Chapter 11 Work and Kinetic Energy

11.1 The Concept of Energy

The transformation of energy is a powerful concept that enables us to describe a vast number of processes:

Falling water releases stored *gravitational potential energy*, which can become the *kinetic energy* associated with a *coherent motion* of matter. The harnessed *mechanical energy* can be used to spin turbines and alternators, doing *work* to generate *electrical energy*, transmitted to consumers along power lines. When you use any electrical device, the electrical energy is transformed into other forms of energy. In a refrigerator, electrical energy is used to compress a gas into a liquid. During the compression, some of the internal energy of the gas is transferred to the *random motion* of molecules in the outside environment. The liquid flows from a high-pressure region into a low-pressure region where the liquid evaporates. During the evaporation, the liquid absorbs energy from the *random motion* of molecules inside of the refrigerator. The gas returns to the compressor.

"Human beings transform the stored chemical energy of food into various forms necessary for the maintenance of the functions of the various organ system, tissues and cells in the body."[1] A person can do work on their surroundings – for example, by pedaling a bicycle – and transfer energy to the surroundings in the form of increasing random motion of air molecules, by using this catabolic energy.

Burning gasoline in car engines converts *chemical energy*, stored in the molecular bonds of the constituent molecules of gasoline, into coherent (ordered) motion of the molecules that constitute a piston. With the use of gearing and tire/road friction, this motion is converted into kinetic energy of the car; the automobile moves.

Stretching or compressing a spring stores *elastic potential energy* that can be released as kinetic energy.

The process of vision begins with stored *atomic energy* released as electromagnetic radiation (light), which is detected by exciting photoreceptors in the eye, releasing chemical energy.

When a proton fuses with deuterium (a hydrogen atom with a neutron and proton for a nucleus), helium-three is formed (with a nucleus of two protons and one neutron) along with radiant energy in the form of photons. The combined *internal energy* of the proton and deuterium are greater than the internal energy of the helium-three. This difference in internal energy is carried away by the photons as light energy.

---

1 George B. Benedek and Felix M.H. Villars, *Physics with Illustrative Examples from Medicine and Biology, Volume 1: Mechanics,* Addison-Wesley, Reading, 1973, p. 5-116

There are many such processes in the manmade and natural worlds, involving different forms of energy: kinetic energy, gravitational energy, thermal energy, elastic energy, electrical energy, chemical energy, electromagnetic energy, nuclear energy and more. The total energy is always conserved in these processes, although different forms of energy are converted into others.

Any physical process can be characterized by two states, initial and final, between which energy transformations can occur. Each form of energy E_i, where "i" is an arbitrary label identifying one of the N forms of energy, may undergo a change during this transformation,

$$\Delta E_i \equiv E_{\text{final},i} - E_{\text{initial},i} . \tag{11.1.1}$$

Conservation of energy means that the sum of these changes is zero,

$$\Delta E_1 + \Delta E_2 + \cdots + \Delta E_N = \sum_{i=1}^{N} \Delta E_i = 0 . \tag{11.1.2}$$

Two important points emerge from this idea. First, we are interested primarily in changes in energy and so we search for relations that describe how each form of energy changes. Second, we must account for all the ways energy can change. If we observe a process, and the sum of the changes in energy is not zero, either our expressions for energy are incorrect, or there is a new type of change of energy that we had not previously discovered. This is our first example of the importance of conservation laws in describing physical processes, as energy is a key quantity conserved in all physical processes. If we can quantify the changes of different forms of energy, we have a very powerful tool to understand nature.

We will begin our analysis of conservation of energy by considering processes involving only a few forms of changing energy. We will make assumptions that greatly simplify our description of these processes. At first we shall only consider processes acting on bodies in which the atoms move in a coherent fashion, ignoring processes in which energy is transferred into the random motion of atoms. Thus we will initially ignore the effects of friction. We shall then treat processes involving friction between rigid bodies. We will later return to processes in which there is an energy transfer resulting in an increase or decrease in random motion when we study the First Law of Thermodynamics.

Energy is always conserved but we often prefer to restrict our attention to a set of objects that we define to be our *system*. The rest of the universe acts as the *surroundings*. We illustrate this division of system and surroundings in Figure 11.1.

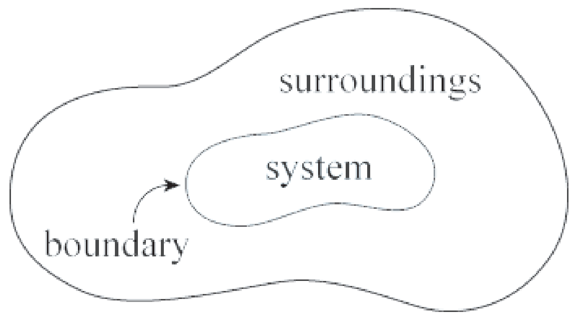

Figure 11.1 A diagram of a system and its surroundings with boundary

Since energy is conserved, any energy that leaves the system must cross through the boundary and enter the surroundings. Consider any physical process that occurs between an initial state and a final state, in which energy transformations occur. Our statement of conservation of energy becomes

$$\Delta E_{total} = \Delta E_{\text{system}} + \Delta E_{\text{surroundings}} = 0\,, \tag{11.1.3}$$

where

$$\Delta E_{\text{system}} \equiv E_{\text{final,system}} - E_{\text{initial,system}} \tag{11.1.4}$$

is the change in energy of the system and

$$\Delta E_{\text{surroundings}} \equiv E_{\text{final,surroundings}} - E_{\text{initial,surroundings}} \tag{11.1.5}$$

is the change in energy of the surroundings.

