}$. The gravitation force acting on the body is constant and equal to $\left|\vec{\mathbf{F}}_{\text{grav}}\right| = mg = 1.000\,\text{N}$. The body will pull the spring down, and the spring will stretch. Since the body is in static equilibrium, the spring stretches and exerts a force pulling the body up that is equal in magnitude to the gravitation force pulling the body down,

$$\left|\vec{\mathbf{F}}_{\text{spring}}\right| = \left|\vec{\mathbf{F}}_{\text{grav}}\right|. \tag{7.4.5}$$

Note that this is not a Third Law Action-Reaction pair. A reference point is attached to the spring, and we mark off $1\,\text{N}$ on the scale at the position of this reference point. A second body of twice the mass is now suspended from the spring and we label the position of the reference point by $2\,\text{N}$. Since the spring force is linearly proportional to the stretched length we can now mark off intervals equal to the distance between $1\,\text{N}$ and $2\,\text{N}$ on the scale. We can now attach an unknown body to the spring and measure the spring force. Again using Newton's Third Law, we can then deduce the force that the spring exerts on the body. Notice that we are not measuring the gravitation force acting on the body; we are instead measuring the force with which the body pulls the spring, which is by Newton' s Third Law is equal in magnitude to the force with which the spring pulls the on the body. Because the body is static, this force must be equal in magnitude to the gravitational force acting on the body.

Example 3

In Figure 7.5, the unknown body exerts a force of $4\,\text{N}$ on the spring scale and since the body is in static equilibrium this is equal in magnitude to the gravitation force,

$\left|\vec{\mathbf{F}}_{grav}\right| = mg = 4.000 \text{ N}$. Using $g = 9.803 \text{m} \cdot \text{s}^{-2}$, we therefore conclude that the body has mass

$$m = \frac{\left|\vec{\mathbf{F}}_{grav}\right|}{g} = \frac{4.000 \text{ N}}{9.803 \text{ m} \cdot \text{s}^{-2}} = 0.4080 \text{ kg} . \qquad (7.4.6)$$

Concept Question 7.1:

If you allowed the spring scale and mass to undergo free fall, would the spring stretch? Hint: is the body still in static equilibrium?

Solution: If the body and spring scale undergo free fall, the body is no longer in static equilibrium since it is accelerating. The only force acting on the falling body is the gravitation force. The spring will not stretch and hence does not exert a force on the free falling body.

Example 4 In Example 1 we found that the sum of the forces due to the ropes on the post was given by Eq. (7.1.10), $\vec{\mathbf{F}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 = (40 \text{ N}) \,\hat{\mathbf{i}} + (60 \text{ N}) \,\hat{\mathbf{j}}$. Find the direction and magnitude of the force of the ground on the post that will make the vector sum of forces on the post equal to zero. Assume that the post is in static equilibrium.

Solution: Denote the force of the ground on the post by $\vec{\mathbf{F}}_3$. Therefore the horizontal component of the third force of the post must be equal to

$$\vec{\mathbf{F}}_{hor} = -\vec{\mathbf{C}} = -(\vec{\mathbf{A}} + \vec{\mathbf{B}}) = (-40 \text{ N}) \,\hat{\mathbf{i}} + (-60 \text{ N}) \,\hat{\mathbf{j}} .$$

Because the post is in static equilibrium $\vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3 = \vec{\mathbf{0}}$, the force of the ground on the post is given by

$$\vec{\mathbf{F}}_3 = -(\vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2) = (-40 \text{ N}) \,\hat{\mathbf{i}} + (-60 \text{ N}) \,\hat{\mathbf{j}} . \qquad (7.4.7)$$
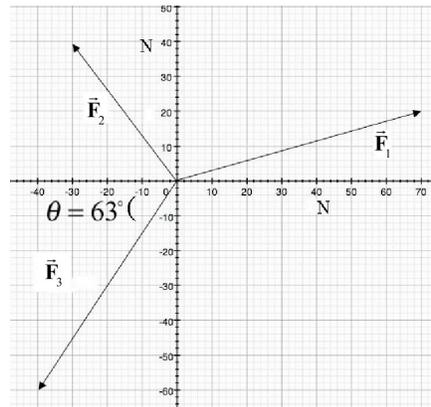
The magnitude is

$$\left|\vec{\mathbf{F}}_3\right| = \sqrt{(-40 \text{ N})^2 + (-60 \text{ N})^2} = 72 \text{ N} . \qquad (7.4.8)$$