11.2 Kinetic Energy

The first form of energy that we will study is an energy associated with the coherent motion of molecules that constitute a body of mass m; this energy is called the *kinetic energy* (from the Greek "*kinetikos*," moving). Let us consider a car moving along a straight road (along which we will place the x-axis). For an observer at rest with respect to the ground, the car has velocity $\vec{\mathbf{v}} = v_x\,\hat{\mathbf{i}}$. The speed of the car is the magnitude of the velocity, $v \equiv \left| v_x \right|$.

Definition: **Kinetic Energy**

The *kinetic energy* K of a non-rotating body of mass m moving with speed v is defined to be the positive scalar quantity

$$K \equiv \frac{1}{2} m v^2 \tag{11.2.1}$$

The kinetic energy is proportional to the square of the speed. The SI units for kinetic energy are $[\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}]$. This combination of units is defined to be a joule and is denoted by $[\mathrm{J}]$, thus $1\ \mathrm{J} \equiv 1\ \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$. (The SI unit of energy is named for James Prescott Joule; the proper pronunciation of the unit "joule" is "jowl", but the far more common usage is the incorrect "jool".)

The above definition of kinetic energy does not refer to any direction of motion, just the speed of the body. We will see that it is sometimes convenient to express Equation (11.2.1) in the equivalent form

$$K \equiv \frac{1}{2} m \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \tag{11.2.2}$$

Let's consider a case in which our car changes velocity. For our initial state, the car moves with an initial velocity $\vec{\mathbf{v}}_0 = v_{x,0}\ \hat{\mathbf{i}}$ along the x-axis. For the final state (at some later time), the car has changed its velocity and now moves with a final velocity $\vec{\mathbf{v}}_f = v_{x,f}\ \hat{\mathbf{i}}$. Therefore the change in the kinetic energy is

$$\Delta K = \frac{1}{2} m v_f^{\,2} - \frac{1}{2} m v_0^{\,2} \tag{11.2.3}$$

11.2.1 Example: Change in Kinetic Energy of a Car

(a) Suppose car *A* increases its speed from 10 to 20 mph and car *B* increases its speed from 50 to 60 mph. Both cars have the same mass m. What is the ratio of the change of kinetic energy of car *B* to the change of kinetic energy of car *A*? Which car has a greater change in kinetic energy?

Answer: The ratio of the change in kinetic energy of car *B* to car *A* is

$$\begin{aligned}\frac{\Delta K_B}{\Delta K_A} &= \frac{\frac{1}{2} m (v_{B,f})^2 - \frac{1}{2} m (v_{B,0})^2}{\frac{1}{2} m (v_{A,f})^2 - \frac{1}{2} m (v_{A,0})^2} = \frac{(v_{B,f})^2 - (v_{B,0})^2}{(v_{A,f})^2 - (v_{A,0})^2} \\ &= \frac{(60\ \mathrm{mph})^2 - (50\ \mathrm{mph})^2}{(10\ \mathrm{mph})^2} = 11/3\end{aligned} \tag{11.2.4}$$

Thus car *B* has a much greater increase in its kinetic energy than car *A*.

(b) What is the ratio of the change in kinetic energy of car *B* to car *A* as seen by an observer moving with the initial velocity of car *A*?

Answer: Car *A* now increases its speed from rest to 10 mph and car *B* increases its speed from 40 to 50 mph. The ratio is now

$$\begin{aligned}\frac{\Delta K_B}{\Delta K_A} &= \frac{\frac{1}{2}m(v_{B,f})^2 - \frac{1}{2}m(v_{B,0})^2}{\frac{1}{2}m(v_{A,f})^2 - \frac{1}{2}m(v_{A,0})^2} = \frac{(v_{B,f})^2 - (v_{B,0})^2}{(v_{A,f})^2 - (v_{A,0})^2} \\ &= \frac{(50\text{ mph})^2 - (40\text{ mph})^2}{(10\text{ mph})^2} = 9\end{aligned} \tag{11.2.5}$$

The ratio is greater than that found in part a). Note that from the new reference frame both car *A* and car *B* have smaller increases in kinetic energy.

11.3 Kinematics and Kinetic Energy in One Dimension

Constant Acceleration Motion

Let's consider a uniform-accelerated motion of a *rigid body* in one dimension. We begin the discussion by treating our object as a point mass. Suppose at $t = 0$ the object has an initial velocity component in the x-direction given by $v_{x,0}$. If the acceleration is in the direction of the displacement of the body then the body will increase its speed. If the acceleration is opposite the direction of the displacement then the acceleration will decrease the body's speed. The displacement of the body is given by

$$\Delta x = v_0\, t + \frac{1}{2} a_x\, t^2 . \tag{11.3.1}$$

The product of acceleration and the displacement is

$$a_x \Delta x = a_x (v_0\, t + \frac{1}{2} a_x\, t^2) . \tag{11.3.2}$$

The acceleration is given by

$$a_x = \frac{\Delta v_x}{\Delta t} = \frac{(v_{x,f} - v_{x,0})}{t} . \tag{11.3.3}$$

Therefore

$$a_x \Delta x = \frac{(v_{x,f} - v_{x,0})}{t}\left(v_0\, t + \frac{1}{2}\frac{(v_{x,f} - v_{x,0})}{t} t^2\right). \qquad (11.3.4)$$

Equation (11.3.4) becomes

$$a_x \Delta x = (v_{x,f} - v_{x,0})(v_0) + \frac{1}{2}(v_{x,f} - v_{x,0})(v_{x,f} - v_{x,0}) = \frac{1}{2} v_{x,f}{}^2 - \frac{1}{2} v_{x,0}{}^2. \qquad (11.3.5)$$

If we multiply each side of Equation (11.3.5) by the mass m of the object this kinematical result takes on an interesting interpretation for the motion of the object. We have

$$m a_x \Delta x = \frac{1}{2} m v_{x,f}{}^2 - m \frac{1}{2} v_{x,0}{}^2 = K_f - K_i. \qquad (11.3.6)$$

Recall that for one-dimensional motion, Newton's Second Law is $F_x = m a_x$; for the situation considered here, Equation (11.3.6) becomes

$$F_x \, \Delta x = K_f - K_i. \qquad (11.3.7)$$

Non-constant Acceleration:

If the acceleration is not constant, then we can divide the displacement into N intervals indexed by $j = 1$ to N. It will be convenient to denote the displacement intervals by Δx_j, the corresponding time intervals by Δt_j and the velocities at the beginning and end of each interval as $v_{x,j-1}$ and $v_{x,j}$. Note that the velocity at the beginning and end of the first interval, $j = 1$ is then v_0 and the velocity at the end of the last interval, $j = N$ is $v_{x,N} = v_{x,j}$.

Consider the sum of the products of the average acceleration $(a_{x,j})_{ave}$ and displacement Δx_j in each interval,

$$\sum_{j=1}^{j=N} (a_{x,j})_{ave} \Delta x_j \,. \qquad (11.3.8)$$

The average acceleration over each interval is equal to

$$(a_{x,j})_{ave} = \frac{\Delta v_{x,j}}{\Delta t_j} = \frac{(v_{x,j+1} - v_{x,j})}{\Delta t_j}, \qquad (11.3.9)$$

and so the contribution in each integral can be calculated as above and we have that

$$(a_{x,j})_{ave}\,\Delta x_j = \frac{1}{2}v_{x,j}^{\;2} - \frac{1}{2}v_{x,j-1}^{\;\;2}. \tag{11.3.10}$$

When we sum over all the terms only the last and first terms survive, all the other terms cancel in pairs, and we have that

$$\sum_{j=1}^{j=N}(a_{x,j})_{ave}\,\Delta x_j = \frac{1}{2}v_{x,N}^{\;2} - \frac{1}{2}v_{x,0}^{\;2}. \tag{11.3.11}$$

In the limit as $N \to \infty$ *and* $\Delta x_j \to 0$ for all j (both conditions must be met!), the limit of the sum is the definition of the integral of the acceleration with respect to the position, So Eq. (11.3.11)

$$\lim_{\substack{N\to\infty \\ \Delta x_j \to 0}} \sum_{j=1}^{j=N}(a_{x,j})_{ave}\,\Delta x_j \equiv \int_{\text{initial}}^{\text{final}} a_x\,dx. \tag{11.3.12}$$

So in the limit as $N \to \infty$ *and* $\Delta x_j \to 0$ for all j, with $v_{x,N} \to v_{x,f}$, Eq. (11.3.11) becomes

$$\int_{\text{initial}}^{\text{final}} a_x\,dx = \frac{1}{2}(v_{x,f}^{\;\;2} - v_{x,0}^{\;\;2}) \tag{11.3.13}$$

This integral result is consequence of the definition that $a_x \equiv dv_x / dt$. Notice how Eq. (11.3.13) compares to the integral of acceleration with respect to time

$$\int_{\text{initial}}^{\text{final}} a_x\,dt = v_{x,f} - v_{x,0}. \tag{11.3.14}$$

Multiplying both sides of Eq. (11.3.14) by the mass m yields

$$\int_{\text{initial}}^{\text{final}} ma_x\,dx = \frac{1}{2}mv_{x,f}^{\;\;2} - \frac{1}{2}mv_{x,0}^{\;\;2} \equiv K_f - K_0 \tag{11.3.15}$$

When we introduce Newton's Second Law in the form $F_x^{\text{total}} = m a_x$, then Eq. (11.3.15) becomes

$$\int_{\text{initial}}^{\text{final}} F_x^{\text{total}}\,dx = K_f - K_i. \tag{11.3.16}$$

The integral of the total x-component of the force with respect to displacement in Equation (11.3.16) applies to the motion of a point-like object. (For extended bodies we shall see in Chapter 9 that Equation (11.3.16) applies to the center of mass motion because the total external force on a rigid body causes the center of mass to accelerate.)

11.4 Work done by Constant Forces

We will begin our discussion of the concept of work by analyzing the motion of an object in one dimension acted on by constant forces. Let's consider an example of this type of motion: pushing a cup forward with a constant force along a desktop. When the cup changes speed (and hence kinetic energy), the sum of the forces acting on the cup must be non-zero according to Newton's Second Law. There are three forces involved in this motion: the applied pushing force $\vec{\mathbf{F}}_{applied}$; the contact force $\vec{\mathbf{C}} \equiv \vec{\mathbf{N}} + \vec{\mathbf{f}}_k$; and gravity $\vec{\mathbf{F}}_{grav} = m\vec{\mathbf{g}}$. The force diagram on the cup is shown in Figure 11.2.

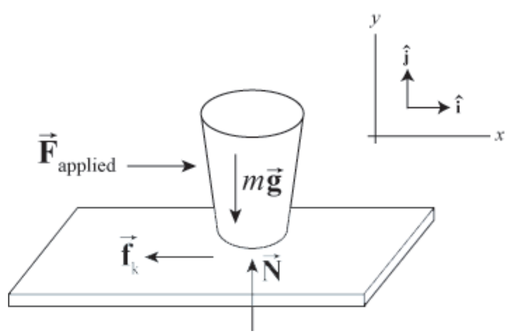

Figure 11.2 Force diagram for cup.