The angle the force $\vec{\mathbf{F}}_3$ makes with the negative x-axis given by

$$\theta = \tan^{-1}\left(\frac{F_{y3}^T}{F_{x3}^T}\right) = \tan^{-1}\left(\frac{-60 \text{ N}}{-40 \text{ N}}\right) = 63° . \qquad (7.4.9)$$

as shown in the figure below.

7.5 Contact Forces

Pushing, lifting and pulling are *contact forces* that we experience in the everyday world. Rest your hand on a table; the atoms that form the molecules that make up the table and your hand are in contact with each other. If you press harder, the atoms are also pressed closer together. The electrons in the atoms begin to repel each other and your hand is pushed in the opposite direction by the table.

According to Newton's Third Law, the force of your hand on the table is equal in magnitude and opposite in direction to the force of the table on your hand. Clearly, if you push harder the force increases. Try it! If you push your hand straight down on the table, the table pushes back in a direction perpendicular (normal) to the surface. Slide your hand gently forward along the surface of the table. You barely feel the table pushing upward, but you do feel the friction acting as a resistive force to the motion of your hand. This force acts tangential to the surface and opposite to the motion of your hand. Push downward and forward. Try to estimate the magnitude of the force acting on your hand.

The total force of the table acting on your hand, $\vec{F}_{contact} \equiv \vec{C}$, is called the *contact force*. This force has both a normal component to the surface, $\vec{N}$, called the *normal force,* and a tangential component to the surface, $\vec{f}$, called the *friction force* (Figure 7.8).
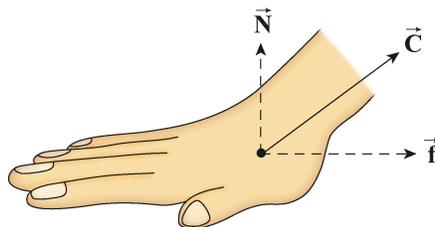

Figure 7.8: Normal and tangential components of the contact force

By the law of vector decomposition for forces,

$$\vec{C} \equiv \vec{N} + \vec{f}. \qquad\qquad (7.5.1)$$

Any force can be decomposed into component vectors so the normal component, $\vec{N}$, and the tangential component, $\vec{f}$, are not independent forces but the vector components of the contact force perpendicular and parallel to the surface of contact.

In Figure 7.9, the total forces acting on your hand are shown. These forces include the contact force, $\vec{C}$, of the table acting on your hand, the force of your forearm, $\vec{F}_{forearm}$, acting on your hand (which is drawn at an angle indicating that you are pushing down on your hand as well as forward), and the gravitational interaction, $\vec{F}_{gravity}$, between the earth and your hand.
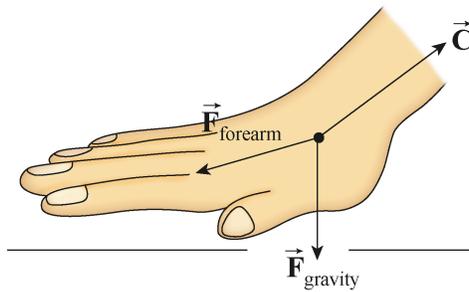

Figure 7.9: Total force on hand moving towards the left

Is there a force law that mathematically describes this contact force? Since there are so many individual electrons interacting between the two surfaces, it is unlikely that we can add up all the individual forces. So we must content ourselves with a macroscopic model for the force law describing the contact force. One point to keep in mind is that the magnitudes of the two components of the contact force depend on how hard you push or pull your hand.