Let's choose our coordinate system so that the $+x$-direction is the direction of the forward motion of the cup. The pushing force can then be described by

$$\vec{\mathbf{F}}_{\text{applied}} = F_{\text{applied,x}}\,\hat{\mathbf{i}}\,. \tag{11.4.1}$$

Definition: **Work done by a Constant Force**

Suppose a body moves from an initial point x_0 to a final point x_f so that the displacement of the point the force acts on is positive $\Delta x \equiv x_f - x_0 > 0$. The *work done by a constant force* $\vec{\mathbf{F}}_{\text{applied}} = F_{\text{applied,x}}\,\hat{\mathbf{i}}$ *acting on the body* is the product of the component of the force $F_{\text{applied,x}}$ and the displacement Δx,

$$W_{\text{applied}} = F_{\text{applied,x}}\Delta x\,. \tag{11.4.2}$$

Work is a scalar quantity; it is not a vector quantity. The SI unit for work is

$$[1\ \text{N}\cdot\text{m}] = [1\ \text{kg}\cdot\text{m}\cdot\text{s}^{-2}][1\ \text{m}] = [1\ \text{kg}\cdot\text{m}^2\cdot\text{s}^{-2}] = [1\ \text{J}]. \tag{11.4.3}$$

Note that work has the same dimension and the same SI unit as kinetic energy. Since our applied force is along the direction of motion, both $F_{\text{applied,x}} > 0$ and $\Delta x > 0$. The work done is just the product of the magnitude of the applied force and the distance through which that force acts and is positive. In the definition of work done by a force, the force can act at any point on the body. The displacement that appears in Equation (11.4.2) is not the displacement of the body but the displacement of the point of application of the force. For point-like objects, the displacement of the point of application of the force is equal to the displacement of the body. For the most part in the remainder of the chapter we shall consider point-like objects or we shall model an extended body as a point-like object however not always as the following question illustrates.

Question 7.4.1 Suppose you are initially standing and you start walking by pushing against the ground with your feet and your feet do not slip. How much work is done on you by the static friction force?

Answer: When you apply a contact force against the ground, the ground applies an equal and opposite contact force on you. The tangential component of this constant force is the force of static friction acting on you. Since your foot is at rest while you are pushing against the ground, there is no displacement of the point of application of this static friction force. Therefore static friction does zero work on you while you are accelerating. You may be surprised by this result but if you think about energy transformation, chemical energy stored in your muscle cells is being transformed into kinetic energy of motion and thermal energy.

We can extend the concept of work to forces that oppose the motion, like friction. In our example of the moving cup, the kinetic friction force is

$$\vec{\mathbf{f}}_{\text{friction}} = f_x\,\hat{\mathbf{i}} = -\mu_k N\,\hat{\mathbf{i}} = -\mu_k mg\,\hat{\mathbf{i}} \tag{11.4.4}$$

where $N = mg$ from consideration of the $\hat{\mathbf{j}}$-components of force in Figure 11.2 and the model $f_k = \mu_k N$ for kinetic friction have been used.

Here the component of the force is in the opposite direction as the displacement. The work done by the friction force is negative,

$$W_{friction} = -\mu_k mg\Delta x\,. \tag{11.4.5}$$

Since the gravitation force is perpendicular to the motion of the cup, the gravitational force has no component along the line of motion. Therefore the gravitation force does zero work on the cup when the cup is slid forward in the horizontal direction. The normal force is also perpendicular to the motion, and hence does no work.

We see that the pushing force does positive work, the friction force does negative work, and the gravitation and normal forces do zero work.

11.4.2 Example: Cup on a horizontal table

Push a cup of mass 0.2 kg along a horizontal table with a force of magnitude 2.0 N for a distance of 0.5 m. The coefficient of friction between the table and the cup is $\mu_k = 0.1$. Calculate the work done by the pushing force and the work done by the friction force.

Answer: The work done by the pushing force is

$$W_{\text{applied}} = F_{\text{applied},x}\Delta x = (2.0\ \text{N})(0.5\ \text{m}) = 1.0\ \text{J}\ . \qquad (11.4.6)$$

The work done by the friction force is

$$W_{\text{friction}} = -\mu_k m g \Delta x = -(0.1)(0.2\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2})(0.5\ \text{m}) = -\,0.10\ \text{J}\ . \qquad (11.4.7)$$

Note that the result in Equation (11.4.6) is known to only one significant figure since μ_{k} is given to only one significant figure. However, the precision in Equation (11.4.7) and Equation (11.4.7), $0.1\ \text{J}$, is the same.

11.4.3 Example: Cup on a table, applied force at an angle

Suppose we push the cup in the previous example with a force of the same magnitude but at an angle $\theta = 30^o$ upwards with respect to the table. Calculate the work done by the pushing force. Calculate the work done by the kinetic friction force. The force diagram on the cup is shown in Figure 11.3.

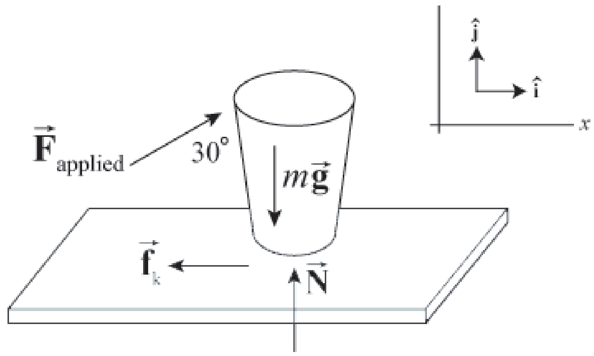

Figure 11.3 Force diagram on cup.

Solution: The x-component of the pushing force is now

$$F_{\text{applied},x} = F_{\text{applied}}\cos(\theta) = (2.0\text{ N})(\cos(30^\circ)) = 1.7\text{ N}. \tag{11.4.8}$$

The work done by the pushing force is

$$W_{\text{applied}} = F_{\text{applied},x}\,\Delta x = (1.7\text{ N})(0.5\text{ m}) = 8.7\times 10^{-1}\text{ J}. \tag{11.4.9}$$

The kinetic friction force is

$$\vec{\mathbf{f}}_k = f_{k,x}\,\hat{\mathbf{i}} = -\mu_k N\,\hat{\mathbf{i}}. \tag{11.4.10}$$

In this case, the magnitude of the normal force is not simply the same as the weight of the cup. We need to find the y-component of the applied force,

$$F_{\text{applied},y} = F_{\text{applied}}\sin(\theta) = (2.0\text{ N})(\sin(30^o) = 1.0\text{ N}. \tag{11.4.11}$$

To find the normal force, we apply Newton's Second Law in the y-direction,

$$F_{\text{applied},y} + N - mg = 0. \tag{11.4.12}$$

Then the normal force is

$$N = mg - F_{\text{applied},y} = (0.2\text{ kg})(9.8\text{ m}\cdot\text{s}^{-2}) - (1.0\text{ N}) = 9.6\times 10^{-1}\text{ N}. \tag{11.4.13}$$

The work done by the kinetic friction force is

$$W_{\text{friction}} = -\mu_k N\Delta x = -(0.1)(9.6\times 10^{-1}\text{ N})(0.5\text{ m}) = 4.8\times 10^{-2}\text{ J}. \tag{11.4.14}$$

or 5.0×10^{-2} J to one significant figure. Strictly speaking, the result in Equation (11.4.13) should be rounded to $0.1\,\text{N}$, which would give the same result as Equation (11.4.14) to one figure.

11.4.4 Example: Work done by Gravity Near the Surface of the Earth

Consider a point-like body of mass m near the surface of the earth falling directly towards the center of the earth. The gravitation force between the body and the earth is nearly constant, $\vec{\mathbf{F}}_{grav} = m\vec{\mathbf{g}}$. Let's choose a coordinate system with the origin at the surface of the earth and the $+y$-direction pointing away from the center of the earth Suppose the body starts from an initial point y_0 and falls to a final point y_f closer to the earth. How much work does the gravitation force do on the body as it falls?

Answer: The displacement of the body is negative, $\Delta y \equiv y_f - y_0 < 0$. The gravitation force is given by

$$\vec{\mathbf{F}}_{\text{gravity}} = m\vec{\mathbf{g}} = F_{\text{gravity},y}\ \hat{\mathbf{j}} = -mg\ \hat{\mathbf{j}}. \tag{11.4.15}$$

The work done on the body is then

$$W_{\text{gravity}} = F_{\text{gravity},y}\Delta y = -mg\Delta y\,. \tag{11.4.16}$$

For a falling body, the displacement of the body is negative, $\Delta y \equiv y_f - y_0 < 0$; therefore the work done by gravity is positive,

$$W_{\text{gravity}} = F_{\text{gravity},y}\Delta y = -mg\Delta y > 0\,. \tag{11.4.17}$$

The gravitation force is pointing in the same direction as the displacement of the falling object so the work should be positive.

When an object is rising while under the influence of a gravitation force, $\Delta y \equiv y_f - y_0 > 0$. The work done by the gravitation force for a rising body is negative,

$$W_{gravity} = F_{gravity,y}\Delta y = -mg\Delta y < 0 \tag{11.4.18}$$

because the gravitation force is pointing in the opposite direction from that in which the object is displaced.

It's important to note that the choice of the positive direction as being away from the center of the earth ("up") does not make a difference. If the downward direction were chosen positive, the falling body would have a positive displacement and the gravitational force as given in Equation (11.4.17) would have a positive downward component; the product $F_{\text{gravity},y}\Delta y$ would still be positive.

11.5 Work done by Non-Constant Forces

Consider a body moving in the x-direction under the influence of a non-constant force in the x-direction, $\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}}$. The body moves from an initial position x_0 to a final position x_f. In order to calculate the work done by a non-constant force, we will divide up the displacement of the point of application of the force into a large number N of small displacements Δx_j where the index j marks the j^{th} displacement and takes integer values from 1 to N, as in Section 7.3. Let $(F_{x,j})_{ave}$ denote the average value of the x-component of the force in the displacement interval $[x_{j-1}, x_j]$. For the j^{th} displacement interval we calculate the contribution to the work

$$\Delta W_j = (F_{x,j})_{ave}\Delta x_j \tag{11.5.1}$$

This contribution is a scalar so we add up these scalar quantities to get the total work

$$W_N = \sum_{j=1}^{j=N} \Delta W_j = \sum_{j=1}^{j=N} (F_{x,j})_{ave}\Delta x_j\ . \tag{11.5.2}$$

The sum in Equation (11.5.2) depends on the number of divisions N and the width of the intervals Δx_j. In order to define a quantity that is independent of the divisions, we take the limit as $N \to \infty$ *and* $\left|\Delta x_j\right| \to 0$ for all j. The work is then

$$W = \lim_{\substack{N\to\infty \\ \left|\Delta x_j\right|\to 0}} \sum_{j=1}^{j=N} (F_{x,j})_{ave}\Delta x_j = \int_{x=x_0}^{x=x_f} F_x dx \tag{11.5.3}$$

This last expression is the definition of the integral of the x-component of the force with respect to the parameter x. In Figure 11.5 we graph the x-component of the force as a function of the parameter x. The work integral is the area under this curve between $x = x_0$ and $x = x_f$.

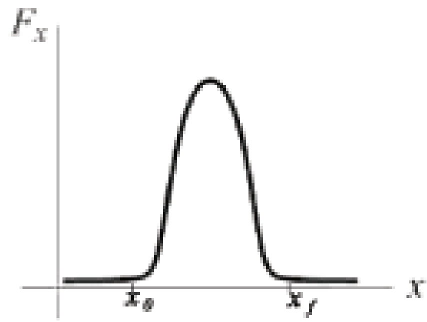

Figure 11.5 Graph of x-component of a sample force as a function of the parameter x.

11.5.1 Example: Work done by the Spring Force

Connect one end of a spring to a body resting on a smooth (frictionless) table and fix the other end of the spring to a wall. Stretch the spring and release the spring-body system. How much work does the spring do on the body as a function of the stretched or compressed length of the spring?