Example 5 Two teams, A and B, are competing in a tug-of-war. Team A is stronger, but neither team is moving because of friction. Draw free-body diagrams for Team A, for Team B, and for the rope. For each force on your free body diagrams, describe the action-reaction force associated with it.
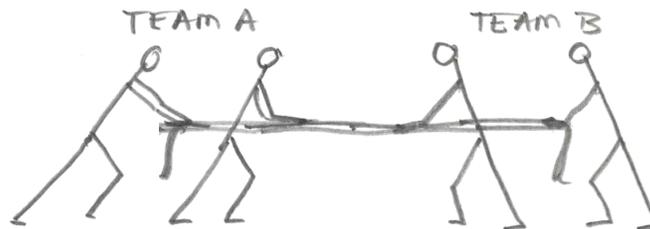

Solution:

The forces acting on Team A are:

1. The force $\vec{\mathbf{F}}_{A, r}$ between Team A and the rope.

2. The gravitation force $(\vec{\mathbf{F}}_{grav})_{A, E} \equiv m_A \vec{\mathbf{g}}$ between Team A and the earth.

3. The contact force $\vec{\mathbf{C}}_{A, G}$ between Team A and the ground.
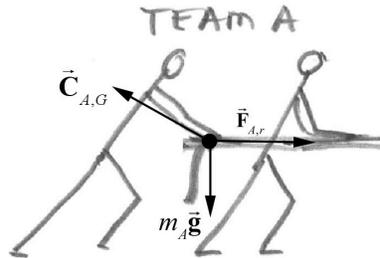
The forces acting on Team B are:

1. The force $\vec{\mathbf{F}}_{B, r}$ between Team B and the rope.

2. The gravitation force $(\vec{\mathbf{F}}_{grav})_{B, E} \equiv m_B \vec{\mathbf{g}}$ between Team B and the earth.

3. The contact force $\vec{\mathbf{C}}_{B, G}$ between Team B and the ground.
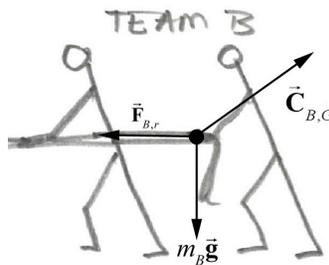
The forces on the rope are

1. The force $\vec{\mathbf{F}}_{r, A}$ between the rope and Team A.

2. The force $\vec{\mathbf{F}}_{r, B}$ between the rope and Team B.

We neglect the gravitation force on the rope.

Since the rope is not moving, these forces sum to zero by Newton's Second Law but they are not Third Law pairs;

$$\vec{\mathbf{F}}_{r,\,A} + \vec{\mathbf{F}}_{r,\,B} = \vec{\mathbf{0}}\,.$$

The Newton's Third Law pairs are

1. The forces between Team A and the rope with $\vec{\mathbf{F}}_{A,\,r} = -\vec{\mathbf{F}}_{r,\,A}$.

2. The forces between Team B and the rope with $\vec{\mathbf{F}}_{B,\,r} = -\vec{\mathbf{F}}_{r,\,B}$.

3. The contact forces between Team A and the ground, $\vec{\mathbf{C}}_{A,\,G} = -\vec{\mathbf{C}}_{G,\,A}$. The contact force of Team A on the ground is not shown on the above force diagrams.

4. The contact forces between Team B and the ground, $\vec{\mathbf{C}}_{B,\,G} = -\vec{\mathbf{C}}_{G,\,B}$. The contact force of Team B on the ground is not shown on the above force diagrams.

5. The gravitation forces between the earth and Team A, $(\vec{\mathbf{F}}_{grav})_{A,\,E} \equiv m_A\,\vec{\mathbf{g}} = -(\vec{\mathbf{F}}_{grav})_{E,\,A}$. The gravitation force on the earth is not shown in the figures above.

6. The gravitation forces between the earth and Team B, $(\vec{\mathbf{F}}_{grav})_{B,\,E} \equiv m_B\,\vec{\mathbf{g}} = -(\vec{\mathbf{F}}_{grav})_{E,\,B}$. The gravitation force on the earth is not shown in the figures above.

The Normal Component of the Contact Force and Weight

Hold an object in your hand. You can feel the "weight" of the object against your palm. But what exactly do we mean by "weight"? Consider the force diagram on the object in Figure 7.10. Let's choose the +-direction to point upward.
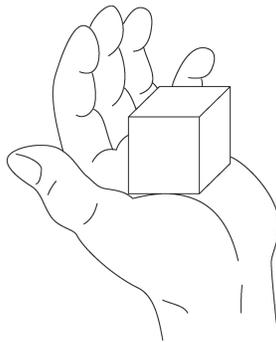

Figure 7.10: Object resting in hand

There are two forces acting on the object. One force is the gravitation force between the earth and the object, and is denoted by $\vec{\mathbf{F}}_{object,\,earth} = \vec{\mathbf{F}}_{gravity} \equiv m\vec{\mathbf{g}}$ where $\vec{\mathbf{g}}$, also known as the gravitational acceleration, is a vector that points downward and has magnitude

$g = 9.8 \text{ m} \cdot \text{s}^{-2}$. The other force on the object is the contact force between your hand and the object. Since we are not pushing the block horizontally, this contact force on your hand points perpendicular to the surface, and hence has only a normal component, $\vec{\mathbf{N}}$. Let N denote the magnitude of the normal force. The force diagram on the object is shown in Figure 7.11.
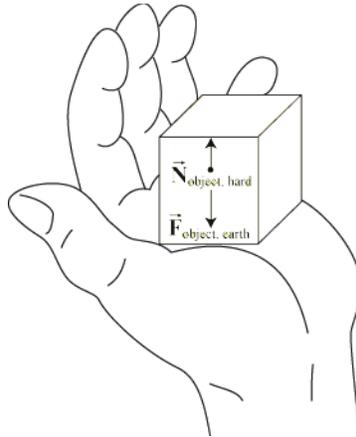

Figure 7.11: Force diagram on object

Since the object is at rest in your hand, the vertical acceleration is zero. Therefore Newton's Second Law states that

$$\vec{\mathbf{F}}_{\text{gravity}} + \vec{\mathbf{N}} = \vec{\mathbf{0}}.$$ (7.5.2)

Since we chose the positive direction to be upwards,

$$N - mg = 0,$$ (7.5.3)

which can be solved for the magnitude of the normal force

$$N = mg.$$ (7.5.4)

This result may give rise to a misconception that the normal force is always equal to the mass of the object times the magnitude of the gravitational acceleration at the surface of the earth. The normal force and the gravitation force are two completely different forces. In this particular example, the normal force is equal in magnitude to the gravitational force and but opposite in direction, which sounds like an example of the Third Law. But is it? No!

First, note that the normal force $\vec{\mathbf{N}}$ in the above example is between your hand and the object, $\vec{\mathbf{N}}_{object,\,hand}$. The gravitation force $m\vec{\mathbf{g}}$ is the force between the earth and the object, $\vec{\mathbf{F}}_{object,\,earth}$.

In order to see all the action reaction pairs we must consider all the bodies involved in the interaction. The extra body is your hand. The force diagram on your hand is shown in Figure 7.12.
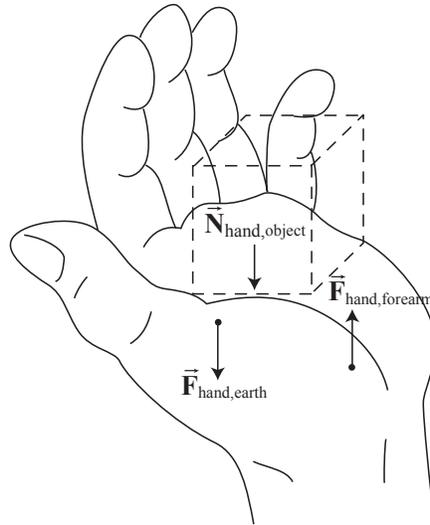

Figure 7.12: Force diagram on hand

The forces shown include the gravitational force between your hand and the earth, $\vec{\mathbf{F}}_{hand,\,earth}$ that points down, the normal force between the object and your hand, $\vec{\mathbf{N}}_{hand,\,object}$, which also points down, and there is a force $\vec{\mathbf{F}}_{hand,\,forearm}$ applied by your forearm to your hand that holds your hand up.