Answer: We first begin by choosing a coordinate system with origin at the position of the body when the spring is at rest in the equilibrium position. We choose the $\hat{\mathbf{i}}$ unit vector to point in the direction the body moves when the spring is being stretched and the coordinate x to denote the position of the body with respect to the equilibrium position, as in Figure 11.6 (which indicates that in general the position x will be a function of time). The spring force on the body is given by

$$\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} = -kx\,\hat{\mathbf{i}}\,. \tag{11.5.4}$$

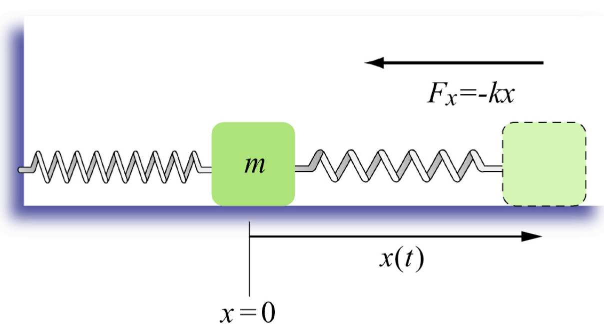

Figure 11.6 Equilibrium position and position at time t

In Figure 11.7 we show the graph of the x-component of the spring force as a function of x for both positive values of x corresponding to stretching, and negative values of x corresponding to compressing of the spring. Note that x_0 and x_f can be positive, zero, or negative.

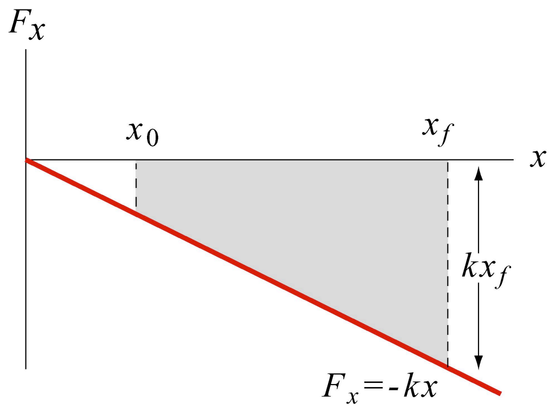

Figure 11.7 The x-component of the spring force as a function of x.

The work done is just the area under the curve for the interval x_0 to x_f,

$$W = \int_{x=x_0}^{x=x_f} F_x dx = \int_{x=x_0}^{x=x_f} (-kx) dx. \tag{11.5.5}$$

This integral is straightforward; the work done by the spring force on the body is

$$W = \int_{x=x_0}^{x=x_f} (-kx) dx = -\frac{1}{2}k(x_f^{\,2} - x_0^{\,2}). \tag{11.5.6}$$

When the absolute value of the final distance is less than the absolute value of the initial distance, $|x_f| < |x_0|$, the work done by the spring force is positive. This means that if the spring is less stretched or compressed in the final state than in the initial state, the work done by the spring force is positive. The spring force does positive work on the body when the spring goes from a state of 'greater tension' to a state of 'lesser tension'. We shall see in Chapter 8 that the positive work done by the spring force decreases the *potential energy* stored in the spring.

11.6 Work-Kinetic Energy Theorem

There is a direct connection between the total work done on a point-like object and the change in kinetic energy the point-like object undergoes If the total work done on the object is non-zero, this implies that an unbalanced force has acted on the object, and the object will have undergone acceleration. For an object undergoing one-dimensional motion the left hand side of Equation (11.3.16) is the work done on the object by the component of the sum of the forces in the direction of displacement ,

$$W_{\text{total}} = \int_{\text{initial}}^{\text{final}} F_x^{\text{total}}\, dx = K_f - K_i = \Delta K. \quad (11.6.1)$$

When the total work done on a object is positive, the object will increase its speed, and negative work done on a object causes a decrease in speed. When the total work done is zero, the object will maintain a constant speed. In fact, the work-energy relationship is quite precise; the total work done by the net applied force on a object is identically equal to the change in kinetic energy of the object,

$$W_{\text{total}} = \Delta K = \frac{1}{2}mv_f^{\,2} - \frac{1}{2}mv_0^{\,2}. \quad (11.6.2)$$

11.6.1 Example: Gravity and the Work-Energy Theorem

Suppose a ball of mass $m = 0.2\ \text{kg}$ starts from rest at a height $y_0 = 15\ \text{m}$ above the surface of the earth and falls down to a height $y_f = 5.0\ \text{m}$ above the surface of the earth. What is the change in the kinetic energy? Find the final velocity using the work-energy theorem.

Answer:

As only one force acts on the ball, the change in kinetic energy is the work done by gravity,

$$\begin{aligned} W_{\text{grav}} &= -mg(y_f - y_0) \\ &= (-2.0\times 10^{-1}\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2})(5\ \text{m} - 15\ \text{m}) = 2.0\times 10^{1}\ \text{J} \end{aligned} \quad (11.6.3)$$

The ball started from rest, $v_{y,0} = 0$. So the change in kinetic energy is

$$\Delta K = \frac{1}{2}mv_{y,f}^{\,2} - \frac{1}{2}mv_{y,0}^{\,2} = \frac{1}{2}mv_{y,f}^{\,2}. \quad (11.6.4)$$

We can solve Equation (11.6.4) for the final velocity using Equation (11.6.3)

$$v_{y,f} = \sqrt{\frac{2\Delta K}{m}} = \sqrt{\frac{2W_{grrav}}{m}} = \sqrt{\frac{2(2.0\times 10^{1}\ \text{J})}{0.2\ \text{kg}}} = 1.4\times 10^{1}\ \text{m}\cdot\text{s}^{-1} \quad (11.6.5)$$

For the falling ball in a constant gravitation field, the positive work of the gravitation force on the body corresponds to an increasing kinetic energy and speed. For a rising body in the same field, the kinetic energy and hence the speed decrease since the work done is negative.