There are also forces on the earth due to the gravitational interaction between the hand and object and earth. We show these forces in Figure 7.13: the gravitation force between the earth and your hand $\vec{\mathbf{F}}_{earth,\,hand}$, and the gravitational force between the earth and the object, $\vec{\mathbf{F}}_{earth,\,object}$.
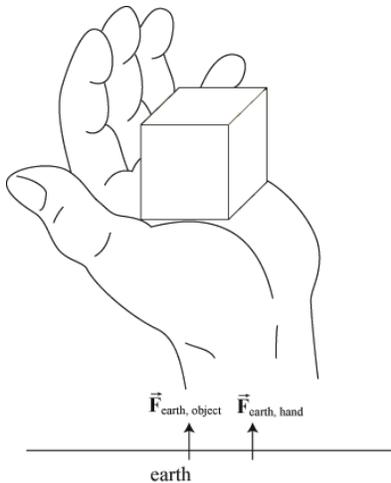
Figure 7.13: Gravitational forces on earth due to object and hand

There are three Third Law pairs. The first is associated with the normal force,

$$\vec{\mathbf{N}}_{hand, object} = -\vec{\mathbf{N}}_{object, hand} \tag{7.5.5}$$

The second is the gravitational force between the mass and the earth,

$$\vec{\mathbf{F}}_{object, earth} = -\vec{\mathbf{F}}_{earth, object} \cdot \tag{7.5.6}$$

The third is the gravitational force between your hand and the earth,

$$\vec{\mathbf{F}}_{hand, earth} = -\vec{\mathbf{F}}_{earth, hand} \cdot \tag{7.5.7}$$

As we see, none of these three law pairs associates the "weight" of the block on the hand with the force of gravity between the block and the earth.

When we talk about the "weight" of an object, we often are referring to the effect that object has on a scale or on the feeling we have when we hold that object. These effects are actually effects of the normal force. We say that an object "feels lighter" if there is an additional force holding the object up. For example, you can rest an object in your hand, but use your other hand to apply a force upwards on the object to make it feel lighter in your supporting hand.

This leads us to the use of the word "weight," which is often used in place of the gravitation force that the earth, exerts on an object, and we will always refer to this as the *gravitation force* instead of "weight." When you jump in the air, you feel "weightless" because there is no normal force acting on you, even though the earth is still exerting a gravitation force on you; clearly, when you jump, you do not turn this force off! When astronauts are in orbit around the earth, televised images show the astronauts floating in the spacecraft cabin; the condition is described, rightly, as being "weightless." The

gravitation force, while still present, has diminished slightly since their distance from the center of the earth has increased. (It's about 3% weaker in a low orbit than on the surface of the earth.)

Static and Kinetic Friction

There are two distinguishing types of friction when surfaces are in contact with each other. The first type is when the two objects in contact are moving relative to each other; the friction is called *kinetic friction* or *sliding friction,* $\vec{\mathbf{f}}_{\text{kinetic}}$.

Based on experimental measurements, the force of kinetic friction, $\vec{\mathbf{f}}_{\text{kinetic}}$, between two surfaces, is independent of the relative speed of the surfaces, the area of contact, and only depends on the magnitude of the normal component of the contact force. The force law for kinetic friction between the two surfaces can be modeled by

$$f_{\text{kinetic}} = \mu_{\text{k}} N, \tag{7.5.8}$$

where μ_{k} is called the *coefficient of kinetic friction.* The direction of kinetic friction on surface A due to the contact with a second surface B is always opposed to the relative direction of motion the surface A with respect to the surface B.

The second type is when the two surfaces are static relative to each other; the friction is called *static friction,* $\vec{\mathbf{f}}_{\text{static}}$. Push your hand forward along a surface; as you increase your pushing force, the friction force feels stronger and stronger. Try this! Your hand will at first stick until you push hard enough, then your hand slides forward. The magnitude of the static friction force, f_{static}, depends on how hard you push.