11.6.2 Example: Final Kinetic Energy of Moving Cup

A person pushes a cup of mass 0.2 kg along a horizontal table with a force of magnitude 2.0 N at an angle of 30° with respect to the horizontal for a distance of 0.5 m as in Example 7.4.2. The coefficient of friction between the table and the cup is $\mu_k = 0.1$. If the cup was initially at rest, what is the final kinetic energy of the cup after being pushed 0.5 m? What is the final speed of the cup?

Answer:

The total work done on the cup is the sum of the work done by the pushing force and the work done by the friction force, as given in Equations (11.4.9) and (11.4.14),

$$\begin{aligned} W_{\text{total}} &= W_{\text{applied}} + W_{\text{friction}} = (F_{\text{applied},x} - \mu_k N)(x_f - x_0) \\ &= (1.7\text{ N} - 9.6\times 10^{-2}\text{ N})(0.5\text{ m}) = 8.0\times 10^{-1}\text{ J} \end{aligned} . \tag{11.6.6}$$

According to our work-kinetic energy theorem,

$$W_{\text{total}} = \Delta K = \frac{1}{2}mv_f^{\ 2} - \frac{1}{2}mv_0^{\ 2}. \tag{11.6.7}$$

The initial velocity is zero so the change in kinetic energy is just

$$\Delta K = \frac{1}{2}mv_{y,f}^{\ 2} - \frac{1}{2}mv_{y,0}^{\ 2} = \frac{1}{2}mv_{y,f}^{\ 2}. \tag{11.6.8}$$

Thus the work-kinetic energy theorem enables us to solve for the final kinetic energy,

$$K_f = \frac{1}{2}mv_f^{\ 2} = \Delta K = W_{\text{total}} = 8.0\times 10^{-1}\text{ J} \tag{11.6.9}$$

We can solve for the final speed,

$$v_{y,f} = \sqrt{\frac{2K_f}{m}} = \sqrt{\frac{2W_{\text{total}}}{m}} = \sqrt{\frac{2(8.0\times 10^{-1}\text{ J})}{0.2\text{ kg}}} = 2.9\text{ m}\cdot\text{s}^{-1}. \tag{11.6.10}$$

11.7 Power Applied by a Constant Force

Suppose that an applied force $\vec{F}_{\text{applied}}$ acts on a body during a time interval Δt, and displacement of the point of application of the force is in the x-direction by an amount Δx. The work done, ΔW, during this interval is

$$\Delta W = F_{\text{applied},x}\,\Delta x\,. \tag{11.7.1}$$

where $F_{\text{applied},x}$ is the x-component of the applied force. (Equation (11.7.1) is the same as Equation (11.4.2).)

The *average power* of this applied force is defined to be the rate at which work is done, so that

$$P_{ave} = \frac{\Delta W}{\Delta t} = \frac{F_{\text{applied},x}\,\Delta x}{\Delta t} = F_{\text{applied},x}\,v_{x,ave}\,. \tag{11.7.2}$$

So the average power delivered to the body is equal to the component of the force in the direction of motion times the component of the average velocity of the body. Power is a scalar quantity and can be positive, zero, or negative depending on the sign of work. The SI units of power are called watts $[\mathrm{W}]$ and $[1\ \mathrm{W}] = [1\ \mathrm{J}\cdot\mathrm{s}^{-1}]$.

Definition: **Instantaneous Power**

The *instantaneous power* at time t is defined to be the limit of the average power as the time interval $[t, t+\Delta t]$ approaches zero,

$$P = \lim_{\Delta t\to 0}\frac{\Delta W}{\Delta t} = \lim_{\Delta t\to 0}\frac{F_{\text{applied},x}\,\Delta x}{\Delta t} = F_{\text{applied},x}\left(\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}\right) = F_{\text{applied},x}\,v_x\,. \tag{11.7.3}$$

The instantaneous power of a constant applied force is the product of the component of the force in the direction of motion and the instantaneous velocity of the moving object.

11.7.1 Example: Gravitational Power for a Falling Object

Suppose a ball of mass $m = 0.2\ \mathrm{kg}$ starts from rest at a height $y_0 = 15\ \mathrm{m}$ above the surface of the earth and falls down to a height $y_f = 5.0\ \mathrm{m}$ above the surface of the earth. What is the average power exerted by the gravitation force? What is the instantaneous power when the ball is at a height $y_f = 5.0\ \mathrm{m}$ above the surface of the Earth? Make a graph of power vs. time. You may ignore the effects of air resistance.

Answer:

There are two ways to solve this problem. Both approaches require calculating the time interval Δt for the ball to fall. Set $t_0 = 0$ for the time the ball was released. We can solve for the time interval $\Delta t = t_f$ that it takes the ball to fall using the equation for a freely falling object that starts from rest,

$$y_f = y_0 - \frac{1}{2} g t_f^{\ 2} . \tag{11.7.4}$$

Thus the time interval for falling is

$$t_f = \sqrt{\frac{2}{g}(y_0 - y_f)} = \sqrt{\frac{2}{9.8 \text{ m} \cdot \text{s}^{-2}}(15 \text{ m} - 5 \text{ m})} = 1.4 \text{ s} . \tag{11.7.5}$$

First Approach:

We can calculate the work done by gravity,

$$\begin{aligned} W_{\text{grav}} &= -mg(y_f - y_0) \\ &= (-2.0 \times 10^{-1} \text{ kg})(9.8 \text{ m} \cdot \text{s}^{-2})(5 \text{ m} - 15 \text{ m}) = 2.0 \times 10^{1} \text{ J} \end{aligned} \tag{11.7.6}$$