If you rest your hand on a table without pushing horizontally, the static friction is zero. As you increase your push, the static friction increases until you push hard enough that your hand slips and starts to slide along the surface. Thus the magnitude of static friction can vary from zero to some maximum value when the pushed object begins to slip,

$$0 \le f_{\text{static}} \le (f_{\text{static}})_{\text{max}}. \tag{7.5.9}$$

Is there a mathematical model for the magnitude of the maximum value of static friction between two surfaces? Through experimentation, we find that this magnitude is, like kinetic friction, proportional to the magnitude of the normal force

$$(f_{\text{static}})_{\text{max}} = \mu_{\text{s}} N. \tag{7.5.10}$$

Here the constant of proportionality is μ_s, the *coefficient of static friction*. This constant is slightly greater than the constant μ_k associated with kinetic friction, $\mu_s > \mu_k$. This small difference accounts for the slipping and catching of chalk on a blackboard, fingernails on glass, or a violin bow on a string.

The direction of static friction on an object is always opposed to the direction of the applied force (as long as the two surfaces are not accelerating). In Figure 7.14a, the static friction is shown opposing a pushing force acting on an object. In Figure 7.14b, when a pulling force is acting on an object, static friction is now pointing the opposite direction from the pulling force.
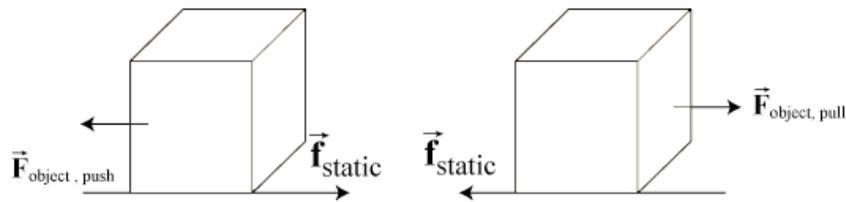

Figure 7.14a and Figure 7.14b: Pushing and pulling forces and the direction of static friction.

Although the force law for the maximum magnitude of static friction resembles the force law for sliding friction, there are important differences:

1. The direction and magnitude of static friction on an object always depends on the direction and magnitude of the total applied forces acting on the object, where the magnitude of kinetic friction for a sliding object is fixed.

2. The magnitude of static friction has a maximum possible value. If the magnitude of the total applied force along the direction of the contact surface exceeds the magnitude of the maximum value of static friction, then the object will start to slip (and be subject to kinetic friction.) We call this the *just slipping* condition.

7.6 Tension in a Rope

How do we define "tension" in a rope? A rope consists of long chains of atoms characteristic of the particular material found in the rope. When a rope is pulled, we say it is under *tension*. The long chains of molecules are stretched, and inter-atomic electrical forces between atoms in the molecules prevent the molecules from breaking apart. A detailed microscopic description of the behavior of the atoms is possible, but would be difficult and unnecessary for our purpose. Instead, we proceed to develop a macroscopic model for the behavior of ropes under tension.

We begin by considering a rope of mass m_r that is attached to a block of mass m_b on one end, and pulled by an applied force, $\vec{\mathbf{F}}_{applied}$, from the other end (Figure 7.15). For simplicity let's assume that the mass of the rope is so small, $m_r \simeq 0$, that we can take the rope is to be horizontal.

Figure 7.15: Forces acting on block and rope

Let's choose an x-y coordinate system with the $+\hat{\mathbf{j}}$ unit vector pointing upward in the normal direction to the surface, and the $+\hat{\mathbf{i}}$ unit vector pointing in the direction of the motion of the object. There are three forces acting on the rope; the applied pulling force, $\vec{\mathbf{F}}_{applied}$, the gravitational force $m_r \vec{\mathbf{g}} \simeq 0$, and the force of the object acting on the rope, $\vec{\mathbf{F}}_{rope,\,block}$. There are four forces acting on the object, the force on the object due to the rope $\vec{\mathbf{F}}_{block,rope}$, the static friction force $\vec{\mathbf{f}}_s$, the normal force $\vec{\mathbf{N}}$, and the gravitational force $m_b \vec{\mathbf{g}}$, forces acting on the object and the rope are shown in Figure 7.16.
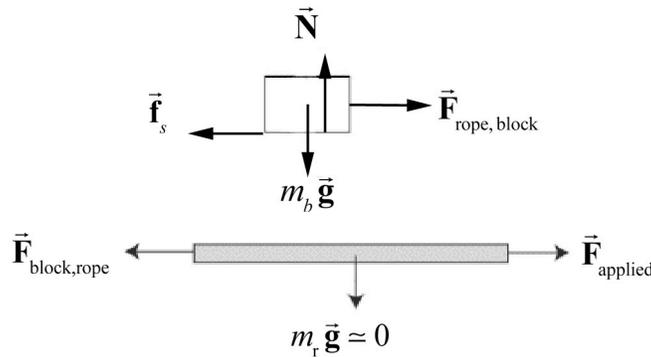

Figure 7.16: Force diagrams on block and rope

The total forces on the rope and the block must each sum to zero. The static equilibrium condition for the rope is:

$$F_{applied} - F_{rope,block} = 0 \qquad\qquad (7.6.1)$$

The static equilibrium conditions for the block are:

in the $+\hat{\mathbf{i}}$-direction:

$$F_{block,rope} - f_{static} = 0, \qquad\qquad (7.6.2)$$

in the $+\hat{\mathbf{j}}$ -direction:

$$N - m_b g = 0. \tag{7.6.3}$$

We now apply Newton's Third Law, the action-reaction law,

$$\vec{\mathbf{F}}_{\text{rope, block}} = -\vec{\mathbf{F}}_{\text{block, rope}}, \tag{7.6.4}$$

which becomes, in terms of our magnitudes,

$$F_{\text{rope, block}} = F_{\text{block, rope}}. \tag{7.6.5}$$

Our static equilibrium conditions now become

$$F_{\text{applied}} = F_{\text{rope, block}} = F_{\text{block,rope}} = f_{\text{static}}. \tag{7.6.6}$$

Thus the applied pulling force is transmitted through the rope to the object since it has the same magnitude as the force of the rope on the object.

$$F_{\text{applied}} = F_{\text{block,rope}}. \tag{7.6.7}$$

In addition we see that the applied force is equal to the static friction,

$$F_{\text{applied}} = f_{\text{static}}. \tag{7.6.8}$$

Static Tension in a Rope

We have seen that in static equilibrium the pulling force transmits through the rope. Suppose we make an imaginary slice of the rope at a distance x from the end that the object is attached to the object (Figure 7.17). The rope is now divided into two sections, labeled left and right.
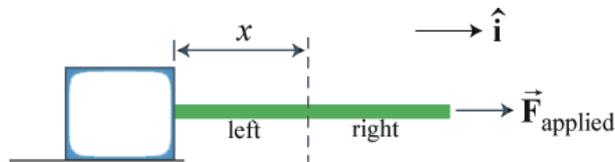

Figure 7.17: Imaginary slice through the rope

Aside from the Third Law pair of forces between the object and the rope, there is now a Third Law pair of forces between the left section of the rope and the right section of the rope. We denote this force acting on the left section by $\vec{\mathbf{F}}_{\text{left, right}}(x)$. The force on the right section due to the left section is denoted by $\vec{\mathbf{F}}_{\text{right,left}}(x)$. Newton's Third Law

requires that each force in this action-reaction pair is equal in magnitude and opposite in direction.

$$\vec{\mathbf{F}}_{\text{left, right}}(x) = -\vec{\mathbf{F}}_{\text{right, left}}(x) \qquad (7.6.9)$$

The force diagram for the left and right sections are shown in Figure 7.18.
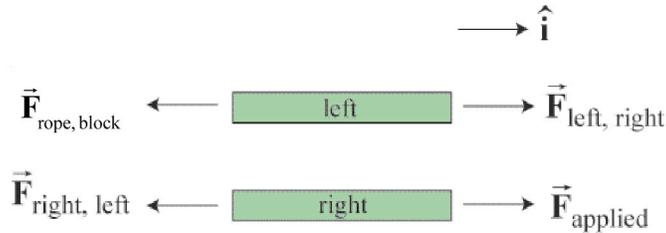