Then the average power is

$$P_{\text{ave}} = \frac{\Delta W}{\Delta t} = \frac{2.0 \times 10^{1} \text{ J}}{1.4 \text{ s}} = 1.4 \times 10^{1} \text{ W} . \tag{11.7.7}$$

Second Approach:

We calculate the gravitation force and the average velocity. The gravitation force is

$$F_{y,\text{grav}} = -mg = -(2.0 \times 10^{-1} \text{ kg})(9.8 \text{ m} \cdot \text{s}^{-2}) = -2.0 \text{ N} . \tag{11.7.8}$$

The average velocity is

$$v_{y,\text{ave}} = \frac{\Delta y}{\Delta t} = \frac{5 \text{ m} - 15 \text{ m}}{1.4 \text{ s}} = -7.0 \text{ m} \cdot \text{s}^{-1} . \tag{11.7.9}$$

The average power is therefore

$$\begin{aligned} P_{ave} &= F_y \, v_{y,\text{ave}} = (-mg) v_{y,\text{ave}} \\ &= (-2.0 \text{ N})(-7.0 \text{ m} \cdot \text{s}^{-1}) = 1.4 \times 10^{1} \text{ W} \end{aligned} . \tag{11.7.10}$$

In order to find the instantaneous power at any time, we need to find the instantaneous velocity at that time. The ball takes a time $t_f = 1.4\ \text{s}$ to reach the height $y_f = 5.0\ \text{m}$. The velocity at that height is given by

$$v_y = -gt_f = -(9.8\ \text{m}\cdot\text{s}^{-2})(1.4\ \text{s}) = -1.4\times10^{1}\ \text{m}\cdot\text{s}^{-1}. \tag{11.7.11}$$

So the instantaneous power at any time $t_f = 1.4\ \text{s}$ is

$$\begin{aligned} P &= F_y v_y = (-mg)(-gt_f) = mg^2 t_f \\ &= (0.2\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2})^2(1.4\ \text{s}) = 2.7\times10^{1}\ \text{W} \end{aligned} \tag{11.7.12}$$

If this problem were done symbolically, the answers given in Equation (11.7.11) and Equation (11.7.12) would differ by a factor of two; the answers have been rounded to two significant figures.

The instantaneous power grows linearly with time so the graph of power vs. time is shown in Figure 11.13. From the figure, it should be seen that the instantaneous power at any time is half of the average power between $t = 0$ and that time.

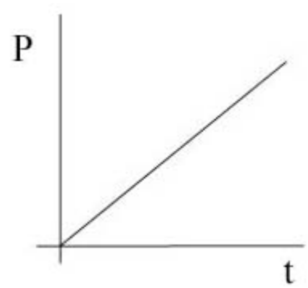

Figure 11.13 Graph of power vs. time.

7.9.2 Example: Power Pushing a Cup

A person pushes a cup of mass $0.2\ \text{kg}$ along a horizontal table with a force of magnitude $2.0\ \text{N}$ at an angle of 30^o with respect to the horizontal for a distance of $0.5\ \text{m}$, as in Example 7.4.3. The coefficient of friction between the table and the cup is $\mu_k = 0.1$. What is the average power of the pushing force? What is the average power of the kinetic friction force?

Answer:

We will use the results from Example 7.8.2 above, but keeping extra significant figures in the intermediate calculations. The work done by the pushing force is

$$W_{\text{applied}} = F_{\text{applied},x}\,(x_f - x_0) = (1.732\ \text{N})(0.50\ \text{m}) = 8.660\times 10^{-1}\ \text{J}\,. \qquad (11.7.13)$$

The final speed of the cup is $v_{x,f} = 2.860\ \text{m}\cdot\text{s}^{-1}$. Assuming constant acceleration, the time during which the cup was pushed is

$$t_f = \frac{2\left(x_f - x_0\right)}{v_{x,f}} = 0.3496\,\text{s}\,. \qquad (11.7.14)$$

The average power of the pushing force is then, with $\Delta t = t_f$,

$$\left(P_{\text{applied}}\right)_{\text{ave}} = \frac{\Delta W_{\text{applied}}}{\Delta t} = \frac{8.660\times 10^{-1}\ \text{J}}{0.3496\ \text{s}} = 2.340\ \text{W}\,, \qquad (11.7.15)$$

or $2.3\,\text{W}$ to two significant figures.

The work done by the friction force is

$$\begin{aligned} W_{\text{friction}} &= f_{\text{k}}\left(x_f - x_0\right) \\ &= -\mu_{\text{k}} N (x_f - x_0) = -\left(9.6\times 10^{-2}\ \text{N}\right)(0.50\ \text{m}) = -\left(4.8\times 10^{-2}\ \text{J}\right). \end{aligned} \qquad (11.7.16)$$

The average power of kinetic friction is

$$\left(P_{\text{friction}}\right)_{\text{ave}} = \frac{\Delta W_{\text{friction}}}{\Delta t} = \frac{-4.8\times 10^{-2}\ \text{J}}{0.3496\ \text{s}} = -1.373\times 10^{-1}\ \text{W} \qquad (11.7.17)$$

or $-1.4\times 10^{-1}\ \text{W}$ to two significant figures.

Time Rate of Change of Kinetic Energy and Power

The time rate of change of the kinetic energy for a body of mass m moving in the x-direction is

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2} m v_x^{\,2}\right) = m\frac{dv_x}{dt} v_x = m\, a_x v_x\,. \qquad (11.7.18)$$

By Newton's Second Law, $F_x = m\,a_x$, and so Equation (11.7.18) becomes

$$\frac{dK}{dt} = F_x v_x = P\,. \tag{11.7.19}$$

the instantaneous power delivered to the body is equal to the time rate of change of the kinetic energy of the body.