Figure 7.18: Force diagram for the left and right sections of rope

Definition: **Tension in a Rope**

The *tension* $T(x)$ in a rope at a distance x from one end of the rope is the magnitude of the action -reaction pair of forces acting at the point x,

$$T(x) = \left| \vec{\mathbf{F}}_{\text{left, right}}(x) \right| = \left| \vec{\mathbf{F}}_{\text{right, left}}(x) \right| . \qquad (7.6.10)$$

Special case: For a rope of negligible mass in static equilibrium, the tension is uniform and is equal to the applied force,

$$T = F_{\text{applied}} . \qquad (7.6.11)$$

Example 6: Tension in a massive rope

Suppose a uniform rope of mass m_r and length l is attached to an object of mass m_1 lying on a table. The rope is pulled from the side opposite the block with an applied force $\vec{\mathbf{F}}$. The coefficient of kinetic friction between the block and the surface is μ_k. Find the tension in the rope as a function of distance from the block.

Solution: The key point to realize is that the rope is now massive. Suppose we make an imaginary slice of the rope at a distance x from the end that the object as we did in Figure 7.17. The mass of the right slice is given by

$$m_{\text{right}} = \frac{m_r}{l}(l - x) . \qquad (7.6.12)$$

The mass of the left slice is

$$m_{left} = \frac{m_r}{l} x .$$

(7.6.13)

We now apply Newton's Second Law to the right slice of the rope

$$F_{applied} - T(x) = \frac{m_r}{l}(l - x)a_{xr} .$$

(7.6.14)

where $T(x)$ is the tension in the rope at a distance x from the object. We also apply Newton's Second Law to the left slice of the rope

$$T(x) - F_{rope,block} = \frac{m_r}{l} x a_{xl} .$$

(7.6.15)

where $T(x)$ is the tension in the rope at a distance x from the object.

The forces acting on the block are shown in Figure 7.16. Newton's second Law is now

in the $+\hat{\mathbf{i}}$-direction:
$$F_{block,rope} - f_k = m_b a_{xb} ,$$

(7.6.16)

in the $+\hat{\mathbf{j}}$ -direction:
$$N - m_b g = 0 .$$

(7.6.17)

Eq. (7.6.17) implies that $N = m_b g$ and so the kinetic friction fore acting on the block is

$$f_k = \mu_k N = \mu_k m_b g .$$

(7.6.18)

We now substitute Eq. (7.6.5) and Eq. (7.6.18) into Eq. (7.6.16) which becomes

$$F_{rope,\,block} = \mu_k m_b g + m_b a_{xb} ,$$

(7.6.19)

We substitute Eq. (7.6.19) into Eq. (7.6.15) yielding

$$T(x) - (\mu_k m_b g + m_b a_{xb}) = \frac{m_r}{l} x a_{xl} .$$

(7.6.20)

Because the rope and block move together, the accelerations are equal which we denote by the symbol a

$$a \equiv a_{xb} = a_{xl} \,. \tag{7.6.21}$$

Then Eq. (7.6.20) becomes

$$T(x) = \mu_k m_b g + (m_b + \frac{m_r}{l}x)a \,. \tag{7.6.22}$$

Check our result:

We could have used Eq. (7.6.14) to find the tension

$$T(x) = F_{applied} - \frac{m_r}{l}(l - x)a \,. \tag{7.6.23}$$

Recall from the free body diagram in Figure 7.15 that the applied force satisfies

$$F_{applied} - f_s = (m_r + m_b)a \,. \tag{7.6.24}$$

Therefore substitute Eq. (7.6.24) into Eq. (7.6.23) yielding

$$T(x) = f_s + (m_r + m_b)a - \frac{m_r}{l}(l - x)a = f_s + (m_b + \frac{m_r}{l}x)a \,. \tag{7.6.25}$$

in agreement with Eq. (7.6.22). We expect this result because the tension is accelerating both the left slice and the block and is opposed by the friction force.

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